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A Note On Force Of Mortality

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Abstract: There are three widely used assumptions in actuarial science regarding the behavior of survivorship function, l_{x+t}, in the interval $0 \leq t \leq 1$, where x is an integral age. These assumptions are, namely, uniform distribution of death, Balducci assumption and constant force of mortality. All these assumptions lead to a continuous l_{x+t} for all $x \geq 0$ and $0 \leq t \leq 1$ but the corresponding forces of mortality are discontinuous at integral ages. The discontinuties in the forces of mortality at all integral ages do not make much sense. This paper suggests a new model for the distribution of deaths between two adjacent integral ages which takes care of the existing discontinuities. A necessary and sufficient condition for the existence of such model is derived and performance measures are discussed. Application of the model is shown and compared with the existing three popular models.

1. Introduction

In actuarial literature there exists many assumptions regarding the distribution of deaths in unit intervals. The trio, namely, uniform distribution of death, Balducci assumption and constant force of mortality are the few in vogue. The problem of discontinuities in the force of mortality at the integer ages (or unit intervals) are inherent characterstic of these three popular models. These models had beeen proposed by Wittstein [3]. Let us consider the models one by one. Note that throughout the text x will be treated as an integral age.

In the uniform distribution of death model, the deaths between integer ages are assumed to be distributed linearly. Under this assumption:

$$l^u_{x+t} = l_x - t.d_x, \tag{1}$$

$$\begin{aligned} {}_tq^u_x &= t.q_x, \\ \mu^u_{x+t} &= \frac{q_x}{1 - t.q_x}, \\ {}_tp^u_x.\mu^u_{x+t} &= q_x, \qquad \text{where} \quad 0 \le t \le 1. \end{aligned} \tag{2}$$

Notable among these results are (i) l^u_{x+t} is a piecewise straight line on $0 \le t \le 1$ and continuous throughout and (ii) μ^u_{x+t} is an increasing function on $0 \le t \le 1$, convex upaward and discontinuous at integral ages. For more discussions see Batten [1].

The Balducci assumtion, named after an Italian actuary, does not have a simple physical description like uniform distribution of death. However, it is a reasonable indicator of the survivorship functions in many instances. This assumption leads to the following results:

$$l^b_{x+t} = \frac{l_x . l_{x+1}}{l_{x+1} + t.d_x}, \tag{3}$$

$${}_{1-t}q^b_x = (1-t).q_x,$$

$$\mu^b_{x+t} = \frac{q_x}{1-(1-t).q_x}, \tag{4}$$

$${}_{1-t}p^b_{x+t}.\mu^b_{x+t} = q_x, \qquad \text{where} \quad 0 \le t \le 1.$$

Notable among these results are (i) l^b_{x+t} is a piecewise hyperbola in $0 \le t \le 1$ and continuous throughout and (ii) μ^u_{x+t} is a decreasing function on $0 \le t \le 1$, convex upaward and discontinuous at intergeral ages. For detail discussion see Gershenson [2].

The third mortality assumption, constant force of mortality, has a sense intermediate of the other two already discussed. The assumption leads to the following results:

$$l^c_{x+t} = l_x.e^{-\mu_x t}$$

$$= l_x . p^t_x, \tag{5}$$

$${}_{1-t}q^c_x = 1 - e^{-\mu_x(1-t)},$$

$$\mu^c_{x+t} = \mu_x, \qquad \text{where} \quad 0 \le t \le 1. \tag{6}$$

The discontinuities in the force of mortality at integral ages and its piecewise linearity over unit interval is clear. Like Balducci model, l^c_{x+t} is a piecewise hyperbola and continuous throughout. For more details see Batten [1].

2. Model development

The usual actuarial symbols used are:

l_x = the number of persons at age x;

d_x = the number of persons expected to die between x to x+1;

q_x = the probability that a person aged x dies before age x+1;

μ_x = the annualized force of mortality at age x.

Suppose the deaths are distributed at a rate $f(t)$ in the interval x to $x+1$ and l^h_{x+t} is the number of persons at age $x+t$ under current assumption, where $0 \leq t \leq 1$. Then

$$l^h_{x+t} = l_x - f(t)d_x. \tag{6}$$

Eq(6) must satisfy the following boundary conditions in order to have continuous survivorship function and continuous force of mortality at all ages. They are:

(1) $l^h_{x+t}|_{t=0} = l_x$,

(2) $l^h_{x+t}|_{t=1} = l_{x+1}$,

(3) $\mu^h_{x+t}|_{t=1} = \mu^h_{x+1+t}|_{t=0}$.

Conditions (1), (2) & (3) ensures the continuity of l^h_{x+t} and μ^h_{x+t} for all $x \geq 0$ and $0 \leq t \leq 1$. The above conditions imply that the $f(t)$ must satisfy the following conditions obtained from eq(6):

(i) $f(0) = 0$,

(ii) $f(1) = 1$,

(iii) $\dfrac{f'(1)}{f'(0)} = \dfrac{d_{x+1}}{d_x}$.

From condition (iii) it is clear that the $f(t)$ is dependent on the integral age x.

Now let us divert our attention to the case where $f(t)$ is based on age x and denote it by $f_x(t)$. In order to satisfy the above three conditions the function $f_x(t)$ must be more than a second degree polynomial in t. Assuming it to be a polynomial of degree three we have

$$f_x(t) = a_x + b_x t + c_x t^2. \tag{7}$$

Now we want to express $f_x(t)$ in terms of life table function using the conditions (i), (ii) and (iii). This is discussed in the following theorem.

Theorem1. In terms of life table function, l^s_{x+t} is given by

$$l^h_{x+t} = l_x - [(t - \frac{t^2}{2})B_x + \frac{t^2}{2}B_{x+1}],$$

where $B_x = 2 \sum_{i=0}^{w-x-1} (-1)^i d_{x+i}$.

Proof: Using conditions (i), (ii) and (iii) in eq(7) we have

$$f_x(t) = (1 - c_x)t + c_x t^2,$$

where c_x and c_{x+1} have the recursive relation

$$c_x = (1 - c_{x+1})\frac{d_{x+1}}{d_x} - 1. \tag{8}$$

After some algebra and using eq(8) as the link between $f_x(t)$'s we have:

$$f_x(t) = -f_{x+1}(t)\frac{d_{x+1}}{d_x} + 2t + \left(\frac{d_{x+1}}{d_x} - 1\right)t^2$$

$$
\begin{aligned}
&= f_{x+2}(t)\frac{d_{x+2}}{d_x} + 2t\left(1 - \frac{d_{x+1}}{d_x}\right) + \left(2\frac{d_{x+1}}{d_x} - \frac{d_{x+2}}{d_x} - 1\right)t^2 \\
&= - f_{x+3}(t)\frac{d_{x+3}}{d_x} + 2t\left(1 - \frac{d_{x+1}}{d_x} + \frac{d_{x+2}}{d_x}\right) + \left(2\frac{d_{x+1}}{d_x} - 2\frac{d_{x+2}}{d_x} + \frac{d_{x+3}}{d_x} - 1\right)t^2 \\
&= f_{x+4}(t)\frac{d_{x+4}}{d_x} + 2t\left(1 - \frac{d_{x+1}}{d_x} + \frac{d_{x+2}}{d_x} - \frac{d_{x+3}}{d_x}\right) + \\
&\quad \left(2\frac{d_{x+1}}{d_x} - 2\frac{d_{x+2}}{d_x} + 2\frac{d_{x+3}}{d_x} - \frac{d_{x+4}}{d_x} - 1\right)t^2 \\
&\quad \vdots \\
&= f_w(t)\frac{d_w}{d_x} + 2t\left(1 - \frac{d_{x+1}}{d_x} + \frac{d_{x+2}}{d_x} - \frac{d_{x+3}}{d_x} + \ldots (-1)^{w-x-1}\frac{d_{w-1}}{d_x}\right) + \\
&\quad \left(2\frac{d_{x+1}}{d_x} - 2\frac{d_{x+2}}{d_x} + 2\frac{d_{x+3}}{d_x} - 2\frac{d_{x+4}}{d_x} + \ldots + (-1)^{w-x-1}2\frac{d_{w-1}}{d_x} + \right. \\
&\quad \left. (-1)^{w-x}\frac{d_w}{d_x} - 1\right)t^2,
\end{aligned}
$$

where w is the limiting age, ie, $x < w$. Since $d_w = 0$, we have

$$
\begin{aligned}
f_x(t) &= 2t\left(1 - \frac{d_{x+1}}{d_x} + \frac{d_{x+2}}{d_x} - \frac{d_{x+3}}{d_x} + \ldots (-1)^{w-x-1}\frac{d_{w-1}}{d_x}\right) + \\
&\quad \left(2\frac{d_{x+1}}{d_x} - 2\frac{d_{x+2}}{d_x} + 2\frac{d_{x+3}}{d_x} - 2\frac{d_{x+4}}{d_x} + \ldots + (-1)^{w-x-1}2\frac{d_{w-1}}{d_x} - 1\right)t^2 \\
&= \frac{2t}{d_x}\sum_{i=0}^{w-x-1}(-1)^i d_{x+i} - \frac{2t^2}{d_x}\sum_{i=0}^{w-x-1}(-1)^i d_{x+i} + t^2 \\
&= t\frac{B_x}{d_x} - t^2\frac{B_x}{d_x} + t^2.
\end{aligned}
$$

Using the fact that $B_x + B_{x+1} = 2d_x$, we have

$$\begin{aligned} l^h_{x+t} &= l_x - f_x(t)d_x \\ &= l_x - [(t - \frac{t^2}{2})B_x + \frac{t^2}{2}B_{x+1}], \qquad \text{where} \quad 0 \le t \le 1. \end{aligned} \tag{9}$$

$\sqcup$

Lemma 1. The function l^h_{x+t} is a decreasing function in $0 \le t \le 1$ iff $B_x > 0$ and $B_{x+1} > 0$ for all integer $x > 0$.

Proof: Clearly,

$$l^{h'}_{x+t} = -[(1-t)B_x + tB_{x+1}]. \tag{10}$$

Now to prove the lemma it suffices to show that the right of side of the eq(10) is negative for all $0 \le t \le 1$ iff $B_x > 0$ and $B_{x+1} > 0$.

To determine whether $l^{h'}_{x+t}$ is positive, negative or none for $0 \le t \le 1$, we need to know the sign nature of B's. Since B's are any real number we may consider the following four possibilities:

(1) $B_x < 0$ and $B_{x+1} < 0$,

(2) $B_x < 0$ and $B_{x+1} > 0$,

(3) $B_x > 0$ and $B_{x+1} < 0$,

(4) $B_x > 0$ and $B_{x+1} > 0$.

Suppose condition (1) holds, then for all $0 \le t \le 1$, $l^{h'}_{x+t}$ is positive. If condition (2) holds, then for t sufficiently close to zero, $l^{h'}_{x+t}$ is positive. Similarly, if condition

(3) holds then for t sufficiently close to one, $l^{h'}_{x+t}$ is positive. Condition (4) is the only situation where $l^{h'}_{x+t}$ is negative for all $0 \leq t \leq 1$, which proves the lemma.

□

Lemma 2. If the statement in lemma 1 holds then the survivorship function l^{h}_{x+t} in eq(9) is always positive in x and t.

Proof: It can be shown easily that $B_{x+i}+B_{x+1+i} = 2d_{x+i}$ for all $x \geq 0$. Taking the sum on both sides of the equation $B_{x+i}+B_{x+1+i} = 2d_{x+i}$ over i, $i = 0,1,2,\ldots, w-x-1$, it may be shown that

$$l_x = \frac{B_x}{2} + \sum_{i=1}^{w-x-1} B_{x+i}.$$

Therefore from eq(9) we have

$$\begin{aligned}
l^{h}_{x+t} &= l_x - \left[(t-\frac{t^2}{2})B_x + \frac{t^2}{2}B_{x+1}\right] \\
&= \frac{B_x}{2} + \sum_{i=1}^{w-x-1} B_{x+i} - \left[(t-\frac{t^2}{2})B_x + \frac{t^2}{2}B_{x+1}\right] \\
&= (\frac{1}{2} - t + \frac{t^2}{2})B_x + (1-\frac{t^2}{2})B_{x+1} + \sum_{i=2}^{w-x-1} B_{x+i} \\
&\geq 0, \qquad \text{if lemma 1 holds.}
\end{aligned}$$

□

3. Performance measures

The following results are easily obtainable from the assumptions of the new model.

$$
{}_tp_x^h = 1 - \left[(t - \frac{t^2}{2})Q_x + \frac{t^2}{2}p_x Q_{x+1}\right],
$$

$$
{}_tq_x^h = (t - \frac{t^2}{2})Q_x + \frac{t^2}{2}p_x Q_{x+1}, \tag{11}
$$

$$
{}_{1-s}q_{x+t}^h = \frac{1-s}{2}\frac{(1+s-2t)Q_x + (1-s+2t)p_x\, Q_{x+1}}{1 - \left[(t - \frac{t^2}{2})Q_x + \frac{t^2}{2}p_x Q_{x+1}\right]},
$$

$$
\begin{aligned}
\mu_{x+t}^h &= \frac{(1-t)B_x + B_{x+1}}{l_x - \left[(t - \frac{t^2}{2})B_x + \frac{t^2}{2}B_{x+1}\right]} \\
&= \frac{(1-t)Q_x + t\, p_x Q_{x+1}}{1 - \left[(t - \frac{t^2}{2})Q_x + \frac{t^2}{2}p_x Q_{x+1}\right]}
\end{aligned} \tag{12}
$$

and $\quad {}_tp_x^h . \mu_{x+t}^h = (1-t)Q_x + t.\, p_x Q_{x+1},$

where $\quad Q_y = \frac{B_y}{l_y} = 2\sum_{i=0}^{w-y-1}(-1)^i\frac{d_{y+i}}{l_y}, \qquad 0 \le t \le 1 \quad \text{and} \quad 0 \le t \le s \le 1.$

4. Application of the model

Consider the set of data given in table 1. It is a ficticious set of data made to satisfy the result of lemma 1. This data set will be used to compare the shapes of survivorship functions, l_{x+t}, and force of mortality, μ_{x+t}, under different assumptions.

Table 1

x	l_x	d_x
0	100	11
1	89	17
2	72	23
3	49	20
4	29	17
5	12	12

The table 2 shows the survivorship functions, l_{x+t}'s, for x=0, 1, 2, 3, 4, 5 and $0 \leq t \leq 1$ under the four assumptions. The survivorship functions are obtained by using eqs (1), (3), (5) and (9), and the data set of table 1.

Table 2

x	l^h_{x+t}	l^u_{x+t}	l^b_{x+t}	l^c_{x+t}
0	$100 - 4t - 7t^2$	$100 - 11t$	$\frac{8900}{89+11t}$	$100(\frac{89}{100})^t$
1	$89 - 18t + t^2$	$89 - 17t$	$\frac{6408}{72+17t}$	$89(\frac{72}{89})^t$
2	$72 - 16t - 7t^2$	$72 - 23t$	$\frac{3528}{49+23t}$	$72(\frac{49}{72})^t$
3	$49 - 30t + 10t^2$	$49 - 20t$	$\frac{1421}{29+20t}$	$49(\frac{29}{49})^t$
4	$29 - 10t - 7t^2$	$29 - 17t$	$\frac{348}{12+17t}$	$29(\frac{12}{29})^t$
5	$12 - 24t + 12t^2$	$12 - 12t$	0	0

The graphs of the survivorship functions for each unit interval under different assumptions are shown in figure 1.

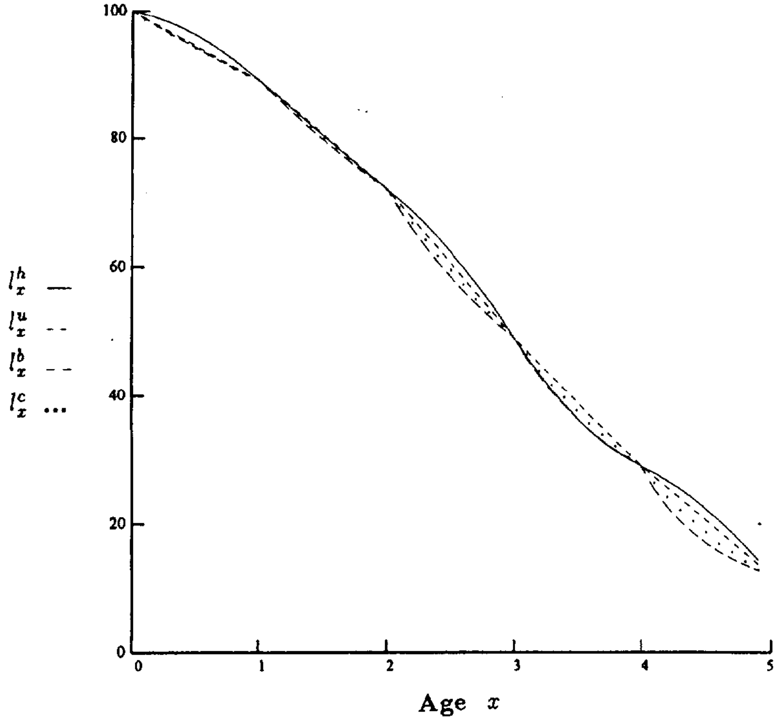

Figure 1. Survivorship function under different assumptions.

It clearly shows that l_{x+t}^h is very flexible with respect to its shape. In each unit interval the curve is a non-increasing function with no particular pattern from interval to interval. The other survivorship functions show a definite pattern in each unit interval all through.

Table 3 shows the forces of mortality obtained under different assumptions using eqs (2), (4), (6) and (12).

Table 3

x	μ^h_{x+t}	μ^u_{x+t}	μ^b_{x+t}	μ^c_{x+t}
0	$\frac{4+14t}{100-4t-7t^2}$	$\frac{11}{100-11t}$	$\frac{11}{89+11t}$	$-\log(\frac{89}{100})$
1	$\frac{18-2t}{89-18t+t^2}$	$\frac{17}{89-17t}$	$\frac{17}{72+17t}$	$-\log(\frac{72}{89})$
2	$\frac{16+14t}{72-16t-7t^2}$	$\frac{23}{72-23t}$	$\frac{23}{49+23t}$	$-\log(\frac{49}{72})$
3	$\frac{30-20t}{49-30t+10t^2}$	$\frac{20}{49-20t}$	$\frac{20}{29+20t}$	$-\log(\frac{29}{49})$
4	$\frac{10+14t}{29-10t-7t^2}$	$\frac{17}{29-17t}$	$\frac{17}{12+17t}$	$-\log(\frac{12}{29})$
5	$\frac{24-24t}{12-24t+12t^2}$	$\frac{12}{12-12t}$	$\frac{1}{t}$	∞

The graph of the forces of mortality under different assumptions for the data set of table 1 is given in figure 2.

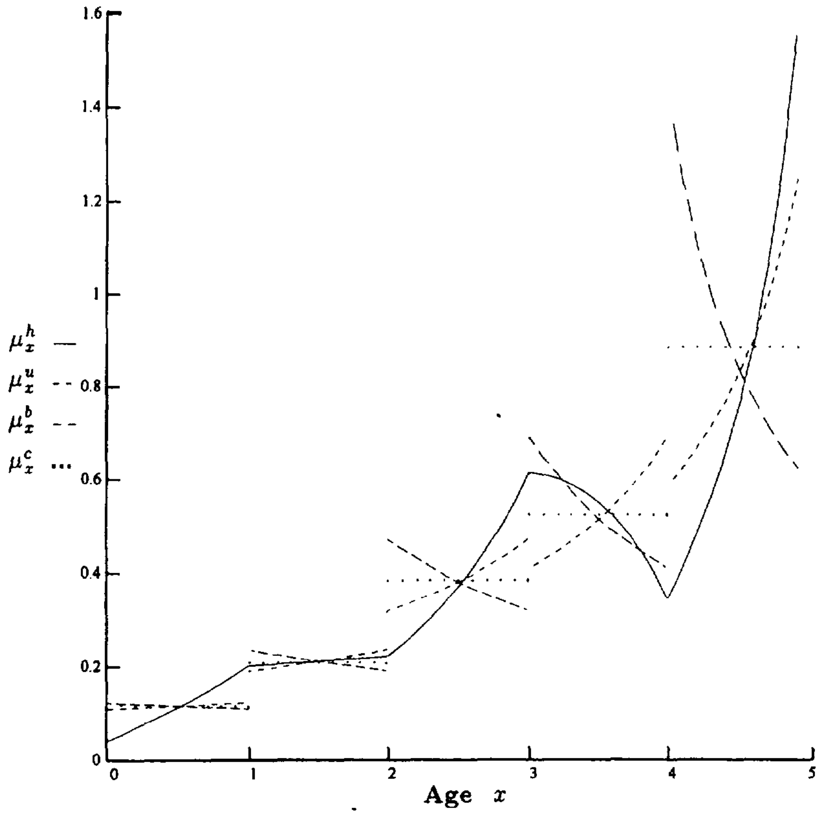

Figure 2. Force of mortality under different assumptions.

The figure 2 shows that all but in the new model the forces of mortality is continous through out. Also unlike the other models already discussed the new model does not show any pattern from interval to interval.

5. Conclusion

A new model for the distribution of deaths over unit interval is described. A necessary and sufficient condition for the existence of such a model is derived. The model is found to be very flexible with respect to its shape and unlike other models dicussed in this paper it does not have any pattern in the survivorship function or in the force of mortality. Also the force of mortality is a continuous function, which was our main concern, for all $x \geq 0$ and $0 \leq t \leq 1$. This makes the model practical and more meaningful in its applications. The only major problem with this model is, it exists under certain constrains. However, the model can be used in a realistic and effective manner whenever exists.

References

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[2] Gershenson, Harry. *Measurement of Mortality*, Chicago: Society of Actuaries, 1961.

[3] Wittstein, 1862. (See page 68 of the text Actuarial Mathematics by Bowers *et al*).