ACTUARIAL RESEARCH CLEARING HOUSE
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"Using Interest Tables to Find Polynomial Zeroes"

INTRODUCTION

It is rare that one can use ideas from such diverse topics as theory of interest, linear algebra, and theory of equations to obtain a solution to a common mathematical problem. However, the perspective that we take in this paper allows us to do just that; namely, we will show how one may find real solutions to polynomial equations by using the tabulated data of the accumulated value of annuities-certain, $s_{\overline{n}|i}$, given in most "Business Mathematics" textbooks.

OUTLINE OF PROCEDURE

Given a polynomial $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n$, with a real root, one can make a suitable change of variable of the form $x = X + M$, where we assume a real zero of $f(x)$ lies in the interval $[M+1, M+2)$, to change $f(x)$ into $g(X)$, where $g(X)$ has a zero in $[1, 2)$.

Consider $b_n(x) = 1 + x + \ldots + x^{n-1}$, it is known from elementary linear algebra [1] that the $b_n(x)$'s, $n = 1, 2, \cdots$, form a basis for the vector space of polynomials in x with coefficients in $\mathbf{R}$; that is, for $\mathbf{R}[x]$. Thus, we can write $g(X)$ as a linear combination of the $b_n(X)$'s, obtaining $L(b_n(X))$.

Notice that if we let $X = 1 + i$, then for $n = 1, 2, \ldots$,

$$b_n(X) = b_n(1+i) = 1 + (1+i) + \ldots + (1+i)^{n-1} = s_{\overline{n}|i}$$

Characteristically, the $s_{\overline{n}|i}$ are listed in "Theory of Interest" textbooks, such as [2], for various values of n and i.

We use the tabulated values of $s_{\overline{n}|i}$ and the linear combination $L(b_n(1+i))$ to obtain the desired root. A complete example follows.

EXAMPLE

Consider the polynomial,

$$f(x) = 50x^3 - 51x^2 - 152x - 453.$$

We want to solve

(1)

$$f(x) = 0$$

by the method outlined above.

Since

$$\begin{aligned} f(3) &= 50 \cdot 27 - 51 \cdot 9 - 152 \cdot 3 - 453 \\ &= 1350 - 459 - 456 - 453 = -18 < 0, and \\ f(4) &= 50 \cdot 64 - 51 \cdot 16 - 152 \cdot 4 - 453 \\ &= 3200 - 816 - 608 - 453 = 1323 > 0, \end{aligned}$$

we can make the transformation $x = X + 2$, to get

$$g(X) = f(X+2) = 50X^3 + 249X^2 + 244X - 561.$$

Notice that

$$\begin{aligned} g(1) &= 50 + 249 + 244 - 561 = -18 < 0, and \\ g(2) &= 50 \cdot 8 + 249 \cdot 4 + 244 \cdot 2 - 561 = 1323 > 0. \end{aligned}$$

Thus, $g(X)$ has the desired property of having a root in $[1, 2)$.

Now

$g(X) = 50b_4(X) + (249 - 50)b_3(X) + (244 - 249)b_2(X) + (-561 - 244)$

(2)

$g(X) = 50b_4(X) + 199b_3(X) - 5b_2(X) - 805 = L(b_4(X))$

That is, the linear combination (2) expresses $g(X)$ as a linear combination of $b_n(X)$'s.

Letting $X = 1 + i$, equation (2) becomes

$$\begin{aligned} g(1+i) &= 50s_{\overline{4}|i} + 199s_{\overline{3}|i} - 5s_{\overline{2}|i} - 805s_{\overline{1}|i} \\ &= 50s_{\overline{4}|i} + 199s_{\overline{3}|i} - 5s_{\overline{2}|i} - 805. \end{aligned}$$

Thus, to solve (1) we can solve

$$g(1+i) = 0,$$

or equivalently,

(4)

$$50s_{\overline{4}|i} + 199s_{\overline{3}|i} - 5s_{\overline{2}|i} - 805 = 0.$$

By using Appendix 1 of [2], we see that for $i = 2\%$, the left hand side of (4) becomes

$$50(4.1216 + 199(3.0604) - 5(2.02) - 805.$$

This expression is equal to $804.9996 - 805$, which can be rounded to 0.

Thus, $1 + i = 1.02$ and $g(1.02) = 0$. This implies that $x = X + 2 = 1.02 + 2 = 3.02$ is the desired solution to (1).

[1] The Theory of Interest, 4th Ed., Kolman, Macmillan, 1986.

[2] Stephen G. Kellison, Elementary Linear Algebra, Irwin, 1970.