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Annuities with Negative Payment Frequency

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A negative with positive results

I recently noticed that allowing m to become negative in the formula for

$$\ddot{a}_{\overline{n}|}^{(-m)}$$

produces an annuity immediate payable m times per year, i.e.

$$\ddot{a}_{\overline{n}|}^{(-m)} = a_{\overline{n}|}^{(m)}.$$

This can be seen by noticing that

$$d^{(-m)} = i^{(m)}$$

This relationship could be put to good use, for example, in the design of financial functions for spreadsheet programs. When a parameter is added to allow variation in the payment frequency, the function is automatically generalized for both due and immediate.

It is interesting that the relationship holds in at least two cases for life annuities.

(a) the usual approximation

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}$$

(b) assuming U.D.D.

$$\ddot{a}_x^{(m)} = \alpha(m) * \ddot{a}_x - \beta(m)$$

where

$$\alpha(m) = \frac{i * d}{i^{(m)} * d^{(m)}} \quad \beta(m) = \frac{i - i^{(m)}}{i^{(m)} d^{(m)}}.$$

Robert A. Alps

The observations

$$d^{(-m)} = i^{(m)}$$

and

$$\ddot{a}^{(-m)}_{\overline{n|}} = a^{(m)}_{\overline{n|}} \tag{1}$$

can also be found on page 573 of [2]. Mr. Alps's conjecture

$$\ddot{a}^{(-m)}_{x} = a^{(m)}_{x}$$

can be derived as follows. Let T denote the time-until-death random variable [1, Chapter 3]. Let

$\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling function and floor function, respectively. Then

$$\ddot{a}^{(k)}_{x} = E(\ddot{a}^{(k)}_{\overline{\lceil kT \rceil / k|}})$$

and

$$a^{(k)}_{x} = E(a^{(k)}_{\overline{\lfloor kT \rfloor / k|}}).$$

It follows from

$$\frac{\lceil (-m)T \rceil}{-m} = \frac{\lfloor mT \rfloor}{m} \tag{2}$$

and (1) that

$$\ddot{a}^{(-m)}_{x} = a^{(m)}_{x}.$$

For each pair of real numbers c and d, let $c \wedge d$ denote the minimum of the numbers. Then

$$\ddot{a}^{(k)}_{x:\overline{n|}} = E(\ddot{a}^{(k)}_{\overline{n \wedge \lceil kT \rceil / k|}})$$

and

$$a^{(k)}_{x:\overline{n|}} = E(a^{(k)}_{\overline{n \wedge \lfloor kT \rfloor / k|}}).$$

It now follows from (1) and (2) that

$$\ddot{a}^{(-m)}_{x:\overline{n|}} = a^{(m)}_{x:\overline{n|}}. \tag{3}$$

Another approach to prove (3) is to express the temporary life annuities as Riemann integrals:

$$\ddot{a}^{(k)}_{x:\overline{n}|} = \int_0^n {}_{\lfloor kt \rfloor/k}E_x \, dt$$

and

$$a^{(k)}_{x:\overline{n}|} = \int_0^n {}_{\lceil kt \rceil/k}E_x \, dt.$$

Then apply the formula $\lceil -\xi \rceil = -\lfloor \xi \rfloor$.

A third proof for (3) is to write the temporary life annuities as Stieltjes integrals:

$$\ddot{a}^{(k)}_{x:\overline{n}|} = \int_0^n {}_tE_x \, d(\lceil kt \rceil / k)$$

and

$$a^{(k)}_{x:\overline{n}|} = \int_0^n {}_tE_x \, d(\lfloor kt \rfloor / k).$$

Remarks

(i) It can be shown that

$$\alpha(-m) = \alpha(m)$$

and

$$\beta(-m) = \beta(m) + m^{-1}.$$

Furthermore, the function $\gamma(.)$ defined in #5.18 on page 152 of [1] satisfies the relation

$$\gamma(-m) = \gamma(m) + m^{-1}.$$

(ii) According to #5.19 on page 153 of [1],

$$\gamma(m) = \alpha(m) - \beta(m) - m^{-1}.$$

Consequently,

$$\gamma(m) = \alpha(m) - \beta(-m)$$

$$= \alpha(-m) - \beta(-m).$$

This result relates to Mr. Alps's case (b).

(iii) For further discussion on the functions $\alpha(.)$, $\beta(.)$ and $\gamma(.)$, we refer the interested reader to [3].

(iv) Because of the formula

$$d^{(-m)} = i^{(m)},$$

we immediately have

$$\ddot{a}^{\{-m\}}_{x:\overline{n}|} = \mathring{a}^{(m)}_{x:\overline{n}|}.$$

See [1, Example 5.12].

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References

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2. Shiu, E.S.W. "Integer Functions and Life Contingencies," *Transactions, Society of Actuaries*, 34 (1982), 571-590; Discussion 591-600.
3. Shiu, E.S.W. "Power Series of Annuity Coefficients," *Actuarial Research Clearing House*, 1987.1, 23-31.