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How Fast Is Mexico Increasing?

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Question #3.36 on page 59 of the Course 161 textbook *Demography Through Problems* states:

> Sixty percent of the population of Mexico is under 25 years of age, and we can suppose that in its life table the fraction under 25 is 35 percent, the mean age [of the Mexican population] is 35 and the variance of ages [of the Mexican population] is 550. How fast is Mexico increasing?

Under the assumption of stable population, the answer given on page 72 of the book is that the continuous rate of increase, r, is 3.09% per annum. In this note we shall point out that the correct answer should be 1.83% per annum. There is a sign error in the book's solution.

The information given in the statement of the problem is translated as follows:

$$0.6 = \frac{\int_0^{25} e^{-rx}\, l(x)\, dx}{\int_0^{\infty} e^{-rx}\, l(x)\, dx},$$

$$0.35 = \frac{\int_0^{25} l(x)\, dx}{\int_0^{\infty} l(x)\, dx} = \frac{{}_{25}L_0}{{}_{\infty}L_0},$$

$$35 = m = \frac{\int_0^{\infty} x\, e^{-rx}\, l(x)\, dx}{\int_0^{\infty} e^{-rx}\, l(x)\, dx}$$

and

$$550 = \sigma^2 = \frac{\int_0^\infty (x-m)^2 e^{-rx}\, l(x)\, dx}{\int_0^\infty e^{-rx}\, l(x)\, dx}.$$

Furthermore, it was suggested in the solution to the problem (page 72 of the book) that

$$10 = m_{-25} = \frac{\int_0^{25} x\, e^{-rx}\, l(x)\, dx}{\int_0^{25} e^{-rx}\, l(x)\, dx}$$

and

$$\left(\frac{25}{3}\right)^2 \approx \sigma_{-25}^2 = \frac{\int_0^{25} (x - m_{-25})^2 e^{-rx}\, l(x)\, dx}{\int_0^{25} e^{-rx}\, l(x)\, dx}.$$

Actually, the approximation

$$\sigma_{-25}^2 \approx 60$$

was used in the book's solution. The problem is to estimate r.

A key idea for solving such problems is the *cumulant expansion formula*. For a given random variable X, let

$$\log_e [E(e^{tX})] = \kappa_1 t + \kappa_2 t^2/2! + \kappa_3 t^3/3! + \ldots ;$$

then κ_1, κ_2 and κ_3 are the mean, variance, and third moment about the mean, respectively, of X. A proof of this result can be found on page 335 of the Course 150 textbook *Actuarial Mathematics;* also see page 100 in Volume 38 (1986) of *T.S.A.*

For $0 \le a < b$, define

$$\gamma(s) = \int_a^b e^{-sx}\, l(x)\, dx .$$

Note that $\gamma(0) = {}_{b-a}L_a$. Applying the cumulant expansion formula with $e^{-rx} l(x)/\gamma(r)$, $a \le x \le b$, as probability density function yields

$$\log_e[\gamma(r-t)/\gamma(r)] \approx \kappa_1 t + \kappa_2 t^2/2!$$

or

$$\gamma(r) = \gamma(r-t)\exp(-\kappa_1 t - \kappa_2 t^2/2).$$

Thus

$$\gamma(r) = \gamma(0)\exp(-\kappa_1 r - \kappa_2 r^2/2). \quad (1)$$

It follows from (1) that

$$\int_0^{25} e^{-rx}\, l(x)\, dx = {}_{25}L_0 \exp(-m_{-25} r - \sigma^2_{-25} r^2/2)$$

and

$$\int_0^{\infty} e^{-rx}\, l(x)\, dx = {}_{\infty}L_0 \exp(-mr - \sigma^2 r^2/2).$$

Combining these two equations by division yields

$$0.6 = 0.35 \exp[(35-10)r + (550-60)r^2/2]$$

or

$$0.539 = 25r + 245r^2,$$

which gives $r = 1.828\%$ as the positive answer.

On the other hand, suppose that we interpret the means and variances as those of the corresponding stationary population (the underlying life table), i.e.,

$$35 = \mathring{e}_0 = \frac{\int_0^{\infty} x\, l(x)\, dx}{\int_0^{\infty} l(x)\, dx},$$

$$550 = \frac{\int_0^{\infty} (x-\mathring{e}_0)^2\, l(x)\, dx}{\int_0^{\infty} l(x)\, dx},$$

$$10 = \varphi = \frac{\int_0^{25} x\, l(x)\, dx}{\int_0^{25} l(x)\, dx}$$

and

$$60 \approx \frac{\int_0^{25} (x - \varphi)^2 \, l(x)\, dx}{\int_0^{25} l(x)\, dx}.$$

Applying the cumulant expansion formula with $l(x)/\gamma(0)$, $a \le x \le b$, as probability density function yields

$$\log_e[\gamma(r)/\gamma(0)] \approx -\kappa_1 r + \kappa_2 r^2/2!$$

or

$$\gamma(r) \approx \gamma(0)\exp(-\kappa_1 r + \kappa_2 r^2/2). \quad (2)$$

Note that the cumulants, κ_1 and κ_2, in (2) are different from those in (1). Applying (2) twice and dividing, we have

$$0.6 \approx 0.35 \exp[(35 - 10)r + (60 - 550)r^2/2]$$

or

$$0.539 \approx 25r - 245r^2,$$

which yields the book's answer $r = 3.09\%$. This is the derivation given on page 66 in the 1989.2 issue of *A.R.C.H.*

Remark

In the solution to #3.31 on page 70 of *Demography Through Problems*, the factor

$$(\sigma_{60}^2 - \sigma^2)$$

should be

$$(\sigma^2 - \sigma_{60}^2).$$

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