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Actuarial Research Clearing House (ARCH)
Re: Rectangularization of the Survivorship Curve - Fact Versus Appearances
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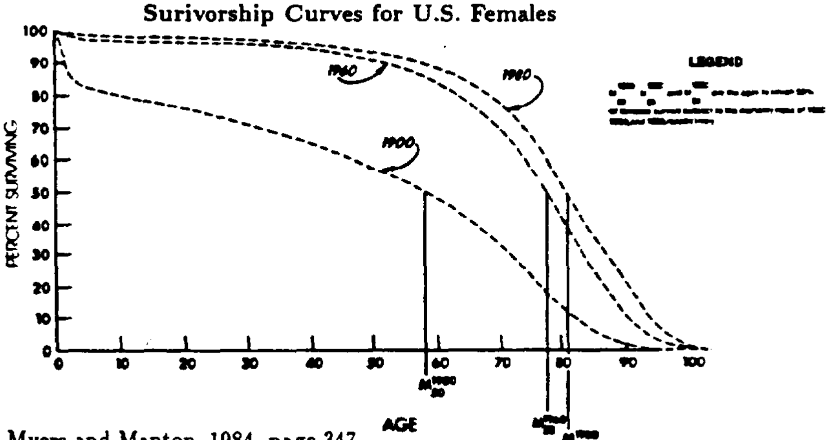
Dear Colleague,

I have an interesting problem on which I seek your assistance. Over a year ago I became curious about the question of whether there was a finite value 'w' for the surivorship curve that would inevitably lead to the 'rectangularization' of that curve as mortality improved.

Discussions about the apparent 'rectangularization' of the survivorship curve have appeared in the literature as early as 1923 (Pearl, 1923). Certainly even a casual glance at the survivorship curves of any developed nation for this century will show obvious evidence of the rectangularization process, as indicated in Figure 1.

FIGURE 1

Surivorship Curves for U.S. Females



SOURCE: Myers and Manton, 1984, page 347.

Some have gone so far as to present mathematical evidence of the validity of the rectangularization theory as indicated in the following table.

TABLE 1

Life Expectancy and Standard Deviation of Age at Death

	MORTALITY TABLE		
	American Experience	1941 CSO	1958 CSO
Experience period	1843-58	1930-40	1950-54
$\mathring{e}_{35}$ (male)	31.78 years	33.44 years	36.69 years
Standard deviation	13.74 years	13.06 years	12.59 years

SOURCE: Rappaport and Plumley, 1978, page 259

One of the stongest supporters of the concept of rectangularization is James F. Fries (see for example, Fries, 1980). Fries maintains that while mean life expectancies have increased remarkably this century, the maximum age of possible human existence has not changed very much, if at all. Hence, he concludes that there is a natural finite limit, w, to human life and that our goal is to strive to get the population survivorship curve as close to w as possible.

These broad-based conclusions have been criticized however. For example, Myers and Manton (1984, page 347) state:

> If one examines the total survival curves from birth onward, they may suggest some rectangularization. This, however, combines the effects of two very different types of mortality reduction, i.e., those due to declines in infant and child mortality and those from chronic disease mortality reductions at later ages. Not only are these phenomena quite different, but they have occurred at different times, i.e., infant and child mortality declined rapidly from 1930 to 1960 to presently low levels, whereas declines in mortality at later ages are more recent, starting after 1960 and associated by many investigators with significant declines in cir-

culatory disease mortality after 1968. The figure also clearly shows a stretching outwards of the curves over time and a shift on the abscissa toward higher ages.

The question remains, therefore, as to whether the survivorship curve is anchored to a finite upper bound, w, which will lead to an inevitable rectangularization of the survivorship curve at advanced ages with continued mortality improvement.

Facts versus Appearances

To respond to this question through empirical analysis, Myers and Manton (1984) suggest that one calculate the standard deviation of the mean life expectancy at advanced ages. If the standard deviation decreases as life expectancy increases, then there is truly evidence of rectangularization. If, on the other hand, the standard deviation increases along with the increase in life expectancy then one would be forced to conclude that there is no finite upper bound, w.

The mathematics required for this calculation is well-defined. For example, the text *Actuarial Mathematics* (Bowers et al., 1986, page 65) indicates that for a given life table, at age x:

$$E[K] = e_x = \sum_{K=0}^{\infty} {}_{K+1}p_x \qquad (3.5.5)$$

and

$$Var[K] = \sum_{K=0}^{\infty} (2K+1)_{K+1}p_x - e_x^2$$

It was decided to apply these formulae to several Canadian Life Tables at a variety of advanced ages. One word of caution is required in interpreting or repeating such analysis however. Virtually every life table is affected by some graduation process. Further, life tables are usually "closed out" by some artificial process. This is true for the life tables upon which this analysis is based. However, the methods used in Canada have been reviewed carefully and, for the ages presented, it is believed that any effects of graduation and "closing off" are not significant enough to warrant further commentary. Any analysis beyond age 80, however, should be viewed with suspicion.

Results of Analysis

The results of the analysis can be presented in tabular form as follows:

Females
Aged

Life	65		70		75		80	
Table	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$
1931	13.22	57.28	10.13	41.88	7.48	29.20	5.42	19.21
1941	13.58	57.83	10.43	42.23	7.69	29.27	5.53	18.88
1951	14.47	59.77	11.12	44.40	8.23	30.97	5.88	20.09
1961	15.57	62.76	12.08	47.09	8.98	33.20	6.40	21.68
1966	16.21	64.95	12.64	49.20	9.44	35.18	6.76	23.32
1971	16.97	69.91	13.36	53.84	10.13	39.05	7.38	26.38
1976	17.50	70.90	13.84	54.66	10.53	39.62	7.65	26.76
1981	18.35	76.32	14.65	59.65	11.28	44.12	8.34	30.48

SOURCE: Canada Life Tables as indicated

Males
Aged

Life	65		70		75		80	
Table	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$
1931	12.48	54.81	9.56	39.52	7.07	27.00	5.11	17.37
1941	12.32	54.28	9.44	38.98	6.98	26.54	5.04	17.03
1951	12.81	57.58	9.92	41.49	7.39	28.46	5.34	18.42
1961	13.06	62.10	10.21	45.71	7.76	32.07	5.83	20.84
1966	13.13	63.51	10.33	46.65	7.87	32.88	5.86	22.15
1971	13.22	63.67	10.40	46.77	7.97	32.53	5.91	21.38
1976	13.45	63.56	10.55	46.74	8.05	32.44	5.94	21.20
1981	14.07	67.47	11.08	50.40	8.50	35.78	6.35	24.05

SOURCE: Canada Life Tables as indicated

As can be seen, life expectancies have improved dramatically over the period of analysis especially for females.

Not so anticipated, however, is the fact that as the life expectancies grew, so too did the associated variances (with some minor exceptions). These results are inconsistent with the hypothesis referred to as the rectangularization of the survivorship curve. In other words, there is no reason to believe that there is a biological limit to life, w, that would constrain increases in life expectancy.

At this point, I added some nice graphs of the results and some implications and sent the paper off for review.

Fortunately, the reviewer was not convinced, expressing a concern that there might be something anomalous about the data, or the shape of the survivorship curve, that could lead to my quantitative conclusions without being able to be sure of my qualitative conclusion (i.e. no finite 'w').

Being convinced that I was right, wanting to prove it, and hoping for a refereed publication, I searched for data that would prove to be more convincing. Rather quickly I came across a paper by J. Wilkin (*TSA* XXXIII, 11) in which he discussed and analyzed mortality rates based on Medicare data. He also argued that this data, at least to ages in the high 90's, could be considered very accurate.

Fortunately, I was able to access this data with updates to the year 1988. In fact, I was able to run this data for all years 1973 to 1988 through the analysis previously presented for the Canadian Life Table data. Based on evidence in the data and arguments by Wilkin, I assumed a value of $q_x = .244$ constant beyond age 100. A summary of the results follows (data are combined, female and male):

Year	AGE							
	75		80		85		90	
	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$
1973	9.37	40.39	7.00	28.10	5.09	19.15	3.72	13.70
1978	10.18	45.28	7.70	31.88	5.65	21.93	4.11	15.55
1983	10.45	47.41	7.96	33.60	5.89	23.21	4.30	16.37
1987	10.58	47.68	8.06	33.73	5.94	23.16	4.31	16.15
1988	10.51	46.69	7.96	32.91	5.85	22.48	4.21	15.61

Despite the fact that there were some strange turn-arounds in 1988 (perhaps Alice Wade could be asked to write a paper on that), I believed that the consistent high and positive correlation between $E[K]$ and $Var[K]$ was proof of the non-existence of a finite 'w'. Surely this data would have convinced the referee (and I believe it would have)!

Unfortunately, curiosity killed the cat. I felt the need to run the data one final time, this time artifically closing the table with $q_{101} = 1.00$. I did so, with the following results:

Year	AGE							
	75		80		85		90	
	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$	$E[K]$	$Var[K]$
1973	9.34	38.94	6.95	26.38	5.01	16.86	3.55	10.02
1978	10.12	43.06	7.63	29.36	5.53	18.76	3.89	10.92
1983	10.38	44.84	7.88	30.69	5.76	19.63	4.06	11.32
1987	10.51	45.11	7.97	30.84	5.81	19.63	4.08	11.22
1988	10.45	44.33	7.89	30.26	5.73	19.22	3.99	11.00

These data present some interesting results. First, what appears to be a significant change in the value of q_x beyond age 100 (from $q_x = .244$ to $q_{101} = 1.00$) has a relatively small effect on $\mathring{e}_x$ (less than 1% at age 75 to around 5% at age 90). Hence the implications of whether there is a finite 'w' or not may be little more than an academic exercise. As one could have expected, the size of the variance statistic is reduced (by around 5% at age 75 to over 30% at age 90).

Unfortunately (at least for me) the previously existing strong positive correlation between $\mathring{e}_x$ and its variance still exists. It seems intuitively obvious that if there is a finite 'w' (here arbitrarily imposed by setting $q_{101} = 1.00$) and $\mathring{e}_x$ improves (especially at the older ages, like age 90) that the variance statistic would decline. However, the facts disprove the intuition in this case.

In conclusion, dear colleague, I pose several questions.

1. How does one explain the results? What is it about the survivorship curve that produces these statistics (or what is it about these statistics)?

2. Do these data shed any light on the question of whether or not there is a finite 'w', or on the issue of the rectangularization of the surivorship curve? If so, what? If not, why?

3. Finally, is there any way I can rescue a refereed publication out of this work?

Please write to me at my year-book address.

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