ACTUARIAL RESEARCH CLEARING HOUSE
1991 VOL. 3

A LINEAR PROGRAMMING APPROACH TO MAXIMIZING POLICYHOLDER VALUE

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Fellowship Credit Research Paper 90-3

Approved by Society of Actuaries
Education and Examination Research Paper Committee

Effective September 16, 1990

Abstract

This paper explores the use of linear programming as a tool to guide policyholders in getting the most value out of their combined insurance and investment programs. Concentrating on flexible premium universal life within the tax environment of the United States, several linear programming models are developed that can be used (1) at the point of sale, to select the most cost effective policy from those available in the marketplace, and (2) after issue, to maximize the present value of future cash flows on the policy of an insured who is in ill-health. These optimization models utilize to the maximum benefit of the policyholder the options available within the typical universal life contract to vary premium payments, make cash value withdrawals, take loans, and select a level or an increasing death benefit. Considerations in developing similar models for traditional plans of insurance are also briefly discussed.

Table of Contents

Chapter 1

Introduction

1.1 Motivation; Overview of the Paper

Policyholders make less than optimal insurance purchases. While illustrations and cost comparison methods offer some guidance to the prospective purchaser of insurance, drawbacks exist that limit their utility. This is especially true when contracts such as flexible premium universal life are purchased, because existing cost comparison methods give no weight to flexibility that may be inherent in a product's design.

Consider, for example, a universal life product that is among the most competitive on the market when products are compared on a basis that generously funds the contracts. This product might very well compare unfavorably when the comparison is done using lower premiums. How, then, is the prospective policyholder to determine which is the better buy, the policy that is competitive when funded generously or the one that is competitive when treated more as term insurance? Given the proliferation of universal contracts offering persisting policyholders interest rate bonuses, the "best" policy could well be one that is treated like term insurance in the early years, and then is funded more generously in the later years, after a higher interest rate takes effect.

A second drawback of most existing cost comparison methods is that they take into account the time value of money by discounting or accumulating funds at a single interest rate. A more valuable approach recognizes that a policyholder may place money not allocated to an insurance program into one or more of any number of investment alternatives, and accounts for the differing returns and tax treatment of the insurance plan and each investment.

Thus, in comparing contracts that offer a policyholder a degree of flexibility, a cost comparison method ideally should compare performance when each policy is performing optimally when used in conjunction with the universe of available investments, subject to the needs of and any constraints imposed by the policyholder.

This paper develops several linear programming models that may be used to develop optimal insurance and investment programs. The paper concentrates on flexible premium universal life and alternative investments within the federal income tax environment of the United States; however, considerations in designing a model to be used for combinations of traditional insurance are also discussed briefly.

This first chapter concludes with an overview of the existing literature on maximizing policyholder value. The second chapter develops the mathematics necessary to express universal life cash values as a linear function of prior policy transactions, a prerequisite to developing linear programming models to optimize universal life purchases.

The third chapter presents a simple linear programming model (The "Term Model") that solves for the optimal funding strategy for a universal life contract that is to be used strictly as term insurance, given a single after-tax interest rate to discount cash flows. While this model is very basic, it introduces some of the considerations that will need to be taken inte account in the more sophisticated models that follow, and provides some insight into linear programming solutions that might be less obvious within a more complicated setting.

Chapter four develops a linear programming model (the "General Model") to be used to solve for the optimal allocation of funds between a universal life contract and alternative investments. The linear programming objective function in this model may take either one of two forms: Maximize the total after-tax accumulated value of an insurance policy and investments, or alternatively, maximize a future after-tax income stream. This model is quite general in is applicability, taking into account, for example, any intermediate need for funds that may exist, and allowing for loans from the universal life contract as well as withdrawals. The use of this model in selecting the best plan of insurance to purchase, i.e., the use of the model as a cost comparison method, is explored in chapter five.

In chapter six, a linear programming model (the "Impaired Life Model") is developed that solves for the optimal strategy to be used by the owner of an in force universal life contract on an insured who is in ill-health. This model recognizes that within the typical universal life design, there is some opportunity to manipulate the net-amount-at-risk, and to exploit the contract to the maximum benefit of the contract holder.

Chapter seven covers some miscellaneous items, and closes with several suggestions for further research.

1.2 Literature Review

The most common approach to maximizing policyholder value is the use of one or more cost comparison indices during the insurance sales process. Black and Skipper [3] contains a very readable discussion of the strengths and weaknesses of the most commonly used cost comparison methods. The General Model developed in this paper can be used as an extension of the equal outlay cost comparison method[1] that permits investment in several side funds simultaneously and is generalized to allow for arbitrary future withdrawal patterns.

Several authors have investigated the suitability of universal life as an efficient combined insurance and investment vehicle. Chung and Skipper [8] have studied using a universal life contract's credited interest rate as a cost comparison index, but concluded that the correlation between the current rate and the tenth and twentieth year cash surrender values is too weak to justify its use as an index.

Cherin and Hutchins [7] have studied information on universal life and term policies available for sale in 1983, and concluded that the sales loads and expense charges inherent in universal life contracts decrease the internal rate of return sufficiently to render them inferior to a "buy term and invest the

[1]The equal outlay method is often used to compare two or more insurance plans with different premium structures. As the name implies, an equal annual contribution is allocated to each plan; any excess is deposited in a side fund that earns interest. The plan with the greatest cash value plus side fund balance at some future point is assumed to be the preferred purchase.

difference" strategy.[2] D'Arcy and Lee [9] have also studied universal life versus buy term and invest the difference, but concluded that once a policyholder's option for contribution to a deductible individual retirement account (IRA) or similar investment vehicle has been fully utilized, that the tax advantages of universal life outweigh the cost of expense loadings and result in a vehicle that is superior to other investment alternatives, assuming the holding period is sufficiently long.

Lee and D'Arcy [14] have developed a notion of the optimal level premium funding strategy for an increasing death benefit universal life contract. Their approach is to calculate the average after-tax marginal rate of return on each dollar of premium paid annually to a universal life contract, and to fund the contract so long as this rate exceeds the marginal rate of return available on an annual contribution of one dollar to an alternative investment. While the linear programming approach in this paper implicitly recognizes the marginal rate of return on each dollar of premium, it also recognizes that the rate of return will vary with the timing of a payment. Thus, linear programming results in a "more optimal" solution in which payments to the contract may vary by duration.

Schleef has written several papers that have recognized the value of linear programming in constructing cost comparison methods. In [21], Schleef develops a linear programming model that solves for the optimal amount of whole life and term insurance to purchase or cancel each year, given an insurance need, but also taking into account the opportunity to self-insure through savings. In [20], Schleef uses the model developed in [21] to derive a functional relationship between the interest-adjusted cost index and the rate of return earned on a whole life policy. In [22], he uses linear programming to derive a version of the interest-adjusted surrender cost index, and compares this index to the traditional interest-adjusted surrender cost index and to Linton's yield for sixty-eight whole life contracts.

One theme of Schleef's work reappears in this paper, namely, that cost

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[2] Since this study was done, however, the universal life marketplace has become significantly more competitive [see 26]. It is not clear that this conclusion would still be valid in today's environment.

comparison methods derived through linear programming techniques are of greater value to a prospective purchaser of insurance than are more traditional cost comparison methods, because unlike traditional methods, linear programming methods determine the best purchase while utilizing a given policy's flexibility to the maximum advantage of the purchaser. As mentioned earlier, this theme is especially valid in the case of flexible premium universal life. In addition to this paper's emphasis on universal life, there are other distinctions between Schleef's work and the linear programming models contained in this paper. Schleef's models concentrate on the optimal timing of insurance purchases and surrenders; this paper focuses on optimal funding strategies. Schleef's objective functions maximize the present value of future cash flows, discounted at an after-tax interest rate; in contrast, the objective functions in the General Model developed in this paper maximize after-tax accumulated values or after-tax retirement income streams. An advantage of the accumulation approach is that it is more amenable to the treatment of investment in several alternative vehicles when the rate earned by each investment is distinct, and when the tax impact of one or more of the investments cannot be reduced to a simple after-tax discount or accumulation rate.

Linear and quadratic programming has long been recognized as a valuable tool in developing the proper asset allocation strategy of the institutional investor. This application is so common that it is used as an example in a several operations research textbooks; see, for example, [4] or [5]. Generally, such models solve for an allocation that attains a given desired rate of return while minimizing the variability the of return. Within the actuarial literature, Tilley [25] has used linear programming in investment allocation decisions to match assets and liabilities.[3]

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[3]Tilley's paper also contains APL code to solve linear programming problems using the Simplex Method. For Simplex Method code in C, Fortran, and Pascal, see [15], [16] and [17].

The Simplex Method operates along the boundary of a linear programming problem's feasible region by making successive movements from one cornerpoint to an adjacent cornerpoint that is at least as optimal. In contrast, recent advancements in linear programming theory have included the development of several new algorithms that operate instead largely within the interior the feasible region. The most promising of these interior point methods, Karmarkar's

Other branches of operations research have been used to model optimal insurance purchase decisions. For example, Babbel and Ohtsuka [1], combining decision analysis and utility theory, have developed a model that shows that, in contrast to studies indicating buy term and invest the difference strategies to be superior to purchases of whole life, that rational decision makers will often opt for purchasing a combination of term and whole life. The Babbel and Ohtsuka model places a value on options available in whole life that are not generally recognized by proponents of term insurance. Hakansson [12] and other economists have developed models to maximize the utility of insurance to the consumer. These models are often quite theoretical and do not find direct application to the insurance sales process.

Algorithm, may prove to be significantly more efficient than the Simplex Method for certain classes of problems. For a comparison of the Simplex Method and Karmarkar's Algorithm, see [23]. For Karmarkar's algorithm code in APL and Basic, see [24] and [18].

As this paper is being written, spreadsheet software has begun to support solutions to linear programming problems. Borland International's Quattro Pro Version 1.0 is capable of solving linear programming problems containing up to 256 variables and 90 constraints. For Lotus 1-2-3 users, [19] contains a Lotus macro to solve linear programming problems using 1-2-3's existing matrix manipulation functions; however, the maximum suggested size (20 variables and 20 constraints) is too small to allow the macro to be applied to the problems in this paper.

A variety of software designed specifically to solve linear programming problems is also available. Such software generally will recognize when the format of a linear programming problem's constraints allow a computational shortcut to be used. Thus, from the standpoint of computational efficiency, one would expect this software to outperform the relatively simplistic codes listed above. Some computational issues are discussed in section 7.2.

Chapter 2

Cash Value as a Linear Function of Prior Transactions

Eckley [11] has developed commutation functions that may be used to express the account value of a universal life policy as a linear function of the face amount, premium payments, and expense charge deductions. This chapter provides an alternative derivation of the coefficients needed to express cash values as linear functions, and provides extensions for handling loan activity and linearity when a policy is within the Internal Revenue Service Section 7702 cash value corridor.[1]

The goal of the chapter is to express the cash value at some future time as a linear function of the prior premium payments, face amounts, loan activity, withdrawals, and expense charge deductions. Logically, premium payments, withdrawals, and expense charge deductions may be grouped: What is of interest is the change in cash value at time u, due to a fixed increment or decrement to the cash value at time t, irrespective of the source. The linear coefficient for the change in cash value at time u due to a one dollar increase in cash value at time t will be denoted by ${}_u\Delta CVC_t$. The impact on cash value at time u due to one dollar of in force face amount and one dollar of loan outstanding at time t will be denoted by ${}_u\Delta CVF_t$ and ${}_u\Delta CVL_t$, respectively.

2.1 Coefficients for the Impact on Cash Value of Prior Changes in Cash Value

A common formula (see Eckley [11]) used for accumulating a level death benefit universal life account value from the beginning of one month to the beginning of the next month is:

$$CV_{t+\frac{1}{12}} = \left(CV_t - \frac{q_t}{12}\left(\frac{F_t}{(1+i_g)^{1/12}} - CV_t\right)\right)(1+i_c)^{1/12} \qquad (2.1.1)$$

[1]All universal life contracts in this paper will be assumed to meet the definition of life insurance through the guideline premium/cash value corridor test.

where: CV_t = Cash value at time t

F_t = Face amount in force from time t to time t + 1/12

i_c = Annual credited effective interest rate

i_g = Guaranteed interest rate

q_t = Annual cost of insurance rate per dollar net-amount-at-risk in effect at time t

This formula assumes that there are no changes in account value other than for the deduction of the current month's cost-of-insurance and for the crediting of interest. If the cash value at the beginning of the month changes, due perhaps to a premium payment, a withdrawal,[2] or the deduction of an expense charge, the impact of this change on the cash value at the beginning of the next month may be written as:

$$CV_{t+\frac{1}{12}}+\Delta CV_{t+\frac{1}{12}} = \left(CV_t+\Delta CV_t-\frac{q_t}{12}\left(\frac{F_t}{(1+i_g)^{1/12}}-(CV_t+\Delta CV_t)\right)\right)(1+i_c)^{1/12} \tag{2.1.2}$$

Subtracting (2.1.1) from (2.1.2) and rearranging terms yields:

$$\frac{\Delta CV_{t+\frac{1}{12}}}{\Delta CV_t} = \left(1+\frac{q_t}{12}\right)(1+i_c)^{1/12} \tag{2.1.3}$$

Restricting t to integer values and continuing the process of accumulating the cash value through the remaining eleven months of the policy year, define:

$${}^{t}\Delta CVC_t = \frac{\Delta CV_{t+1}}{\Delta CV_t} = \left(1+\frac{q_t}{12}\right)^{12}(1+i_c) \tag{2.1.4}$$

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[2] In the case of a withdrawal, this assumes that the face amount stays constant. In fact, on a level death benefit policy, to minimize the potential for net-amount-at-risk manipulation on the part of an unhealthy insured, a withdrawal will generally result in a dollar-for-dollar reduction in the face amount. This aspect of the impact of the withdrawal on future cash values will be handled using the face amount coefficients developed in the next section by explicitly recognizing the reduction in face amount that accompanies the withdrawal.

${}^L\Delta CVC_t$ may be interpreted as the change in the cash value of a level death benefit policy one year later as a result of a one dollar change in cash value at time t. Similarly, continuing this process further into the future, define:[3]

$$ {}^L_u\Delta CVC_t = \frac{\Delta CV_u}{\Delta CV_t} = \prod_{j=t}^{u-1} {}^L\Delta CVC_j \tag{2.1.5} $$

which may be interprsted as the change in cash value at time u as a result of a one dollar change in cash value at time t.[4]

Suppose that a total of one dollar is added to or deducted from the cash value on a modal basis, with frequency m. Using (2.1.3), the impact of these m transactions on the cash value at the end of the year may be written as:[5]

$$ {}^L\Delta CVC_t^{(m)} = \frac{1}{m}\sum_{j=1}^{m} [(1+\frac{q_t}{12})^{12}(1+i_c)]^{\frac{j}{m}} \tag{2.1.6} $$

and the impact on the cash value at time u may be written as:

$$ {}^L_u\Delta CVC_t^{(m)} = {}^L_u\Delta CVC_{t+1} \cdot {}^L\Delta CVC_t^{(m)} \tag{2.1.7} $$

A common formula for the monthly accumulation of an increasing death benefit

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[3]Notice that ${}^L_u\Delta CVC_t$ for arbitrary values of t and u may be obtained by division from a table of the values of ${}^L_j\Delta CVC_0$. A transition to a commutation function approach may be made by letting $D_0 = 1$ and $D_j = 1/{}^L_j\Delta CVC_0$ for j = 1,2,3,..., where D denotes the usual commutation function.

[4]The use of subscripts here is a mild departure from International Actuarial Notation. While the lower right subscript properly refers to an initial time frame (t) for the base symbol, under the International Notation, the lower left subscript would refer not to an ultimate time frame (u), but to an amount of elapsed time (u-t). In this paper, it will be more natural to think in terms of "the impact on cash value at age 65 of a premium paid at age 50" than in terms of "the impact in fifteen years of a premium paid at age 50." Defining the lower left subscript in terms of an ultimate time frame will eliminate the need for some mental addition.

[5]Alternatively, under a commutation function approach, define $D_j^{(12)} = D_{j+1}\cdot {}^L\Delta CVC_j^{(12)}$.
Notice that, as geometric progressions, the summations in sections 2.1 through 2.3 may be reduced to fractions.

option universal life policy is:

$$CV_{t+\frac{1}{12}} = \left(CV_t - \frac{q_t}{12}\left(\frac{F_t+CV_t}{(1+i_g)^{1/12}} - CV_t\right)\right)(1+i_c)^{1/12} \tag{2.1.8}$$

Using this formula, the following analogous factors may be derived for the increasing death benefit option case:

$${}^I\Delta CVC_t = \left(1+\frac{q_t}{12}\left(1-\frac{1}{(1+i_g)^{1/12}}\right)\right)^{12}(1+i_c) \tag{2.1.9}$$

$${}_u^I\Delta CVC_t = \prod_{j=t}^{u-1} {}^I\Delta CVC_j \tag{2.1.10}$$

$${}^I\Delta CVC_t^{(m)} = \frac{1}{m}\sum_{j=1}^{m}\left[\left(1+\frac{q_t}{12}\left(1-\frac{1}{(1+i_g)^{1/12}}\right)\right)^{12}(1+i_c)\right]^{\frac{j}{m}} \tag{2.1.11}$$

$${}_u^I\Delta CVC_t^{(m)} = {}_u^I\Delta CVC_{t+1} \cdot {}^I\Delta CVC_t^{(m)} \tag{2.1.12}$$

2.2 Coefficients for the Impact on Cash Value of Prior Face Amounts

From (2.1.1) for the level death benefit case and (2.1.8) for the increasing death benefit case, it is apparent that each dollar of face amount results in a monthly deduction of:

$$\frac{q_t}{12(1+i_g)^{1/12}}$$

At the end of the month, the impact of this deduction has grown by the interest lost; bringing twelve such deductions to the end of the year results in:

$$^{L}\Delta CVF_t = -\frac{q_t}{12}\left(\frac{1+i_c}{1+i_g}\right)^{1/12}\sum_{j=0}^{11}\left[\left(1+\frac{q_t}{12}\right)^{12}(1+i_c)\right]^{\frac{j}{12}} \tag{2.2.1}$$

which may be interpreted as the impact on cash value at the end of the year of one dollar of level death benefit face amount in force at time t,[6] and:

$$^{I}\Delta CVF_t = -\frac{q_t}{12}\left(\frac{1+i_c}{1+i_g}\right)^{1/12}\sum_{j=0}^{11}\left[\left(1+\frac{q_t}{12}\left(1-\frac{1}{(1+i_g)^{1/12}}\right)\right)^{12}(1+i_c)\right]^{\frac{j}{12}} \tag{2.2.2}$$

which is the analogous factor for the increasing death benefit case.

For the impact at time u, define:

$$_{u}^{L}\Delta CVF_t = {}_{u}^{L}\Delta CVC_{t+1}\cdot {}^{L}\Delta CVF_t \tag{2.2.3}$$

$$_{u}^{I}\Delta CVF_t = {}_{u}^{I}\Delta CVC_{t+1}\cdot {}^{I}\Delta CVF_t \tag{2.2.4}$$

2.3 Coefficients for the Impact on Cash Value of Prior Loan Activity

Suppose that a loan is taken out at the beginning of a policy year. In administering a universal life policy, it is common to split the cash value into two funds. The first, consisting of unloaned cash value, is credited with premiums, charged with mortality and expense deductions, and earns interest at the current interest rate. The second, initially set to the amount of the outstanding loan, is segregated and earns interest at a different rate, typically

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[6]Under the commutation function approach, let $C_j = D_{j+1}\cdot {}^{L}\Delta CVF_j$.

fixed and specified in the contract. To take into account the existence of the loan, formula (2.1.1) expressing the monthly processing of a level death benefit universal life contract may be modified as follows:

For t an integer and $s = 0, {}^{1}/_{12}, \ldots, {}^{11}/_{12}$, let

$$CV_{t+s} = UCV_{t+s} + LCV_{t+s} \tag{2.3.1}$$

where:

$$UCV_{t+s+\frac{1}{12}} = (UCV_{t+s} - \frac{q_t}{12}(\frac{F_t}{(1+i_g)^{1/12}} - (UCV_{t+s} + LCV_{t+s})))(1+i_c)^{1/12} \tag{2.3.2}$$

$$LCV_{t+s+\frac{1}{12}} = LCV_{t+s}(1+i_L)^{\frac{1}{12}} \tag{2.3.3}$$

UCV_{t+s} = Unloaned cash value

LCV_{t+s} = Loaned cash value

i_L = Loaned cash value earned interest rate.

At the end of the year, if the loan is repaid, the cash value is recombined into a single fund. If the loan is not repaid in full, then a transfer is made between the unloaned fund and the loaned fund so that at the commencement of the new policy year, the loaned fund will again equal the amount of the outstanding loan.

Letting $s = 0$ in (2.3.1) and subtracting (2.1.1), the difference in cash value after one month between a policy with an amount LCV_t in the loaned fund and an otherwise identical policy without a loan is:

$$LCV_t[(1+i_L)^{1/12} - (1+i_c)^{1/12}]$$

Were the loan to be repaid after only one month, the impact of this differential could be brought forward into the future as if it were a single deduction[7] from the cash value. When the loan remains outstanding for the entire policy year, however, one must account for the differential for each month, which increases as the loaned fund is credited with interest. Thus, the net impact on the cash value at the end of the year of a one dollar loan outstanding for the entire year may be written as:

$$ {}^{L}\Delta CVL_t = \sum_{j=0}^{11} [(1+i_L)^{1/12}-(1+i_c)^{1/12}]\,(1+i_L)^{j/12}[(1+\frac{q_t}{12})^{12}(1+i_c)]^{(11-j)/12} \tag{2.3.4} $$

and the impact at time u may be written as:

$$ {}^{L}_{u}\Delta CVL_t = {}^{L}_{u}\Delta CVC_{t+1} \cdot {}^{L}\Delta CVL_t \tag{2.3.5} $$

The analogous factors for the increasing death benefit case are:

$$ {}^{I}\Delta CVL_t = \sum_{j=0}^{11} [(1+i_L)^{1/12}-(1+i_c)^{1/12}]\,(1+i_L)^{j/12}[(1+\frac{q_t}{12}(1-\frac{1}{(1+i_g)^{1/12}}))^{12}(1+i_c)]^{(11-j)/12} \tag{2.3.6} $$

$$ {}^{I}_{u}\Delta CVL_t = {}^{I}_{u}\Delta CVC_{t+1} \cdot {}^{I}\Delta CVL_t \tag{2.3.7} $$

2.4 Cash Value as a Linear Function; Product Design Considerations

The coefficients derived above allow the cash value of a universal life policy at any point to be expressed as a linear function of prior premium payments, face amounts, loan activity, withdrawals, and expense deductions. Let α denote whether a policy is a level death benefit or an increasing death benefit

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[7] It is common to think of loaned cash value as earning a rate less than the rate earned on unloaned cash value; certainly, this has been the experience given the high interest environment that has existed since the introduction of universal life. However, since i_L, the interest rate credited on loaned cash value, is typically greater than i_g, the guaranteed interest rate, it is possible that in a low interest environment, the rate credited on loaned cash value would be greater than the rate credited on unloaned cash value. In such a case, this "deduction" would in fact be a credit.

policy. Then[8]:

$$CV_t(P_j, F_j, L_j, W_j \mid j=1,\ldots,t) = \sum_{j=1}^{t} \begin{aligned}[t] & (1-E\%)\ {}^{s}_{t}\Delta CVC_{j-1}^{(m_p)} \cdot P_j \\ & + \left({}^{s}_{t}\Delta CVF_{j-1} - \frac{EF}{1000}\ {}^{s}_{t}\Delta CVC_{j-1}^{(m_{EF})} \right) F_j \\ & + {}^{s}_{t}\Delta CVL_{j-1} \cdot L_j \\ & - {}^{s}_{t}\Delta CVC_{j-1} \cdot W_j \\ & - {}^{s}_{t}\Delta CVC_{j-1}^{(m_{EP})} \cdot EP_j \end{aligned} \tag{2.4.1}$$

where: CV_t = Cash value at the end of year t

P_j = Premiums paid during year j, mode m_p

F_j = Face amount in force during year j

L_j = Loan outstanding during year j

W_j = Withdrawal taken at the beginning of year j

E% = Percent of premium charge

EF = Per thousand of face charge, deducted mode m_{EF}

EP = Per policy charge, deducted mode m_{EP}

In using the formula above, it should be remembered that on level death benefit policies, each withdrawal will generally cause a reduction in the face amount of the policy. To handle this, F_t can be expressed as a function of the initial face amount and all prior withdrawals. Changes in death benefit option will also generally cause an adjustment to the face amount, with a change from a level

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[8]As presented, this formula effectively converts the "time line" approach used for subscripting in the derivation of the various linear coefficients to the policy year approach that is used in the models developed in chapters three through five. Under the convention adopted, premiums can occur modally throughout the policy year, withdrawals are taken at the beginning of the policy year, loans remain outstanding throughout the policy year, and cash values quoted are end-of-year.

Formula (2.4.1) will not be repeated in this paper; rather, $CV_t(P_j, F_j, L_j, W_j \mid j=1,\ldots,t)$ will be used whenever it is desired to express cash value as a linear function of prior transactions.

death benefit policy to an increasing death benefit policy resulting in a reduction of the face amount by the amount of cash value in the policy at the time of the change, and conversely, a change from an increasing death benefit policy to a level death benefit policy resulting in the face amount being increased by the amount of the cash value at the time of the change. In the event of a death benefit option change, future face amounts can be expressed as a function of the cash value at the time of the change. In such a case, linear coefficients accounting for the impact of transactions before the death benefit option change on cash value after the death benefit option change will need to be a hybrid of annual level and increasing factors. The handling of a per thousand charge after a change in death benefit option will depend upon the product design.

Using factors such as ${}^{s}_{u}\Delta CVC_{t}^{(12)}$ to account for the impact on cash value of modal per thousand or per policy expense deductions implies that these deductions are made at the beginning of the month. In fact, many product designs deduct these charges at the end of the month. Summing from $j=0$ to $m-1$ in (2.1.6) and (2.1.11) will produce the factors necessary for the end-of-month case.

There are several forms of expense charges in existence that present some difficulty when one attempts to express cash value as a linear function of prior transactions. An example of one such "problem" product design is the policy which has a monthly expense charge that is expressed as the "lesser of $X.XX or the amount of excess interest credited for the month." As a practical matter, the problem of the potentially variable expense charge can be overcome if it is possible to restrict the calculation of cash values to situations in which the intermediate cash values will always be large enough to require the deduction of the full expense charge. A similar problem occurs in policies with an "interest corridor," in which only the guaranteed interest rate is credited to the first $Y of cash value (typically five hundred or one thousand dollars). In this situation, if the calculation of cash values can be restricted to situations in which intermediate cash values will always exceed the corridor amount, then cash values may be calculated using linear coefficients by crediting the entire cash value with the rate for amounts in excess of the corridor, and imputing a monthly expense charge equal to the amount of interest lost on the corridor amount.

Among other product designs that require special consideration are those with "competitive enhancements" or "policyholder persistency bonuses." For example, in some policies, the credited interest rate increases after the policy has been in force a specified number of years. For portfolio rate products, this increase may be handled simply by changing the rate used in the calculation of linear coefficients at the appropriate duration. New money crediting strategies, however, pose a greater challenge. Under many new money methods, the rate earned on a policy is unique to the policy and depends upon the amount and timing of prior payments. Such designs may be amenable to an approach which treats separately different generations of cash flow. For example, the cash value existing up to the time of an interest rate increase may be brought forward using one set of ΔCVC factors in which the interest rate used to calculate the factors gradually increases to the new rate, as this cash value is presumed to roll over to the new rate. New premiums, on the other hand, may be brought forward using a set of factors based solely upon the new rate. The ease with which loans and withdrawals may be handled will depend upon the specifics of policy administration.

Under another form of persistency bonus, if a policy stays in force a given number of years, the cash value will be recalculated as if a higher interest rate had been credited from issue. This design requires one set of factors to calculate cash values up to the time of the retroactive interest bonus, and a second set of factors, derived using the higher interest rate and applied since issue, to calculate cash values after the crediting of the interest bonus.

2.5 Nonlinearity and the IRS Section 7702 Cash Value Corridor

A universal life policy that meets the definition of life insurance through the guideline premium/cash value corridor test requires an increase in death benefit when the face amount (for a level death benefit policy) or the face amount plus cash value (for an increasing death benefit policy) is less than the cash value times the corridor percent for the appropriate attained-age. At the point where a cash value rich policy hits the corridor, cash values are no longer expressible as linear functions of prior transactions. The impact on future cash values of each additional dollar of premium is reduced as it incurs the added cost of purchasing the required additional death benefit. As more premium is received, the policy hits the corridor in successively earlier months, thus resulting in successively greater penalties.

In the examples that illustrate the linear programming models in chapters three through five, cash value corridor considerations will not come into play. The linear programming model developed in chapter 6 will take the cash value corridor into account, and in fact will sometimes exploit corridor effects to the advantage of the policyholder. Approaches to handling cash value corridor effects are discussed in section 7.3.

It should be noted that if a policy is in the corridor at time t, and if the nature of the problem precludes the policy leaving the corridor, future cash values may be expressed as a linear function of CV_t and the subsequent transactions. Derivation of the linear coefficients uses the formula for accumulating cash values within the corridor:

$$CV_{t+\frac{1}{12}} = (CV_t - \frac{q_t}{12}(\frac{CORR_t \cdot CV_t}{(1+i_g)^{1/12}} - CV_t))(1+i_c)^{1/12} \qquad (2.5.1)$$

where: $CORR_t$ = IRS Section 7702 corridor percent at time t

and results in factors such as:

$$^{CORR}\Delta CVC_t = (1 - \frac{q_t}{12}(\frac{CORR_t}{(1+i_g)^{1/12}} - 1))^{12}(1+i_c) \qquad (2.5.2)$$

Notice that since the current death benefit for a policy in the corridor is determined solely by the beginning-of-month cash value, there is no need for separate factors for level and increasing death benefit options, and no need for a face amount factor.

Chapter 3

A Simple Model

As an introduction to the use of linear programming as a means of maximizing policyholder value, this chapter presents a very simple model (the "Term Model") that answers the following question: Given an after-tax discount rate reflecting a policyholder's appraisal of the time value of money, what funding strategy minimizes the present value of future premiums? In constructing this model, it is assumed that the policyholder has a fixed period insurance need and has no desire for a cash surrender value; thus, the universal life policy is effectively being used as term insurance.

Two examples using this model will be studied. In the first example, the change in the optimal funding strategy will be examined as the discount rate is varied, under the assumption that the insured has an unlimited source of funds in any year. In the second example, the optimal funding strategy will be looked at when an annual limit is placed on the amount of money that the insured has available to fund his contract.

3.1 Objective Function and Constraints

The list below defines the notation used in the objective function and the constraints equations for this model. [P] to the left of a variable indicates a model input parameter, [D] indicates a variable that is largely descriptive and is used internally within the model, and [S] indicates a variable that is of fundamental importance to the user as part of the linear programming solution. It should be noted that while many [D] items are also solved for as part of the process of minimizing or maximizing the objective function, these items are not of primary interest, serving, rather, an internal accounting function.

[S] P_t = Premiums paid at the beginning of year t

[D] $CV_t(P_j \mid j=1,\ldots,t)$ = Cash value at the end of year t[1]

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[1]In this chapter, it is assumed that a policy has a fixed face amount and has no withdrawal or loan activity. Thus, given an expense structure, cash values are a function solely of the annual premiums.

[P] SC_t = Surrender charge in effect during year t

[P] v = Present value discount factor

[P] Per = Length of time policy is to stay in force

[P] $FundsAvail_t$ = Maximum cash available to fund contract in year t

Assuming premiums are paid annually, the objective of the model is to:

$$MINIMIZE \quad S = \sum_{t=1}^{Per} v^{t-1} P_t$$

subject to the following two sets of constraints:

(T1) The net cash value is not less than zero throughout the period the policy is to stay in force:[2]

For t = 1 to Per: $CV_t(P_j \mid j=1,\ldots,t) \geq SC_t$

(T2) For each year t, the policyholder pays no more than $FundsAvail_t$ to fund the contract:

For t = 1 to Per: $P_t \leq FundsAvail_t$

If in any year the insured has no restriction on the funds available, the constraint for that year may be eliminated.

3.2 Examples

Assume that a forty-five year old policyholder with a twenty year insurance

[2]In this model and in the General Model developed in chapter four, it is assumed that end-of-year testing for cash value sufficiency is adequate to keep the policy from lapsing. Most universal life administrative systems will check for lapsation on a monthly basis. Whether end-of-year testing is in fact the most appropriate check to ensure that a policy stays in force will depend on several things, including the pattern of the surrender charge (which may also vary monthly) and the premium payment mode.

Some universal life plans may in the early years substitute a minimum premium requirement for a surrender charge test. For such plans, the constraints requiring that cash value exceed the surrender charge would be replaced by constraints requiring that cumulative premiums paid exceed the lower bounds specified for the years in which the minimum premium requirement is in effect.

Under some plan designs, the surrender charge will vary with the first year premium. In such a case, SC_t should be modelled as a function of premium.

need purchases a $100,000 level death benefit policy from Company A.[3] Table 3-1 shows the premiums over the twenty year period that will optimize the objective function as the present value interest rate is varied, as well as the corresponding progression of cash values, under the assumption that the policyholder has unlimited funds available in any year.

As one might expect, at low present value interest rates, up to 10.2925%, the strategy that optimize the objective function is to purchase the policy with a single premium. The cash value that results initially grows with time, as interest credits exceed cost of insurance deductions; however, in year eleven the cash value begins to decrease with time, so that precisely by the end of year twenty it is exhausted. At high present value interest rates, above 11.5174%, funding of the contract is deferred as long as possible: Each year's premium payment is precisely enough to keep the cash value at the end of that year from dropping below the policy's surrender charge.

At interest rates in between, the optimal strategy involves deferring funding for several years, and then funding the contract with the single premium required to bring the policy through age 64. The interest rate boundaries are defined by the factors ${}^{L}\Delta CVC_{45}-1$, ${}^{L}\Delta CVC_{46}-1, \ldots, {}^{L}\Delta CVC_{64}-1$. (See the appendix.) ${}^{L}\Delta CVC_{t}-1$ represents an "interest rate" that is the sum of two components, the interest rate credited to the policy, and the savings in year t cost-of-insurance deductions that occur with every dollar increase in cash value, due to a decrease in net-amount-at-risk.[4] Though the interest rate for the Company A policy is level over time, the cost-of-insurance rates increase with time, resulting in ${}^{L}\Delta CVC_{t}$ factors that increase with time. As the present value interest rate increases and crosses each ${}^{L}\Delta CVC_{t}-1$ boundary, the optimal strategy shifts to

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[3]Three illustrative universal life products, herein referred to as the products of Company A, Company B, and Company C, are used in the examples in this paper. Details on the rate structure of these sample products are given in the appendix.

[4]Since the objective function does not involve life contingencies, the cost of coverage is effectively minimized for the case in which the insured survives the term period. One should recognize, though, that the "savings" that occurs through any prefunding has a cost, namely, the reduction in the net death benefit should the insured in fact die.

Present Value Rate From	Up To		Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10	Year 11	Year 12	Year 13	Year 14	Year 15	Year 16	Year 17	Year 18	Year 19	Year 20
0.0000%	10.2925%	Premium (BOY)	$4,765																			
		CV (EOY)	$4,661	$4,840	$5,014	$5,181	$5,338	$5,481	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.2925%	10.3165%	Premium (BOY)	$993	$4,161																		
		CV (EOY)	$750	$4,840	$5,014	$5,181	$5,338	$5,481	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.3165%	10.3420%	Premium (BOY)	$993	$168	$4,404																	
		CV (EOY)	$750	$700	$5,014	$5,181	$5,338	$5,481	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.3420%	10.3694%	Premium (BOY)	$993	$168	$197	$4,643																
		CV (EOY)	$750	$700	$650	$5,181	$5,338	$5,481	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.3694%	10.4004%	Premium (BOY)	$993	$168	$197	$227	$4,874															
		CV (EOY)	$750	$700	$650	$600	$5,338	$5,481	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4004%	10.4334%	Premium (BOY)	$993	$168	$197	$227	$260	$5,094														
		CV (EOY)	$750	$700	$650	$600	$550	$5,481	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4334%	10.4719%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$5,299													
		CV (EOY)	$750	$700	$650	$600	$550	$500	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4719%	10.5171%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$5,484												
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.5171%	10.5675%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$5,639											
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.5675%	10.6259%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$5,757										
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.6259%	10.6906%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$5,826									
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.6906%	10.7624%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$5,835								
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.7624%	10.8387%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$5,770							
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.8387%	10.9213%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$700	$5,620						
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$150	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.9213%	11.0145%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$700	$780	$5,369					
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$150	$100	$4,745	$4,205	$3,499	$2,593	$1,443	$0
11.0145%	11.1186%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$700	$780	$870	$4,995				
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$150	$100	$50	$4,205	$3,499	$2,593	$1,443	$0
11.1186%	11.2344%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$700	$780	$870	$969	$4,474			
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$150	$100	$50	$0	$3,499	$2,593	$1,443	$0
11.2344%	11.3665%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$700	$780	$870	$969	$1,127	$3,723		
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$150	$100	$50	$0	$0	$2,604	$1,443	$0
11.3665%	11.5174%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$700	$780	$870	$969	$1,127	$1,246	$2,758	
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$150	$100	$50	$0	$0	$0	$1,443	$0
11.5174% and up		Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$625	$700	$780	$870	$969	$1,127	$1,246	$1,382	$1,535
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$200	$150	$100	$50	$0	$0	$0	$0	$0

Table 3-1 Term Model Solutions Under Unlimited Funding As Present Value Interest Rate Varies
Solutions at present value interest rate boundaries are not unique.

BOY = Beginning-of-year, EOY = End-of-year

Present Value Rate From	Up To		Year 1	Year 2	Year 3	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10	Year 11	Year 12	Year 13	Year 14	Year 15	Year 16	Year 17	Year 18	Year 19	Year 20
0.0000%	10.3590%	Premium (BOY)	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$10	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$758	$1,571	$2,444	$3,382	$4,389	$5,472	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.3590%	10.3723%	Premium (BOY)	$993	$1,000	$1,000	$1,000	$1,000	$1,000	$23	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$1,562	$2,435	$3,372	$4,378	$5,459	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.3723%	10.3889%	Premium (BOY)	$993	$606	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$944	$1,752	$2,619	$3,547	$4,541	$5,605	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.3889%	10.4034%	Premium (BOY)	$993	$168	$1,000	$1,000	$1,000	$1,000	$1,000	$426	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$1,483	$2,322	$3,219	$4,179	$5,205	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4034%	10.4224%	Premium (BOY)	$993	$168	$650	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$1,120	$1,921	$2,776	$3,690	$4,665	$5,701	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4224%	10.4384%	Premium (BOY)	$993	$168	$197	$1,000	$1,000	$1,000	$1,000	$1,000	$822	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$1,402	$2,204	$3,058	$3,966	$4,929	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4384%	10.4599%	Premium (BOY)	$993	$168	$197	$921	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$1,290	$2,079	$2,921	$3,814	$4,761	$5,762	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4599%	10.4836%	Premium (BOY)	$993	$168	$197	$361	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$718	$1,449	$2,224	$3,045	$3,911	$4,822	$5,776	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.4836%	10.5027%	Premium (BOY)	$993	$168	$197	$227	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$230	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$1,318	$2,079	$2,885	$3,734	$4,626	$5,560	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.5027%	10.5295%	Premium (BOY)	$993	$168	$197	$227	$377	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$879	$1,595	$2,358	$3,142	$3,972	$4,836	$5,735	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.5295%	10.5510%	Premium (BOY)	$993	$168	$197	$227	$260	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$638	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$1,232	$1,949	$2,699	$3,482	$4,294	$5,135	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.5510%	10.5812%	Premium (BOY)	$993	$168	$197	$227	$260	$802	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$1,026	$1,722	$2,448	$3,204	$3,987	$4,795	$5,624	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.5812%	10.6134%	Premium (BOY)	$993	$168	$197	$227	$260	$307	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$515	$1,154	$1,821	$2,511	$3,220	$3,946	$4,684	$5,433	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.6134%	10.6391%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$27	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$1,140	$1,808	$2,494	$3,201	$3,925	$4,661	$5,407	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.6391%	10.6743%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$521	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$643	$1,256	$1,888	$2,529	$3,180	$3,836	$4,493	$5,147	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.6743%	10.7033%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$618	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$1,043	$1,650	$2,268	$2,892	$3,517	$4,139	$4,754	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.7033%	10.7421%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$714	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$746	$1,322	$1,905	$2,490	$3,071	$3,646	$4,207	$4,745	$4,205	$3,499	$2,593	$1,443	$0
10.7421%	10.7743%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$753	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$940	$1,482	$2,022	$2,553	$3,071	$3,570	$4,038	$4,205	$3,499	$2,593	$1,443	$0
10.7743%	10.8173%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$879	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$814	$1,343	$1,888	$2,383	$2,882	$3,380	$3,805	$4,205	$3,499	$2,593	$1,443	$0
10.8173%	10.8636%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$449	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$357	$838	$1,309	$1,763	$2,195	$2,598	$2,960	$3,263	$3,499	$2,593	$1,443	$0
10.8636%	10.9006%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$318	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$830	$1,300	$1,753	$2,184	$2,582	$2,942	$3,250	$3,482	$2,593	$1,443	$0
10.9006%	10.9523%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$571	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$383	$806	$1,208	$1,578	$1,913	$2,199	$2,420	$2,559	$2,593	$1,443	$0
10.9523%	10.9932%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$204	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$713	$1,102	$1,464	$1,788	$2,068	$2,285	$2,387	$2,401	$1,443	$0
10.9932%	11.0513%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$655	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$0
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$354	$705	$1,023	$1,298	$1,516	$1,661	$1,715	$1,653	$1,443	$0
11.0513%	11.0965%	Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$256
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$598	$896	$1,137	$1,359	$1,487	$1,521	$1,437	$1,202	$0
11.0965% and up		Premium (BOY)	$993	$168	$197	$227	$260	$295	$335	$381	$432	$491	$555	$680	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000	$1,000
		CV (EOY)	$750	$700	$650	$600	$550	$500	$450	$400	$350	$300	$250	$257	$526	$744	$904	$981	$958	$810	$503	$0

Table 3-2 Term Model Solutions With Funding Limited to $1,000 Per Year, As Present Value Interest Rate Varies
Solutions at interest rate boundaries are not unique.

defer payment of the single premium for one additional year, instead paying the minimum premium necessary to keep the policy in force.[5]

Within the interest rate intervals shown on Table 3-1, the optimal solution is unique; on the boundaries, the solution is not unique. If $\alpha=\{P'_1,\ldots,P'_{20}\}$ is the optimal solution on the interval of present value rates $[i_{n-1},i_n]$ and $\beta=\{P''_1,\ldots,P''_{20}\}$ is the optimal solution on $[i_n,i_{n+1}]$, then for a present value interest rate of i_n, any solution of the form $\delta\alpha + (1-\delta)\beta$ $(0\le\delta\le1)$ will be an optimal solution.

Table 3-2 illustrates the optimal solution as a function of the present value interest rate when the policyholder can afford to pay no more than $1,000 in any given year. At low present value interest rates, up to 10.3590%, since the policy can no longer be purchased with a single premium, it is funded with the $1,000 per year maximum until enough money is paid to bring the policy through age 64. Conversely, at high present value interest rates, above 11.0965%, the policy is funded with the minimum premium required to keep the policy in force each year, with the exception that in years 17 through 20, since this premium is in excess of the $1,000 per year maximum, only $1,000 is paid, and in years 12 through 16, more than the minimum required is paid, in order to prefund the later deficits.

At low interest rates, the optimal solution is to fund the contract through premiums of $1,000 in years 1 through 6, and $ 10 in year 7. The first change in the optimal funding strategy occurs at the present value interest rate at which the policyholder is indifferent to transferring policy funding from year 1 to year 7, that is, at i=10.3590%, when ${}_{5|}^{L}\Delta CVC_{45}=(1+i)^6$.

Since the minimum premium for year 1 is $ 993, year 1 funding cannot drop more than $ 7 from the $ 1,000 per year maximum, and any further funding deferral will be due to a reduction in year 2 premium. When year 1 premium is at $ 993,

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[5]The solution becomes more complicated when the ${}^{L}\Delta CVC_x$ factors are not monotonically increasing, as might be the case, for example, for a policy issued to a male in his early twenties, or to an insured who is assigned a temporary flat extra. In such a case, given an insured aged x at issue and a present value interest rate i, the cost of insurance for policy year n will be funded with premium at the beginning of the year m (m≤n) that maximizes ${}_{n-m}^{L}\Delta CVC_{x+m-1}(1+i)^{m-1}$.

year 2 premium can drop to as low as \$ 168 before the policy will lapse. Indifference to transferring funding from year 2 to year 7 occurs at $i=10.3723\%$, when ${}_{51}^{L}\Delta CVC_{46}=(1+i)^5$, thus defining the next shift in the optimal solution and the second interest rate boundary. However, dropping the year 2 premium from \$1,000 to \$ 404 results in funding at the full \$1,000 level in year 7; any further funding reduction will need to be made up in year 8. This defines the third interest rate boundary, at $i=10.3889\%$, when ${}_{52}^{L}\Delta CVC_{46}=(1+i)^6$. Further interest rate boundaries are similarly defined by the points at which the policyholder is indifferent to transferring funding from period m to period n ($m \leq n$), where either period m funding is done at the minimum level required to keep the policy in force, or period n funding is done at the \$1,000 maximum.

Chapter 4

A General Model

This chapter develops a model to be used by a policyholder to optimally allocate funds between a universal life (UL) contract and several alternative investments. The alternatives considered are a money-market fund (MM), a deductible individual retirement account (IRA), and a flexible premium annuity (FPA). As presented here, the model is constructed to optimally allocate funds at the time a contract is issued, but with minor modifications it could be used to optimize the allocation of future funds on an insurance and investment program that is already in effect. In the next chapter, the use of this model as a cost comparison method will be studied.

4.1 Notation

A rather substantial amount of notation is required for the model. To begin, the following notation is used for insurance policy elements:

[S]	P_t	= Premium paid during year t
[S]	F_t	= Face amount in force during year t
[S]	W_t	= Withdrawal taken at the beginning of year t
[D]	$W_t^{TAXABLE}$	= Portion of W_t that is taxable
[D]	$W_t^{TAXFREE}$	= Portion of W_t that is tax-free
[S]	LT_t	= Loan taken out at the beginning of year t
[S]	LR_t	= Loan repaid, beginning of year t
[D]	L_t	= Loan outstanding during year t
[P]	i_{LC}	= Interest rate charged on outstanding loans
[P]	$GAPAdj000_t$	= IRS Section 7702 guideline annual premium per thousand of face amount, for a new policy at the insured's attained age in year t
[P]	$GSPAdj000_t$	= Guideline single premium per thousand of face amount, year t

[P] GAPolFee$_1$ = Any additional guideline annual premium at issue required to cover fixed policy costs, such as policy fees or other administrative charges

[P] GSPolFee$_1$ = Any additional guideline single premium at issue required to cover fixed policy costs

[D] GAP$_t$ = Guideline annual premium, year t

[D] ΣGAP$_t$ = Sum of the guideline annual premiums through year t

[D] GSP$_t$ = Guideline single premium, year t

[D] GPL$_t$ = Guideline premium limit, year t

[P] Corr$_t$ = IRS section 7702 corridor factor for the insured's attained-age during year t

[P] 7Pay000 = Modified endowment 7-Pay premium test premium per thousand for a policy at the insured's issue age.

[D] TaxBasis$_t$ = Policy's tax basis, year t

[D] 7702Basis$_t$ = Policy's IRS section 7702 basis, year t

[P] DBInd = 1 for an increasing death benefit policy
0 for a level death benefit policy

[P] SurrPer000$_t$ = Surrender charge in year t, per thousand of face amount issued in year 1

[D] $CV_t(P_j, F_j, L_j, W_j \mid j=1,\ldots t)$
= Cash value, end of year t, as a function of prior transactions

[P] $ULim_t^{UL}$ = Upper limit on the amount of money the policyholder desires to put into the UL contract, year t

[P] $LLim_t^{UL}$ = Lower limit on the amount of money the policyholder desires to put into the UL contract, year t

[P] MinNAR$_t$ = Policyholder's minimum desired net-amount-at-risk, year t

In the model, F_t will be allowed to vary from year to year, to accommodate the dollar-for-dollar reduction in face amount that occurs on level death benefit policies when withdrawals are made. As part of the linear programming solution, the model will solve for the minimum face amount at issue, F_1, so that the policy will always meets its minimum insurance element requirement, as defined by the input parameters $MinNAR_t$, and all other constraints. As developed here, F_t, though it may vary by duration, is not meant to model increases or decreases in face amount other than those due to withdrawals. The model assumes that withdrawals of cash value can be made without invoking a surrender charge;[1] thus, once F_1 is determined, the surrender charge in effect each duration is fixed based on the rates in the input parameters $SurrPer000_t$. For the sake of convenience, surrender charges are assumed to be based on the face amount at issue; most other surrender charge patterns (for example, based on a percentage of first year premium) could be modelled equally easily.

The following notation is used to describe the investments, where $\alpha \epsilon$ {MM, IRA, FPA}:

[S]	D_t^α	= Deposit to investment α, beginning of year t
[P]	$LoadPct^\alpha$	= Percent-of-deposit load charged by investment α
[P]	i^α	= Interest rate credited investment α
[S]	W_t^α	= Withdrawal from investment α, beginning of year t
[D]	${}^{TAXABLE}W_t^{FPA}$	= Portion of W_t^{FPA} that is taxable
[D]	${}^{TAXFREE}W_t^{FPA}$	= Portion of W_t^{FPA} that is tax-free
[D]	$FPABasis_t$	= Flexible premium annuity tax basis, year t
[D]	AB_t^α	= Investment α account balance, end of year t
[P]	$ULim_t^\alpha$	= Upper limit on the amount the policyholder desires or is allowed to contribute to investment α in year t
[P]	$LLim_t^\alpha$	= Lower limit on the amount the policyholder desires or is allowed to contribute to investment α in year t

In keeping track of cash flows, deposits to the individual retirement account will be tax-deductible, interest will accumulate at i^{IRA} and will not be

[1]More common than a surrender charge upon making a withdrawal is a fixed fee for the transaction. The imposition of such a charge cannot be modelled using linear programming techniques, because making a withdrawal then results in a discontinuity in the account value. Such charges can be modelled using integer programming techniques. Generically, a mathematical programming problem in which there is a fixed cost associated with one or more variables only when they are non-zero is called, appropriately enough, as a *fixed charge problem*.

currently taxed, and all withdrawals will be fully taxable. Deposits to the money-market fund are assumed to be made with after-tax funds, and accumulate at an after-tax interest rate i^{MM} ; thus, withdrawals are not taxable. Flexible premium annuity deposits are assumed to be made from after-tax funds which accumulate tax-deferred at i^{FPA}. In contrast to the individual retirement account and the money-market fund, annuity taxation rules necessitate keeping track of the taxable and non-taxable component of each withdrawal.

Some miscellaneous notation follows:

[P]	n	= Number of years the insurance and investment program is to remain in force
[P]	FitRate	= Policyholder's marginal federal income tax rate
[P]	$TPInd_t$	= Tax penalty indicator for IRA or FPA withdrawals, equals 1 if a withdrawal at the beginning of year t would be deemed to be premature, 0 otherwise[2]
[P]	$IRAMinW\%_t$	= IRA minimum withdrawal percent, year t.[3]
[P]	FCF_t	= Policyholder's (fixed) estimate of the after-tax funds that will be available to fund (if positive) or that will need to be withdrawn from (if negative) the insurance and investment program at the beginning of year t

[2]A withdrawal from an IRA or annuity is generally considered premature if it is made before the owner reaches age 59 1/2; however, there are exceptions to this rule. The taxable portion of any premature withdrawal is hit with an extra ten percent penalty tax. This model can be used to help determine whether in situations in which funds need to be withdrawn from the investment program, it would be worthwhile taking a premature withdrawal from an IRA or annuity, even with the tax penalty.

[3]Current tax law requires that distributions from an IRA commence by age 70 1/2. $IRAMinW\%_t$ is thus zero if the policyholder is less than age 70 1/2 at the beginning of year t; in years in which the policyholder is 70 1/2 or older, $IRAMinW\%_t$ is the reciprocal of the joint and last survivor life expectancy of the insured and the insured's IRA beneficiary, based upon tables promulgated by the IRS. For technical compliance with IRS regulations, some work would be required to ensure that the $TPInd_t$ and $IRAMinW\%_t$ factors, as well as the timing of any required withdrawals, are consistent with IRS timing conventions. IRS requirements are defined in calendar year terms on an age-last-birthday basis, which probably will not coincide with the insurance policy year and the insured's insurance age.

[S] CAB_n = Combined after-tax universal life cash surrender value and alternative investment account balances, end of year n

4.2 Objective Function and Constraints

The goal of this linear programming model is to maximize the sum of the end of year n after-tax universal life cash surrender value and associated investment account balances:

$$\text{MAXIMIZE:} \quad S = CAB_n$$

The nineteen sets of constraints required for this model, labelled (G1) through (G19), are described below. Constraint sets (G1) through (G12) deal specifically with the universal life policy, constraints (G13) through (G17) deal with the investments, and set (G18) links the insurance and investment cash flows. CAB_n is defined in (G19).

(G1) The following constraints relate the universal life loan outstanding each year to the amounts loaned, amounts repaid, and the interest rate charged on loans:[4]

$$L_1 = LT_1$$

$$\text{For } t = 2 \text{ to } n: \quad L_t = LT_t - LR_t + (1+i_{LC})\, L_{t-1}$$

(G2) The following constraints ensure that the contract remains in force, by requiring that the end-of-year cash value less the loan outstanding during the year (including the interest that will accumulate on the loan) equals or exceeds the surrender charge:

$$\text{For } t = 1 \text{ to } n: \quad CV_t - (1+i_{LC})\, L_t \geq .001 \cdot F_1 \cdot SurrPer000_t$$

[4]Linear programming algorithms determine a solution that is on a cornerpoint of the feasible region defined by the constraint equations. In the formulation of this linear programming model, any solution in which both LT_t and LR_t are greater than zero for a given t will not be a cornerpoint of the feasible region. Thus, one or the other of LT_t and LR_t may be greater than zero, but not both.

It is possible to formulate this linear programming model without defining "loans taken" or "loans repaid" variables, by instead defining and solving for only the loan outstanding each year. The loan taken or repaid at the beginning of year t would then be $L_t - (1 + i_{LC})\, L_{t-1}$. This approach is more direct, but somewhat less natural. This same comment applies as well to money market account processing: It would be more direct, but less natural, to set up the model to solve only for the end-of-year account balances, and then back into the required deposits and withdrawals.

(G3) The following constraints administer the reduction in face amount on level death benefit policies that occurs whenever a withdrawal is made:

$$\text{For } t = 1 \text{ to } n\text{-}1: \quad F_{t+1} = F_t - (1 - \text{DBInd}) \cdot W_{t+1}$$

(G4) The following constraints ensure that the policy face amounts through time are sufficient to meet the minimum insurance need, as defined by the MinNAR_t factors:

$$\text{For } t = 1 \text{ to } n: \quad F_t - (1 - \text{DBInd}) \cdot CV_t \geq \text{MinNAR}_t$$

Should the policyholder ever deem his insurance to be clearly secondary to the use of the policy as an investment vehicle, MinNAR_t could be set to zero from that point forward.

Two comments deserve to be made regarding constraint set (G4). First, in the absence of this set of constraints, the model could exhibit undesirable behavior. The objective is to maximize a cash accumulation, and reducing the amount spent for insurance will, all other things being equal, increase the cash accumulation. Thus, if the cost of paying and immediately withdrawing a dollar of premium is less than the cost of purchasing that one dollar of insurance over the period the policy is to be held, the optimal solution could involve overfunding the contract and then withdrawing money, for the sole purpose of reaping the fictitious "benefit" of reducing insurance costs. Clearly this is not desirable.

Second, while constraint set (G4) effectively places a lower limit on the face amount to be purchased, it does not imply an upper limit. Under some circumstances, the model will purchase more face amount than one might initially expect by simply examining the factors MinNAR_t. For example, suppose the tax advantages combined with the investment return on a universal life contract make it a desirable investment vehicle, compared to other available options. If guideline premium constraints (G11) or cash value corridor constraints (G12) for the minimum face amount necessary to meet the insurance need get in the way of fully allocating available funds to the insurance contract, a larger face amount will be purchased if the cost of the additional protection required to make room for additional premium does not dilute the return to such an extent that the contract is no longer the preferred investment.

(G5) As part of the compliance requirements for the IRS section 7702 definition of life insurance, the following set of constraints defines the initial guideline premiums, adjusts them as necessary for any subsequent reductions in face amount when withdrawals are taken from level death benefit policies, and defines the sum of the guideline annual premiums:

$$GAP_1 = .001 \cdot F_1 \cdot GAPAdj000_1 + GAPolFee_1$$

$$GSP_1 = .001 \cdot F_1 \cdot GSPAdj000_1 + GSPolFee_1$$

$$\Sigma GAP_1 = GAP_1$$

and for t = 2 to n:

$$GAP_t = GAP_{t-1} - .001 \cdot (1 - DBInd) \cdot GAPAdj000_t \cdot W_t$$

$$GSP_t = GSP_{t-1} - .001 \cdot (1 - DBInd) \cdot GSPAdj000_t \cdot W_t$$

$$\Sigma GAP_t = \Sigma GAP_{t-1} + GAP_t$$

(G6) The following constraints define the guideline premium limit each year as the greater of the guideline single premium and the sum of the guideline annual premiums:

For t = 1 to n:

$$GSPExc\Sigma GAP_t - \Sigma GAPExcGSP_t = GSP_t - \Sigma GAP_t$$

where:

[D] $GSPExc\Sigma GAP_t$ = Amount by which the guideline single premium exceeds the sum of the guideline annual premiums, year t

[D] $\Sigma GAPExcGSP_t$ = Amount by which the sum of the guideline annual premiums exceeds the guideline single premium, year t

and:

$$GPL_t = \Sigma GAP_t + GSPExc\Sigma GAP_t$$

This technique will be used several times in this paper for defining a variable that is the maximum (or the minimum) of two other variables. In the linear programming solution, in a given year, either $GSPExc\Sigma GAP_t$ will be greater than zero, or $\Sigma GAPExcGSP_t$ will be greater than zero, but not both. If $GSPExc\Sigma GAP_t$ is greater than zero, then the guideline single premium dominates the sum of the guideline annual premiums in year t, and $GSPExc\Sigma GAP_t$ will be added to ΣGAP_t to get the guideline premium limit, GPL_t. Conversely, if the sum of the guideline annual premiums dominates, then GPL_t will equal ΣGAP_t.

(G7) Constraints are required to split withdrawals during the first five years into taxable and non-taxable components. The rules for doing this split, defined in IRS Section 7702(f)(7), are somewhat complicated. Cash distributions are to be recognized as income up to the amount of the gain in the contract, to the extent of the recapture ceiling, which for a withdrawal at the beginning of year t is the larger of:

A. The tax basis of the contract, $TaxBasis_{t-1}$, less the guideline premium limitation GPL_t after the withdrawal

and B. The cash value, CV_{t-1}, immediately prior to the withdrawal less the face amount, F_t, after the withdrawal, divided by the corridor factor $Corr_t$.

The recapture ceiling can be calculated using the following set of constraints:

For $t = 2,\ldots,5$: $RecapA_t - DummyA_t = TaxBasis_{t-1} - GPL_t$

$$RecapB_t - DummyB_t = CV_{t-1} - F_t/Corr_t$$

$$RecapAExcRecapB_t - RecapBExcRecapA_t = RecapA_t - RecapB_t$$

$$RecapCeiling_t = RecapB_t + RecapAExcRecapB_t$$

where:

[D] $RecapA_t$ = Recapture ceiling due to rule (A) above, year t

[D] $RecapB_t$ = Recapture ceiling due to rule (B) above, year t

[D] $DummyA_t$ = "Dummy" variable, used to hold the absolute value of $TaxBasis_{t-1} - GPL_t$ if rule A defines a negative number

[D] $DummyB_t$ = "Dummy" variable, used to hold the absolute value of $CV_{t-1} - F_t/Corr_t$, if rule B defines a negative number

[D] $RecapAExcRecapB_t$ = Amount by which $RecapA_t$ exceeds $RecapB_t$

[D] $RecapBExcRecapA_t$ = Amount by which $RecapB_t$ exceeds $RecapA_t$

[D] $RecapCeiling_t$ = Recapture ceiling for year t

The gain in the contract can determined by adding the constraints:

$$ULGain_t - ULLoss_t = CV_{t-1} - TaxBasis_{t-1}$$

where:

[D] $ULGain_t$ = Gain in the contract, beginning of year t

[D] $ULLoss_t$ = Loss on the contract, beginning of year t

Finally, splitting withdrawals into taxable and tax-free components can be accomplished by adding the constraints:[5]

$$GainExcRecap_t - RecapExcGain_t = ULGain_t - RecapCeiling_t$$

$$W_t^{TAXABLE} = ULGain_t - GainExcRecap_t$$

$$W_t = W_t^{TAXFREE} + W_t^{TAXABLE}$$

where:

[D] $GainExcRecap_t$ = Amount by which $Gain_t$ exceeds $RecapCeiling_t$

[D] $RecapExcGain_t$ = Amount by which $RecapCeiling_t$ exceeds $Gain_t$

(G8) A set of constraints is required to split withdrawals into taxable and non-taxable components in years six through fifteen. The rules are identical to those stated above for withdrawals taken during the first five years, except that the recapture ceiling is defined as (B) above, rather than the greater of (A) and (B).[6]

(G9) The following constraints are sufficient to split withdrawals in year sixteen on, which are taxed only to the extent that they exceed the tax basis, into their taxable and non-taxable components:

For t = 16 to n:

$$W_t = W_t^{TAXFREE} + W_t^{TAXABLE}$$

$$W_t^{TAXFREE} \leq TaxBasis_{t-1}$$

In arriving at an optimal solution to this linear programming problem, notice

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[5]One aspect of the taxation of withdrawals is not addressed by constraints (G6) and (G7), namely the treatment of distributions "made in anticipation of death benefit reductions" referred to in 7702(f)(7)(E). Under this section, the calculation of taxable income upon a withdrawal that reduces benefits requires an examination of any other withdrawals made in the previous two years and a possible recalculation of taxable income. This aspect of the tax code has not been modelled.

[6]Any taxable withdrawals allowed by the model during the first fifteen policy years will be considered to be premiums returned to the policyholder when determining the "sum of premiums paid" under 7702. Thus, such withdrawals offset premiums in the definition of $7702Basis_t$ in (G11).

Although not allowed by the model, the policyholder could have a contractual right to additional taxable withdrawals, if there is still cash value remaining in the contract once tax-free withdrawals up to the full amount of the tax basis have been taken. For a further discussion of withdrawals during the first fifteen policy years under the guideline premium/cash value corridor test, see [10], including the enlightening discussion by J. Peter Duran.

that withdrawals will automatically be allocated to tax-free withdrawals before taxable withdrawals, due to the desirability of deferring taxable income.[7] Conveniently, this parallels the allocation for tax treatment.

(G10) The following constraints define $TaxBasis_t$ for use in determining taxable income:

$$TaxBasis_1 = P_1$$

$$\text{For } t = 2 \text{ to } n: \quad TaxBasis_t = TaxBasis_{t-1} + P_t - W_t^{TAXFREE}$$

(G11) The following constraints define $7702Basis_t$ and ensure that the policy meets the guideline premium limitations of the definition of life insurance:

$$7702Basis_1 = P_1$$

$$\text{For } t = 2 \text{ to } 15: \quad 7702Basis_t = 7702Basis_{t-1} + P_t - W_t$$

$$\text{For } t \geq 16: \quad 7702Basis_t = 7702Basis_{t-1} + P_t - W_t^{TAXFREE}$$

$$\text{For } t = 1 \text{ to } n: \quad 7702Basis_t \leq GPL_t$$

(G12) The following set of constraints ensures that the policy meets the cash value corridor constraints of the definition of life insurance:

$$\text{For } t = 1 \text{ to } n: \quad CV_t \leq (F_t + DBInd \cdot CV_t) \;/\; Corr_t$$

Rather than increase the death benefit when the cash value becomes sufficiently high, these constraints limit the cash value so that the policy never enters the cash value corridor. This approach is necessary because of the desirability of keeping cash values a linear function of prior transactions; linearity breaks down as a policy enters the corridor. Defining the constraints using end-of-year values is sufficient to ensure that the policy has not entered the corridor at any time during the year, since on cash value rich policies such as policies near the corridor, the end-of-year cash value would be expected to be the largest cash value in effect during the year.

(G13) Lastly, the following constraints prohibit the policy from becoming

[7]Given the time value of money, it would generally be to one's benefit to delay to the extent possible the payment of any income tax. This observation holds in the case of a withdrawal when the tax on the withdrawal does not vary with the timing of the withdrawal (as is true in the General Model, since FitRate is fixed by duration). The proper ordering of taxable versus tax-free withdrawals would also be preserved if FitRate were allowed to monotonically increase by duration.

a modified endowment contract. In testing cumulative premiums against modified endowment limits, the smallest face amount in effect during the first seven years is used. Since face reductions occur only when withdrawals are taken from level death benefit policies, the face amount monotonically decreases with time, and thus F_7 may be used in each year's test:

For t = 1 to 7:

$$\sum_{j=1}^{t} (P_j - W_j^{TAXFREE}) \leq t \cdot (.001 F_7) \cdot 7Pay000$$

Constraints (G14) through (G17) administer the investment alternatives:

(G14) The following constraints define the end-of-year account balances for each investment:

For $\alpha \in$ {MM, IRA, FPA}:

$$AB_1^{\alpha} = (1 + i^{\alpha})\ (1 - LoadPct^{\alpha})\ D_1^{\alpha}$$

and for t = 2 to n:

$$AB_t^{\alpha} = (1 + i^{\alpha})\ (AB_{t-1}^{\alpha} + (1 - LoadPct^{\alpha})\ D_t^{\alpha} - W_t^{\alpha})$$

(G15) The following constraints requires that each investment (including the insurance policy) meet the minimum and maximum contribution limits defined by the policyholder (or by tax law, in the case of contribution limits to IRAs).

For t = 1 to n and for $\alpha \in$ {MM, IRA, FPA}:

$$P_t \geq LLim_t^{UL} \text{ and } W_t = 0$$

$$D_t^{\alpha} \geq LLim_t^{\alpha} \text{ and } W_t^{\alpha} = 0$$

$$P_t \leq ULim_t^{UL}$$

$$D_t^{\alpha} \leq ULim_t^{\alpha}$$

The above constraints involving withdrawals prevent the possibility that withdrawals will occur simultaneously with deposits when the input parameters $LLim_t^{\alpha}$ required a deposit. Constraints involving $LLim_t^{\alpha}$ and withdrawals should only be defined when $LLim_t^{\alpha}$ is greater than zero. $LLim_t^{\alpha}$ and $ULim_t^{\alpha}$ factors can be used to force a diversification of investments. Alternatively, constraints placing a minimum or maximum on the account balance of any particular investment as a percentage of total invested funds could be defined.

(G16) The following constraints ensure that the minimum withdrawal requirements from an IRA are met:

For t = 1 to n:

$$W_t^{IRA} \geq AB_{t-1}^{IRA} \cdot IRAMinW\%_t$$

In years in which withdrawals from IRAs are required, no IRA deposits should be allowed, so $ULim_t^{IRA}$ should be set to zero.

(G17) The following constraints keep track of the tax basis of the flexible premium annuity and split withdrawals into taxable and non-taxable components. For the tax basis, let

$$FPABasis_1 = D_1^{FPA}$$

$$\text{For } t = 2 \text{ to } n\text{:} \quad FPABasis_t = FPABasis_{t-1} + D_t^{FPA} - {}^{TAXFREE}W_t^{FPA}$$

To account for the tax treatment of withdrawals, which during the accumulation phase of an annuity are treated as taxable income (with a possible penalty tax) to the extent of any gain in the contract, let:

For t = 2 to n:

$$FPAGain_t - FPALoss_t = AB_{t-1}^{FPA} - FPABasis_{t-1}$$

$${}^{TAXFREE}W_t^{FPA} - FPAGainExcW_t = W_t^{FPA} - FPAGain_t$$

$$W_t^{FPA} = {}^{TAXFREE}W_t^{FPA} + {}^{TAXABLE}W_t^{FPA}$$

where:

[D] $FPAGain_t$ = Gain in the annuity contract, beginning of year t

[D] $FPALoss_t$ = Loss in the annuity contract, beginning of year t

[D] $FPAGainExcW_t$ = Amount by which the annuity gain at the beginning of year t exceeds the amount of the withdrawal, if it does

(G18) The following constraints set the after-tax cash flow into or out of the insurance policy and the investment alternatives equal to the policyholder's expected cash flow each year, as specified by the input factors FCF_t. In the formula below, adjustments are made to account for tax credits available on IRA deposits, and tax charges (including any penalty) on taxable withdrawals from the insurance policy, the individual retirement account, and the flexible premium annuity:

$$
\begin{aligned}
FCF_t \;=\; & P_t + LR_t + D_t^{MM} + (1 - FitRate)\, D_t^{IRA} + D_t^{FPA} \\
& - (1 - FitRate)\, W_t^{TAXABLE} \\
& - (1 - FitRate - .1\, TPInd_t)\, (W_t^{IRA} + {}^{TAXABLE}W_t^{FPA}) \\
& - (LT_t + W_t^{TAXFREE} + {}^{TAXFREE}W_t^{FPA} + W_t^{MM})
\end{aligned}
$$

(G19) The following constraint defines CAB_n, the end of period n combined after-tax account balances which is maximized under the objective function. In defining CAB_n, it is necessary to split the cash surrender value CV_n of the universal life policy into taxable and tax-free components $CV_t^{TAXABLE}$ and $CV_t^{TAXFREE}$, and to similarly split the flexible premium annuity into components ${}^{TAXABLE}AB_t^{FPA}$ and ${}^{TAXFREE}AB_t^{FPA}$. The mechanics of doing this split have already been developed within the context of the taxation of withdrawals, and will not be repeated here. Any outstanding loan, with interest for the year just ended, also needs to be repaid. Therefore:

$$
\begin{aligned}
CAB_n \;=\; & CV_n^{TAXFREE} - (1 + i_{LC})\, L_n + AB_n^{MM} + {}^{TAXFREE}AB_n^{FPA} \\
& + (1 - FitRate)\, CV_n^{TAXABLE} \\
& + (1 - FitRate - .1\, TPInd)\, (AB_n^{IRA} + {}^{TAXABLE}AB_n^{FPA})
\end{aligned}
$$

Under some circumstances, it may be more desirable to maximize an after-tax income stream for a period of years, rather than a single future cash accumulation. This may be accomplished by limiting the fixed cash flow input factors FCF_t to m years, and for years m + 1 thorough n, replacing the fixed input factors FCF_t by the variable $-VCF_t$, while adding the constraints:[8]

$$
\begin{aligned}
VCF_{m+1} &= VCF_{m+2} \\
VCF_{m+2} &= VCF_{m+3} \\
&\;\;\cdots \\
VCF_n &= CAB_n
\end{aligned}
$$

As expressed above, the n - m + 1 after-tax disbursements will be equal in amount. However, it would be a simple matter to adjust the constraint equations

---

[8]Cash flows into the policy have been defined as positive, and cash flows out as negative. To keep VCF_t for t = m + 1,...,n greater than zero, FCF_t is replaced by $-VCF_t$. It is desirable to keep the sign of each VCF_t value positive, both for consistency with CAB_n (which also is a component of the income stream), and to satisfy non-negativity constraints on linear programming variables.

above to obtain a varying income pattern, e.g., one that increases at a fixed rate from year to year to take into account the effects of inflation. Additional work would be required if it is desired to treat payouts from the flexible premium annuity as coming from annuitization, rather than as withdrawals during the accumulation phase.

In running this model, it is assumed that a death benefit option has been chosen. One might expect the policyholder to be indifferent to the death benefit option, so long as the minimum insurance element, $MinNAR_t$, has been met each year. Assuming this to be the case, it would be desirable to run the General Model twice, once under each death benefit option. The death benefit option selected would then be the one under which the model produces the higher value of the objective function.

4.3 An Example

Suppose a prospective forty-five year old policyholder wants to begin an insurance and retirement savings program. He desires a minimum net-amount-at-risk of $100,000 each year until age 65, and has $ 6,000 of after-tax cash flow that he is willing to allocate in any manner among the following investments:

(1) A universal life policy from Company A

(2) A deductible, no-load IRA that credits interest at 9.0% per year, and has a maximum contribution limit of $ 2,000

(3) A flexible premium annuity that charges a 3% load, and credits interest at 9.5% per year

(4) A no-load, tax-free money-market mutual fund that credits interest at 6.5% per year.

Table 4-1 illustrates the optimal insurance purchase (a $ 100,000 increasing death benefit policy) and allocation of funds so as to maximize the age 65 combined after-tax cash accumulation, assuming the policyholder's marginal income tax rate is 28%.

In the example, the IRA is funded with the maximum allowable contribution of $ 2,000 each year. This provides an annual tax credit of $ 560, which becomes

Table 4-1: Optimal Allocation Among Investments and a $ 100,000 Issue Age 45 Increasing Death Benefit Policy From Company A

Year	UL Premium	IRA Deposit	IRA Tax Credit	FPA Premium	MM Deposit	Net Cash Flow	UL NCV	IRA Balance	FPA Balance	MM Balance
1	4,560	2,000	(560)	0	0	6,000	3,686	2,180	0	0
2	4,560	2,000	(560)	0	0	6,000	8,593	4,556	0	0
3	4,560	2,000	(560)	0	0	6,000	13,962	7,146	0	0
4	4,560	2,000	(560)	0	0	6,000	19,836	9,969	0	0
5	3,224	2,000	(560)	1,336	0	6,000	24,882	13,047	1,419	0
6	2,968	2,000	(560)	1,592	0	6,000	30,132	16,401	3,245	0
7	4,072	2,000	(560)	488	0	6,000	37,006	20,057	4,071	0
8	4,072	2,000	(560)	488	0	6,000	44,521	24,042	4,976	0
9	4,072	2,000	(560)	488	0	6,000	52,734	28,386	5,968	0
10	4,072	2,000	(560)	488	0	6,000	61,708	33,121	7,053	0
11	4,072	2,000	(560)	488	0	6,000	71,513	38,281	8,241	0
12	4,072	2,000	(560)	488	0	6,000	82,226	43,907	9,542	0
13	4,072	2,000	(560)	488	0	6,000	93,934	50,038	10,967	0
14	4,072	2,000	(560)	488	0	6,000	106,730	56,722	12,527	0
15	0	2,000	(560)	4,560	0	6,000	116,502	64,007	18,561	0
16	0	2,000	(560)	4,560	0	6,000	127,149	71,947	25,168	0
17	0	2,000	(560)	0	4,560	6,000	138,696	80,603	27,559	4,856
18	0	2,000	(560)	0	4,560	6,000	151,275	90,037	30,177	10,028
19	0	2,000	(560)	0	4,560	6,000	164,971	100,320	33,043	15,537
20	0	2,000	(560)	0	4,560	6,000	179,877	111,529	36,183	21,403
						20th Year After-Tax Balances:	145,474	80,301	30,518	21,403
									Total =	277,696

Assumptions:

IRA contributions are tax-deductible and limited to $ 2,000 per year.
The IRA is no-load and earns interest at 9.0% per year.
The flexible premium annuity has a 3% load and earns interest at 9.5% per year.
The money market fund is no-load and earns interest at a tax-free rate of 6.5% per year.
The policyholder has $ 6,000 of after-tax cash flow to invest each year.
The policyholder's marginal federal income tax rate is 28%.

available for investment in one of the three other investments. During the first fourteen years, the universal life contract is the second most desirable investment, and it is funded to the maximum extent allowed by the guideline premium limitations of IRS section 7702. Any additional money available is allocated to the flexible premium annuity.

In year fifteen, the universal life contract is no longer preferable to the annuity, because the 6% load charged by Company A on universal life premiums cannot be made up by the contract's superior 10% interest crediting rate when funds are to remain on deposit for six years or less; thus, the annuity becomes the preferred investment, after the IRA, for years fifteen and sixteen. From year seventeen on, the annuity's three percent load similarly dilutes its overall return to such an extent that the no-load tax-free money market fund becomes the investment of choice.

A more typical insurance and investment program design would have been to fund the $ 100,000 increasing death benefit policy with level premiums of $ 1,200 per year, while depositing $ 2,000 per year to the IRA and $ 3,360 per year to the annuity. Such an allocation results in a twentieth year after-tax cash accumulation of $ 273,315. Of the $ 4,381 increase in after-tax cash accumulation under the optimal allocation, $ 4,099 is due to shifting money to or from the insurance contract and one of the alternative investments, and $ 282 is due to the allocation of annuity premiums instead to the tax-free money market fund in years seventeen on. The $4,099 increase may appear rather modest when compared to the total accumulated value of $ 277,696. It is more impressive when viewed as the savings that results on a $ 100,000 policy when the policy is utilized most appropriately within the universe of possible investments.

Chapter 5

The General Model as a Cost Comparison Method

5.1 Comparing Universal Life Plans

The General Model may be used as a cost comparison method, to aid the prospective policyholder in purchasing the most cost effective policy, by following these steps:

1. Select a goal (e.g., maximize an after-tax cash accumulation at age sixty-five or an after-tax income stream commencing at age 65) and develop an objective function to reflect that goal.
2. Select the policies (with investment alternatives) to be compared.
3. Define parameters for the constraints equations.
4. Use linear programming to optimally allocate funds among the insurance policy and the alternative investments.
5. Choose as the optimal purchase the policy and allocation that produces the highest value of the objective function.

This method has several advantages over existing cost comparison methods when the traditional methods are used to compare universal life type contracts.

First, the linear programming cost comparison method defined above fully utilizes universal life's premium flexibility, solving for the premium stream (and other policy transactions) that causes each contract to perform optimally when used in conjunction with other available investments. In contrast, traditional cost comparison methods do not account adequately for the premium flexibility of universal life. Under traditional methods, policies can be compared at any desired premium level, but no recognition is given to the fact that some policies will operate better when funded generously, while others will perform well when funded at a lower level.

Second, the linear programming cost comparison method recognizes that there are numerous alternative investment media available to the policyholder that can be used advantageously in conjunction with the insurance plan. The tax aspects of each investment are recognized and utilized.

Third, existing cost comparison methods either are inadequate when they are used to compare policies funded at different levels due to their inability to account for differences in the resulting net death benefit, or overcome this drawback by adjusting for the difference in death benefit based upon an arbitrary scale of term charges. The linear programming cost comparison method skirts this problem by requiring only that each year, a minimum insurance element (as defined by the input parameters $MinNAR_t$) be met. So long as the minimum is met, the policyholder should not care to what extent the policy is funded, as long as it operates optimally when used in combination with the other investment options.

Tables 5-1A and 5-1B illustrate the results of the General Model when the example illustrated in Table 4-1 is applied to purchases of policies from Companies B and C, respectively, rather than Company A. The value of the objective function (the age 65 combined after-tax insurance and investment account balances) for each potential purchase is summarized below:

Company A	\$ 277,696	(#1)
Company B	\$ 276,470	(#3)
Company C	\$ 277,428	(#2)

Thus, under the assumptions given, a policy from Company A, structured with alternative investments as illustrated in Table 4-1, is the preferred purchase; Company C and Company B come in second and third, respectively. Interestingly enough, had the comparison been done simply by comparing twentieth year cash values under a \$ 1,200 per year level funding scheme, the ranking of policies would have been reversed:

Company A	\$ 39,455	(#3)
Company B	\$ 40,998	(#1)
Company C	\$ 39,790	(#2)

The increases in after-tax cash accumulation over a more typical strategy that allocates a level \$ 1,200 per year to the insurance contract, \$ 2,000 per year to the IRA, and \$ 3,360 to the annuity, are:

Company A	\$ 277,696 - \$ 273,315 = \$ 4,318
Company B	\$ 276,470 - \$ 274,426 = \$ 2,044

Table 5-1A: Optimal Allocation Among Investments and a $ 100,000 Issue Age 45 Increasing Death Benefit Policy From Company B

Year	UL Premium	IRA Deposit	IRA Tax Credit	FPA Premium	MM Deposit	Net Cash Flow	UL NCV	IRA Balance	FPA Balance	MM Balance
1	4,560	2,000	(560)	0	0	6,000	3,700	2,180	0	0
2	613	2,000	(560)	3,947	0	6,000	4,540	4,556	4,193	0
3	0	2,000	(560)	4,560	0	6,000	4,805	7,146	9,434	0
4	0	2,000	(560)	4,560	0	6,000	5,070	9,969	15,174	0
5	0	2,000	(560)	4,560	0	6,000	5,333	13,047	21,459	0
6	0	2,000	(560)	4,560	0	6,000	5,592	16,401	28,341	0
7	0	2,000	(560)	4,560	0	6,000	5,844	20,057	35,877	0
8	0	2,000	(560)	4,560	0	6,000	6,083	24,042	44,128	0
9	0	2,000	(560)	4,560	0	6,000	6,305	28,386	53,164	0
10	0	2,000	(560)	4,560	0	6,000	6,502	33,121	63,058	0
11	0	2,000	(560)	4,560	0	6,000	6,669	38,281	73,892	0
12	0	2,000	(560)	4,560	0	6,000	6,797	43,907	85,755	0
13	0	2,000	(560)	4,560	0	6,000	6,882	50,038	98,745	0
14	0	2,000	(560)	4,560	0	6,000	6,916	56,722	112,969	0
15	0	2,000	(560)	4,560	0	6,000	6,886	64,007	128,545	0
16	0	2,000	(560)	4,560	0	6,000	6,781	71,947	145,600	0
17	0	2,000	(560)	0	4,560	6,000	6,537	80,603	159,432	4,856
18	0	2,000	(560)	0	4,560	6,000	6,186	90,037	174,578	10,028
19	0	2,000	(560)	0	4,560	6,000	5,705	100,320	191,163	15,537
20	0	2,000	(560)	0	4,560	6,000	5,073	111,529	209,323	21,403
						20th Year After-Tax Balances:	5,073	80,301	169,693	21,403
									Total =	276,470

Assumptions:

IRA contributions are tax-deductible and limited to $ 2,000 per year.
The IRA is no-load and earns interest at 9.0% per year.
The flexible premium annuity has a 3% load and earns interest at 9.5% per year.
The money market fund is no-load and earns interest at a tax-free rate of 6.5% per year.
The policyholder has $ 6,000 of after-tax cash flow to invest each year.
The policyholder's marginal federal income tax rate is 28%.

Table 5-1B: Optimal Allocation Among Investments and a $ 100,000 Issue Age 45 Increasing Death Benefit Policy From Company C

Year	UL Premium	IRA Deposit	IRA Tax Credit	FPA Premium	MM Deposit	Net Cash Flow	UL NCV	IRA Balance	FPA Balance	MM Balance
1	1,002	2,000	(560)	3,558	0	6,000	0	2,180	3,779	0
2	207	2,000	(560)	4,353	0	6,000	0	4,556	8,762	0
3	4,560	2,000	(560)	0	0	6,000	4,625	7,146	9,594	0
4	4,560	2,000	(560)	0	0	6,000	9,641	9,969	10,505	0
5	4,560	2,000	(560)	0	0	6,000	15,084	13,047	11,503	0
6	4,560	2,000	(560)	0	0	6,000	20,986	16,401	12,596	0
7	4,560	2,000	(560)	0	0	6,000	27,390	20,057	13,793	0
8	4,560	2,000	(560)	0	0	6,000	34,330	24,042	15,103	0
9	4,560	2,000	(560)	0	0	6,000	41,851	28,386	16,538	0
10	4,560	2,000	(560)	0	0	6,000	49,994	33,121	18,109	0
11	4,560	2,000	(560)	0	0	6,000	59,348	38,281	19,829	0
12	4,560	2,000	(560)	0	0	6,000	69,568	43,907	21,713	0
13	4,340	2,000	(560)	220	0	6,000	80,501	50,038	24,010	0
14	3,935	2,000	(560)	625	0	6,000	92,014	56,722	26,955	0
15	3,935	2,000	(560)	625	0	6,000	104,587	64,007	30,180	0
16	3,935	2,000	(560)	625	0	6,000	118,315	71,947	33,712	0
17	3,935	2,000	(560)	0	625	6,000	133,247	80,603	36,914	666
18	3,935	2,000	(560)	0	625	6,000	149,542	90,037	40,421	1,376
19	0	2,000	(560)	0	4,560	6,000	163,076	100,320	44,261	6,321
20	0	2,000	(560)	0	4,560	6,000	177,793	111,529	48,466	11,589

	UL NCV	IRA Balance	FPA Balance	MM Balance
20th Year After-Tax Balances:	147,841	80,301	37,698	11,589
			Total =	277,428

Assumptions:

IRA contributions are tax-deductible and limited to $ 2,000 per year.
The IRA is no-load and earns interest at 9.0% per year.
The flexible premium annuity has a 3% load and earns interest at 9.5% per year.
The money market fund is no-load and earns interest at a tax-free rate of 6.5% per year.
The policyholder has $ 6,000 of after-tax cash flow to invest each year.
The policyholder's marginal federal income tax rate is 28%.

Company C $ 277,428 - $ 273,556 = $ 3,872

In this example, the least desirable policy under its most optimal strategy performs better than any of the three policies under the typical strategy.

Examining Tables 4-1, 5-1A and 5-2B, it becomes apparent that the optimal strategy for each policy is quite different. In Table 4-1, once the IRA has been fully funded, Policy A becomes the preferred investment during the *first fourteen* years. In contrast, in Table 5-1A, Policy B, with both a higher percent-of-premium load (6%) and a lower interest crediting rate (9%) than the flexible premium annuity, is funded only during the first two years. In Table 5-1B, Policy C, with a relatively small percent-of-premium load (2%) and an interest rate that increases from 9% to 10% in year eleven, is the preferred investment, after the IRA, in years three through eighteen. Actually, premiums paid into Policy C during years one and two would accumulate to more than if paid into the annuity or the money-market fund, but guideline premium limitations restrict the cumulative amount that can be paid into the policy. Given a choice between using this "limited resource" (i.e, premiums up to the guideline premium limit) in the first two years or in later years, the linear programming solution chooses to defer investment in the universal life contract to those periods in which the difference between the after-tax accumulation on a dollar paid into the universal life policy and a dollar paid into the annuity is greatest.

5.2 Purchases of Multiple Universal Life Plans

Suppose a prospective forty-five year old policyholder is interested in purchasing a $ 100,000 universal life contract and funding it with an annual premium of $ 1,200. If he intends to surrender the policy at age 65, which contract would he prefer to purchase, the policy from Company A, Company B, or Company C?

Table 5.2 illustrates the buildup of cash surrender under each of these policy designs over a twenty year period. Since the contracts have identical twentieth year cash values, ignoring the differences in cash values through year nineteen, the prospective purchaser presumably would find each of these three

Policy A: $ 100,000 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$1,200	$72	$262	$99	$965	$750	$215
2	1,200	72	280	194	2,007	700	1,307
3	1,200	72	300	297	3,132	650	2,482
4	1,200	72	320	409	4,350	600	3,750
5	1,200	72	342	530	5,665	550	5,115
6	1,200	72	365	660	7,088	500	6,588
7	1,200	72	391	801	8,626	450	8,176
8	1,200	72	421	953	10,286	400	9,886
9	1,200	72	453	1,117	12,079	350	11,729
10	1,200	72	489	1,295	14,013	300	13,713
11	1,200	72	526	1,486	16,100	250	15,850
12	1,200	72	566	1,693	18,355	200	18,155
13	1,200	72	605	1,916	20,794	150	20,644
14	1,200	72	643	2,158	23,436	100	23,336
15	1,200	72	683	2,420	26,302	50	26,252
16	1,200	72	722	2,704	29,412	0	29,412
17	1,200	72	761	3,013	32,792	0	32,792
18	1,200	72	798	3,349	36,471	0	36,471
19	1,200	72	833	3,715	40,481	0	40,481
20	1,200	72	863	4,114	44,861	0	44,861

Policy B: $ 100,000 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$1,200	$72	$210	$91	$1,010	$750	$260
2	1,200	72	223	182	2,097	700	1,397
3	1,200	72	237	279	3,266	650	2,616
4	1,200	72	251	383	4,526	600	3,926
5	1,200	72	267	496	5,883	550	5,333
6	1,200	72	283	617	7,345	500	6,845
7	1,200	72	302	748	8,919	450	8,469
8	1,200	72	323	889	10,613	400	10,213
9	1,200	72	346	1,040	12,436	350	12,086
10	1,200	72	371	1,203	14,396	300	14,096
11	1,200	72	397	1,378	16,504	250	16,254
12	1,200	72	425	1,566	18,774	200	18,574
13	1,200	72	451	1,769	21,221	150	21,071
14	1,200	72	477	1,988	23,861	100	23,761
15	1,200	72	503	2,225	26,710	50	26,660
16	1,200	72	529	2,480	29,789	0	29,789
17	1,200	72	554	2,756	33,118	0	33,118
18	1,200	72	579	3,054	36,722	0	36,722
19	1,200	72	601	3,377	40,626	0	40,626
20	1,200	72	620	3,728	44,861	0	44,861

Policy C: $ 100,000 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$1,200	$54	$272	$91	$965	$750	$215
2	1,200	54	287	177	2,007	700	1,307
3	1,200	54	303	270	3,132	650	2,482
4	1,200	54	318	369	4,350	600	3,750
5	1,200	54	332	476	5,665	550	5,115
6	1,200	54	350	592	7,088	500	6,588
7	1,200	54	368	716	8,626	450	8,176
8	1,200	54	392	849	10,286	400	9,886
9	1,200	54	417	992	12,079	350	11,729
10	1,200	54	451	1,145	14,013	300	13,713
11	1,200	54	491	1,454	16,100	250	15,850
12	1,200	54	528	1,663	18,355	200	18,155
13	1,200	54	564	1,890	20,794	150	20,644
14	1,200	54	600	2,135	23,436	100	23,336
15	1,200	54	640	2,401	26,302	50	26,252
16	1,200	54	680	2,689	29,412	0	29,412
17	1,200	54	724	3,002	32,792	0	32,792
18	1,200	54	767	3,342	36,471	0	36,471
19	1,200	54	809	3,712	40,481	0	40,481
20	1,200	54	845	4,115	44,861	0	44,861

Table 5-2: Comparison of Values Under Policies from Company A, B and C
Male Age 45, Face Amount = $ 100,000, Annual Premium = $ 1,200

Policy A: $ 48,855 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$716	$43	$128	$60	$606	$366	$239
2	1,140	68	135	161	1,703	342	1,361
3	1,129	68	142	269	2,891	318	2,573
4	1,117	67	150	386	4,178	293	3,885
5	1,105	66	157	513	5,572	269	5,303
6	1,091	65	165	651	7,084	244	6,839
7	1,076	65	173	800	8,722	220	8,502
8	1,058	63	181	962	10,497	195	10,302
9	1,038	62	189	1,137	12,421	171	12,250
10	1,016	61	198	1,327	14,506	147	14,359
11	0	0	211	1,439	15,734	122	15,612
12	0	0	224	1,561	17,072	98	16,975
13	666	40	230	1,757	19,226	73	19,153
14	858	51	233	1,991	21,791	49	21,742
15	858	51	231	2,247	24,613	24	24,588
16	858	51	225	2,530	27,724	0	27,724
17	303	18	218	2,789	30,579	0	30,579
18	0	0	208	3,047	33,418	0	33,418
19	0	0	190	3,331	36,559	0	36,559
20	0	0	161	3,647	40,045	0	40,045

Policy B: $ 51,145 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$484	$29	$107	$36	$384	$384	$0
2	60	4	116	34	358	358	0
3	71	4	124	32	332	332	0
4	83	5	134	30	307	307	0
5	95	6	144	29	281	281	0
6	109	7	155	27	256	256	0
7	125	7	168	25	230	230	0
8	142	9	183	24	205	205	0
9	162	10	200	22	179	179	0
10	184	11	219	21	153	153	0
11	1,200	72	236	104	1,149	128	1,021
12	1,200	72	254	193	2,216	102	2,114
13	534	32	275	231	2,674	77	2,598
14	342	21	298	255	2,953	51	2,902
15	342	21	325	279	3,229	26	3,203
16	342	21	354	302	3,499	0	3,499
17	897	54	382	372	4,333	0	4,333
18	1,200	72	409	472	5,523	0	5,523
19	1,200	72	439	577	6,789	0	6,789
20	1,200	72	471	690	8,136	0	8,136

Policy A and B Combined: $ 100,000 Total Level Death Benefit

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$1,200	$72	$235	$96	$989	$750	$239
2	1,200	72	251	195	2,061	700	1,361
3	1,200	72	267	301	3,223	650	2,573
4	1,200	72	283	417	4,485	600	3,885
5	1,200	72	301	542	5,853	550	5,303
6	1,200	72	320	678	7,339	500	6,839
7	1,200	72	341	826	8,952	450	8,502
8	1,200	72	364	986	10,702	400	10,302
9	1,200	72	389	1,160	12,600	350	12,250
10	1,200	72	417	1,348	14,659	300	14,359
11	1,200	72	447	1,543	16,884	250	16,634
12	1,200	72	477	1,754	19,289	200	19,089
13	1,200	72	505	1,989	21,900	150	21,750
14	1,200	72	531	2,246	24,743	100	24,643
15	1,200	72	556	2,526	27,841	50	27,791
16	1,200	72	579	2,832	31,222	0	31,222
17	1,200	72	600	3,161	34,912	0	34,912
18	1,200	72	617	3,518	38,941	0	38,941
19	1,200	72	630	3,909	43,348	0	43,348
20	1,200	72	633	4,337	48,181	0	48,181

Table 5-3A: Optimal Combination of Policies from Companies A and B
Male Age 45, Total Face Amount = $ 100,000, Total Annual Premium = $ 1,200

Policy A: $ 25,000 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$443	$27	$65	$38	$390	$188	$202
2	1,039	62	68	133	1,432	175	1,257
3	1,022	61	70	235	2,558	163	2,395
4	1,004	60	71	346	3,776	150	3,626
5	987	59	73	467	5,098	138	4,960
6	604	36	75	563	6,153	125	6,028
7	0	0	79	611	6,685	113	6,573
8	0	0	84	664	7,265	100	7,165
9	0	0	89	722	7,897	88	7,809
10	0	0	95	785	8,587	75	8,512
11	0	0	100	853	9,340	63	9,277
12	0	0	105	928	10,163	50	10,113
13	0	0	109	1,010	11,065	38	11,027
14	0	0	112	1,100	12,053	25	12,028
15	0	0	114	1,199	13,139	13	13,126
16	0	0	114	1,308	14,333	0	14,333
17	0	0	112	1,427	15,648	0	15,648
18	0	0	106	1,559	17,101	0	17,101
19	0	0	97	1,705	18,708	0	18,708
20	0	0	82	1,866	20,492	0	20,492

Policy C: $ 75,000 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$757	$45	$205	$55	$563	$563	$0
2	161	33	218	53	525	525	0
3	178	34	233	50	488	488	0
4	196	34	248	48	450	450	0
5	213	34	262	45	412	413	0
6	596	42	279	75	762	375	387
7	1,200	54	293	159	1,774	338	1,437
8	1,200	54	313	249	2,856	300	2,556
9	1,200	54	335	345	4,012	263	3,749
10	1,200	54	363	448	5,243	225	5,018
11	1,200	54	397	619	6,611	188	6,423
12	1,200	54	429	754	8,081	150	7,931
13	1,200	54	462	899	9,665	113	9,553
14	1,200	54	494	1,056	11,373	75	11,298
15	1,200	54	532	1,225	13,213	38	13,175
16	1,200	54	571	1,407	15,194	0	15,194
17	1,200	54	615	1,603	17,328	0	17,328
18	1,200	54	661	1,813	19,627	0	19,627
19	1,200	54	710	2,041	22,103	0	22,103
20	1,200	54	759	2,286	24,776	0	24,776

Policy A and C Combined: $ 100,000 Total Level Death Benefit

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$1,200	$72	$270	$94	$952	$750	$202
2	1,200	96	286	186	1,957	700	1,257
3	1,200	95	302	286	3,045	650	2,395
4	1,200	94	319	394	4,226	600	3,626
5	1,200	93	334	512	5,510	550	4,960
6	1,200	78	354	637	6,916	500	6,416
7	1,200	54	373	770	8,459	450	8,009
8	1,200	54	398	913	10,120	400	9,720
9	1,200	54	424	1,067	11,909	350	11,559
10	1,200	54	458	1,233	13,830	300	13,530
11	1,200	54	497	1,472	15,951	250	15,701
12	1,200	54	535	1,682	18,245	200	18,045
13	1,200	54	571	1,910	20,730	150	20,580
14	1,200	54	606	2,157	23,426	100	23,326
15	1,200	54	645	2,424	26,351	50	26,301
16	1,200	54	685	2,714	29,527	0	29,527
17	1,200	54	726	3,030	32,976	0	32,976
18	1,200	54	768	3,372	36,727	0	36,727
19	1,200	54	807	3,745	40,812	0	40,812
20	1,200	54	841	4,152	45,268	0	45,268

Table 5-38: Optimal Combination of Policies from Companies A and C
Male Age 45, Total Face Amount = $ 100,000, Total Annual Premium = $ 1,200

Policy B: $ 41,700 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$395	$24	$88	$29	$313	$313	$0
2	49	3	94	28	292	292	0
3	58	3	101	26	271	271	0
4	67	4	109	25	250	250	0
5	78	5	117	23	229	229	0
6	89	5	127	22	209	209	0
7	102	6	137	21	188	188	0
8	116	7	149	20	167	167	0
9	132	8	163	18	146	146	0
10	150	9	179	17	125	125	0
11	169	10	196	16	104	104	0
12	574	34	213	48	478	83	394
13	482	29	232	73	772	63	709
14	189	11	253	73	770	42	728
15	189	11	277	72	743	21	722
16	189	11	304	68	685	0	685
17	189	11	334	62	590	0	590
18	189	11	368	51	451	0	451
19	189	11	408	37	258	0	258
20	189	11	454	17	0	0	0

Policy C: $ 58,300 Level Death Benefit Policy

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$805	$46	$159	$62	$662	$437	$225
2	1,151	53	166	152	1,747	408	1,339
3	1,142	53	173	248	2,911	379	2,532
4	1,133	53	180	352	4,163	350	3,813
5	1,122	52	186	463	5,510	321	5,189
6	1,111	52	194	583	6,958	292	6,666
7	1,098	52	201	712	8,516	262	8,253
8	1,084	52	211	850	10,188	233	9,955
9	1,068	51	220	999	11,983	204	11,779
10	1,050	51	233	1,158	13,908	175	13,733
11	1,031	51	248	1,477	16,116	146	15,971
12	626	43	262	1,657	18,096	117	17,979
13	718	44	273	1,864	20,360	87	20,272
14	1,011	50	279	2,118	23,160	58	23,101
15	1,011	50	284	2,398	26,235	29	26,206
16	1,011	50	284	2,706	29,617	0	29,617
17	1,011	50	279	3,044	33,343	0	33,343
18	1,011	50	265	3,417	37,456	0	37,456
19	1,011	50	240	3,830	42,006	0	42,006
20	1,011	50	199	4,287	47,055	0	47,055

Policy B and C Combined: $ 100,000 Total Level Death Benefit

Year	Premium	Expense Charges	Cost of Insurance	Interest Credited	Cash Value	Surrender Charge	Net Cash Value
1	$1,200	$70	$246	$91	$975	$750	$225
2	1,200	56	260	179	2,039	700	1,339
3	1,200	56	275	274	3,182	650	2,532
4	1,200	57	289	377	4,413	600	3,813
5	1,200	57	303	487	5,739	550	5,189
6	1,200	58	320	605	7,166	500	6,666
7	1,200	58	338	733	8,703	450	8,253
8	1,200	59	360	870	10,355	400	9,955
9	1,200	59	383	1,017	12,129	350	11,779
10	1,200	60	412	1,176	14,033	300	13,733
11	1,200	61	444	1,493	16,221	250	15,971
12	1,200	77	475	1,705	18,573	200	18,373
13	1,200	73	505	1,936	21,132	150	20,982
14	1,200	62	532	2,192	23,930	100	23,830
15	1,200	62	561	2,470	26,978	50	26,928
16	1,200	62	588	2,774	30,302	0	30,302
17	1,200	62	613	3,106	33,934	0	33,934
18	1,200	62	634	3,449	37,907	0	37,907
19	1,200	62	648	3,867	42,264	0	42,264
20	1,200	62	652	4,304	47,055	0	47,055

Table 5-3C: Optimal Combination of Policies from Companies B and C
Male Age 45, Total Face Amount = $ 100,000, Total Annual Premium = $ 1,200

policies equally palatable.

A more optimal purchase than buying any single contract would be to purchase two contracts which have a combined face amount of $ 100,000, and a combined annual premium payment of $ 1,200. Letting $\alpha, \beta \in$ {Company A, Company B, Company C}, a linear programming problem may be set up with the goal to:

$$MAXIMIZE: \quad S = CV_{20}^{\alpha} + CV_{20}^{\beta}$$

subject to the constraints that the total policy face amount equals $ 100,000, the total annual premiums equal $ 1,200, and the face amount of each policy meets the minimum face amount requirement of each issuing company:

$$F^{\alpha} + F^{\beta} = 100{,}000$$

$$\text{For } t = 1 \text{ to } 20: \quad P_t^{\alpha} + P_t^{\beta} = 1{,}200$$

$$F^{\alpha} \geq 25{,}000$$

$$F^{\beta} \geq 25{,}000$$

Finally, each contract is required to have a non-negative cash surrender value at the end of each year, must meet guideline premium constraints, and is prohibited from entering the cash value corridor.

Tables 5.3A through Table 5.3C illustrate the optimal face amounts and premium payment patterns for each combination of policies from two companies. The twentieth year cash surrender values of these combinations are summarized below:

Company A and Company B	$ 48,181
Company A and Company C	$ 45,268
Company B and Company C	$ 47,055

Because the linear programming solution plays the mortality, expense and interest elements of each policy off against each other, the resulting cash surrender value of each combination of policies is greater than the $ 44,861 available on any single policy. For example, the policies from Company A and Company B have identical 6% percent of premium expense charges; however, they have different cost-of-insurance scales and different interest crediting rates. Since the twentieth year cash values of the two plans are identical, the total expense recovery from the two plans is similar, but Company A, with an interest crediting rate of 10% compared with the 9% rate for Company B, recovers more money from any expense margin in the cost-of-insurance rates than does Company B, and less money from the interest spread. In the optimal purchase of plans from Companies A and B shown in Table 5-3A, roughly equal face amounts are purchased from Company A ($ 48,455) and Company B ($51,145), but the policy from Company A is generously funded, thus taking advantage of its high interest crediting rate, while the policy from Company B is marginally funded, thus taking advantage of its lower cost-of-insurance rates.

In fact, for the first ten years, the policy from Company B is funded only to the extent necessary to cover the surrender charge. Conversely, the policy from Company A is funded as generously as allowed: The sum of the premiums paid through the first ten years, $ 10,486, matches the guideline premium limit for a $ 48,855 face amount purchase, determined at this point by the guideline single premium. Funding of Policy B in years eleven and twelve, at $ 1,200 per year, is done only because no further funding of policy A is allowed: It is not until year thirteen that the guideline premium limit for Policy A is determined by the sum of the guideline annual premiums. In year thirteen, Policy A is funded with $ 666, the amount by which thirteen guideline annual premiums exceeds the guideline single premium. In years fourteen through sixteen it is funded with $ 858, which is the amount of the annual increment to the guideline premium limit.

Funding for Policy A ceases in year seventeen. This is because any further funding would bring the policy into the cash value corridor. As it stands, the twentieth year cash value for Policy A of $ 40,045 exactly equals the face amount of $ 48,855 divided by the corridor factor of 122% for an attained age forty-five insured.

Purchases involving contracts from Company C result in a smaller gain in cash surrender value ($ 45,268 when combined with Company A; $ 47,055 when combined with Company C) than the $ 48,181 available when contracts from Company

A and Company B are purchased together. One reason for this is the per policy charge on contracts purchased from Company C. Unlike other contract charges, this "overhead" is not reduced proportionally when a smaller policy size is purchased. Many universal life contracts available on the market have per policy charges. At the $ 100,000 total face amount level, it is possible that the overhead associated with the purchase of two policies, each with a per policy charge, would exceed the gain that could be obtained through linear programming by playing the various cost elements off against each other. On the other hand, at larger face amounts the effect of a per policy charge would be diluted, and a linear programming strategy could be expected to provide a more significant gain.

One might expect that the greater the divergence of cost structures on two policies, the larger the gain that could be expected from an optimal two policy purchase. As an example, one might expect a linear programming solution to favorably exploit a combination of a plan with a select-and-ultimate cost-of-insurance structure with a plan with a reverse select-and-ultimate cost-of-insurance structure.

There may be other advantages to purchasing multiple policies from different companies. First, such a purchase provides a limited hedge against one company adjusting the nonguaranteed cost elements of its contract, since future premium payments could be rebalanced to offset the effect of any adverse change (or, for the optimist, to take further advantage of any favorable change). Second, in cases in which an insured intends to heavily utilize the investment features of a contract, but also anticipates a reduced insurance need at some future point in time, purchasing two contracts, one of which is for an amount equal to the future reduction in insurance need and is to be surrendered, can be superior to purchasing a single contract with the intention of electing a future face decrease. Guideline premium limits allow for more generous funding of the contract that remains in force under the former option than is allowed for the single contract that undergoes a face reduction, which has its guidelines adjusted downward under the an attained-age decrement approach specified in the

"Dole-Bentsen colloquy" [10].

Of course, these advantages may be offset not only by the effects of any per policy charges, as discussed above, but also by any effect splitting a single policy purchase in two has on the policyholder's ability to purchase lower cost insurance due to the availability of banded products.

5.3 Comparing Universal Life and Traditional Plans

The linear programming cost comparison method can be used to compare universal life policies to traditional plans of insurance. In making the comparison, a face amount needs to be selected for each traditional plan. Then, premiums less dividends for that plan are subtracted from the total funds available each year, and the remaining funds are allocated optimally among the investment alternatives, so as to maximize the after-tax account accumulations. The strategy above can be used satisfactorily in comparing universal life with either permanent, cash value plans, or term insurance.

In comparing universal life and ordinary life policies, the premium flexibility of universal life will often cause it to be the favored contract. For the sake of illustration, suppose that one can purchase a \$ 100,000 current assumption whole life policy on a life aged 45 from Company A for \$ 1,200 per year. Suppose further that the assumptions underlying the pricing of the whole life policy are identical to those used to price its universal life counterpart, so that the cash values that develop under the two policies are identical. If the prospective purchaser cannot afford the \$ 1,200, the universal life contract is the preferred policy by default, since it may be purchased at a premium level lower than \$ 1,200 per year. If the prospective purchaser can afford more than \$1,200 per year and the universal life contract is a better investment vehicle than any of the alternative investments, additional funds will be allocated to the universal life policy, thus enhancing its total account accumulation relative to the accumulation of the current assumption policy plus its side funds. Conversely, if the universal life contract is a less attractive investment vehicle than one of the investments, it will be funded with less than \$ 1,200 per year, with the balance going to one of the better investments, thus again enhancing its accumulated value relative to that of the current assumption whole life policy.

In comparing universal life to "buy term and invest the difference" strategies, universal life enjoys advantageous tax treatment, because premiums required to cover cost-of-insurance charges are added to the policy cost basis and thus can be used to offset the taxability of an equal portion of the policy's investment income. This is illustrated in Tables 5.4A and 5.4B. Table 5.4A shows the optimal allocation of $ 1,500 per year[1] between a $ 100,000 increasing death benefit universal life policy and a no-load flexible premium deferred annuity that also credits 10% interest. Generally, one would expect the annuity to be preferred over the universal life policy as an investment vehicle, since the credited interest rates are identical, but no load is deducted from contributions to the annuity. In spite of this, however, the universal life policy is funded with premiums larger than the minimum necessary to keep the policy in-force, so that the twentieth year cash surrender value will equal the sum of the premiums paid, thus taking full advantage of the basis offset to investment income.[2] Table 5-4B illustrates an allocation between a $ 100,000 yearly renewable term contract and the flexible premium annuity that results in a total after-tax account balance that is identical to that of the universal life and flexible premium annuity combination.[3] In order to achieve equality, the term rates were set at approximately 88% of the universal life cost-of-insurance rates. Factoring in the universal life contract's percent-of-premium load, in this instance the investment income offset effect of the universal life's

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[1]$ 1,500 per year was chosen as an amount that would be sufficient to fund an attained-age yearly renewable term contract, without making withdrawals of previously accumulated excess funds.

[2]This effect can also be seen in optimal purchase of a policy from Company B illustrated in Table 5-1A.

[3]Assuming a 28% marginal tax rate, the after-tax balance for the universal life policy combined with the annuity is $ 7,369.30 + $ 22,630.70 + (1 - .28) X ($ 53025.15 - $ 22,630.70) = $ 51,884.01. For the term policy combined with the annuity, it is $ 17,605.05 + (1 - .28) X ($ 65214.72 - $ 17,605.05) = $ 51,884.01.

Table 5.4A: Universal Life Policy Combined With a No-load Tax-Deferred Annuity

	$ 100,000 Issue Age 45 Increasing Death Benefit Universal Life Policy from Company A			No-Load Annuity @ 10%		
Year	Premium	Tax Basis	NCV	Premium	Tax Basis	Account Balance
1	1500.00	1500.00	1272.12	0.00	0.00	0.00
2	1500.00	3000.00	2648.68	0.00	0.00	0.00
3	1500.00	4500.00	4138.62	0.00	0.00	0.00
4	1500.00	6000.00	5751.47	0.00	0.00	0.00
5	1369.30	7369.30	7361.06	130.70	130.70	143.77
6	0.00	7369.30	7684.29	1500.00	1630.70	1808.15
7	0.00	7369.30	8003.29	1500.00	3130.70	3638.96
8	0.00	7369.30	8311.27	1500.00	4630.70	5652.86
9	0.00	7369.30	8602.18	1500.00	6130.70	7868.14
10	0.00	7369.30	8866.68	1500.00	7630.70	10304.96
11	0.00	7369.30	9096.26	1500.00	9130.70	12985.45
12	0.00	7369.30	9280.68	1500.00	10630.70	15934.00
13	0.00	7369.30	9411.21	1500.00	12130.70	19177.40
14	0.00	7369.30	9476.59	1500.00	13630.70	22745.14
15	0.00	7369.30	9460.25	1500.00	15130.70	26669.65
16	0.00	7369.30	9343.84	1500.00	16630.70	30986.62
17	0.00	7369.30	9106.34	1500.00	18130.70	35735.28
18	0.00	7369.30	8720.43	1500.00	19630.70	40958.81
19	0.00	7369.30	8153.64	1500.00	21130.70	46704.69
20	0.00	7369.30	7369.30	1500.00	22630.70	53025.16

Table 5.4B: Yearly Renewable Term Policy Combined With a No-load Tax-Deferred Annuity
YRT Premiums are approximately 88% of the Company A UL cost-of-insurance rate

	$ 100,000 Issue Age 45 YRT Policy	No-Load Annuity @ 10%		
Year	Premium	Premium	Tax Basis	Account Balance
1	234.79	1265.21	1265.21	1391.73
2	253.97	1246.03	2511.24	2901.54
3	274.41	1225.59	3736.83	4539.84
4	296.40	1203.60	4940.43	6317.79
5	321.17	1178.83	6119.26	8246.28
6	347.66	1152.34	7271.60	10338.47
7	378.45	1121.55	8393.15	12606.03
8	414.60	1085.40	9478.55	15060.57
9	454.90	1045.10	10523.65	17716.23
10	501.65	998.35	11521.99	20586.04
11	553.34	946.66	12468.66	23685.97
12	610.69	889.31	13357.96	27032.81
13	671.62	828.38	14186.35	30647.31
14	737.47	762.53	14948.88	34550.83
15	811.79	688.21	15637.08	38762.94
16	894.70	605.30	16242.39	43305.06
17	986.85	513.15	16755.54	48200.04
18	1091.81	408.19	17163.72	53469.05
19	1211.62	288.38	17452.10	59133.16
20	1347.05	152.95	17605.05	65214.72

contract's tax basis will support loadings within the universal life contract of up to 21% of the term rates before buying term becomes the preferred strategy.

Recently, several companies have introduced versions of traditional products that provide the policyholder some degree of premium flexibility through combining a base policy with term and paid-up additions riders. To make a fair comparison between a universal life contract and these traditional combinations, a cost comparison method ideally should take into account any flexibility available in designing the traditional plan of insurance. Section 7.4 discusses how linear programming can be applied as a tool to construct optimal combinations of traditional insurance.

5.4 Concerns of the Selling Company

Presumably, anything that is gained by the purchaser of insurance through the use of linear programming cost comparison techniques is lost by one of the other parties to the transaction. Whether insurance companies would embrace linear programming methods would depend upon the extent to which the gain of the policyholder cuts into the profit of the insurer.

Universal life policies are priced to recover expenses and generate profits through the margin between the earned interest rate and credited interest rate, the margin between the cost-of-insurance rate charged to the policyholder and the company's actual mortality experience, and a combination of explicit expense charges. The choice of where expenses are recovered is determined in part by the intended market for the product -- for example, plans intended to attract single premium buyers typically will recover only a small portion of total expenses through the interest margin, since a high credited rate is instrumental in encouraging single premium sales. Policies intended for a broad market require balancing expense recovery between the various elements, and generally a company's profit objective will be exceeded under some possible uses of the policy and fall short under others. Even products priced for the same market which have similar underlying assumptions may differ markedly as to where expenses are recovered and profits generated.

Linear programming techniques exploit imbalances in the expense recovery structure to the maximum benefit of the policyholder.[4] All other things being equal, most likely the policy selected from a set of alternatives as the best choice for a prospective policyholder will be one in which the imbalance in expense recovery is most acute. Whether an insurer would be at risk for losing money on a block of universal life issues would be a function both of the extent to which a policy can be utilized in a manner that favors the policyholder to the detriment of the insurer, and of the market efficiency of the purchasers of insurance. In the two policy purchase example of Section 5.2, the 7.4% increase in twentieth year cash value obtained by combining policies from Companies A and C rather than purchasing a single policy from either company is of magnitude sufficient to entirely eliminate the profit margin priced into many universal life contracts.

Under certain circumstances, the use of the General Model may actually enhance the company's profitability as well as that of the policyholder. For example, in situations in which the tax advantages granted life insurance cause a universal life contract to be the investment vehicle of choice, the General Model may result in the purchase of a larger face amount than would have been contemplated had only a strict need for insurance been considered. In such a

[4]To a lesser extent, this is true as well when traditional cost comparison methods are applied to flexible premium policies. For example, the figures below give the interest-adjusted-net-cost at 8% for policies from Companies A, B and C, for a $100,000 level death benefit policy purchased on a forty-five year old under several level premium scenarios:

	Interest-Adjusted-Net-Cost		
Premium	Company A	Company B	Company C
$ 1,000 ("low")	3.60	3.23	3.59
$ 1,200	2.92	2.92	2.92
$ 1,500 ("high")	1.90	2.47	1.92

Under the low premium scenario, the policy from Company B is preferred, as might be expected since relative to the other policies, Company B recovers a larger portion of its expenses from percent-of-premium charges and the interest margin. Conversely, under the high premium scenario, Company A is preferred, which is to be expected since Company A has the smallest interest margin.

Unfortunately, indices such as the interest-adjusted-net-cost are often presented at interest rates that do not reflect the current environment, thus leading to distortions in the ratings. For example, numerous states mandate that the interest-adjusted-net-cost, calculated at 5%, be provided to prospective purchasers of insurance.

case, a portion of the policyholder's gain is funded with lost tax revenue, rather than at the expense of the company. As another example, the General Model will not allocate discretionary funds to the universal life contract in the few years before surrender if the percent-of-premium load charged would reduce the overall return on those premiums to below what would be available had the money been deposited into one of the alternative investments. This will enhance the profitability of the product if, as is often the case, the percent-of-premium load on the contract is insufficient to cover the sum of premium taxes, commissions, and any percent-of-premium based home office expenses. Such circumstances, however, are the exception rather than the rule.

Chapter 6

Maximizing the Return on an In Force Plan on an Impaired Life

This chapter develops a model (the "Impaired Life Model") to be used by the owner of a universal life policy on an insured who is in ill-health. The model recognizes that several options available within the typical universal life contract allow some manipulation of the net-amount-at-risk, and solves for the set of transactions that optimizes the owner's return, in the sense of maximizing the actuarial present value of future cash flows.

6.1 The Model

The approach taken to developing cash values in the Impaired Life Model differs from the approach taken in the models that have preceded it, in that cash values are explicitly calculated on a monthly basis, by using the relationships in (2.1.1), (2.1.8), or (2.5.1) within the constraint equations. This month-by-month approach is taken for several reasons. First, the insured will be assumed to be very ill; the resulting short expected future lifetime invites a more frequent approach to transaction processing, and makes the number of constraints defined by monthly cash value accumulation equations manageable. Second, a month-by-month approach allows precise recognition of the cash value corridor requirements of IRS Section 7702. As a result of this monthly processing, subscripts referring to time will be in months since the commencement of the model, rather than in years.

To simplify the presentation, it is assumed that the contract has been in force for fifteen years. Under current tax law, such an assumption significantly reduces the amount of overhead required to account for the tax treatment of the policy. First, withdrawals can unambiguously be treated as tax-free until the accumulated withdrawals exceed the tax basis. Second, assuming that the contract is not a modified endowment contract, one does not need to guard against the possibility that the policy could become one. Third, in keeping track of the guideline premium limits of section 7702, it is fairly safe to assume that the limiting factor is the sum of the guideline annual premiums; thus, developing the mechanics required to allow premiums up to the maximum of the guideline single and the sum of the guideline annual premiums is unnecessary. Techniques for handling the taxation of withdrawals within the first fifteen years of a policy have already been developed within the context of the General Model, as have the mechanics of fully accounting for guideline premium limitations; these techniques could be adopted for use in the Impaired Life Model if necessary.

The following notation is used to describe the assumption regarding the future lifetime of the insured in ill-health:[1]

[P]	n	= Maximum number of months the insured could live
[P]	${}_tP$	= Probability the insured will live through month t
[P]	${}_{t\|}q$	= ${}_tP - {}_{t+1}P$
		= Probability that the insured will die in month t+1

The following notation is used to describe characteristics of the insurance policy and the policyowner's status at the time the model commences (i.e., sometime after issue, when the insured has fallen into ill health):

[P]	CV_0	= Initial end-of-month cash value
[P]	F_0	= Initial face amount
[P]	ΣGAP_0	= Initial 7702 sum of guideline annual premiums
[P]	GAP_0	= Initial guideline annual premium
[P]	$TaxBasis_0$	= Policy's tax basis
[P]	TaxDBInd	= 1 if the death benefit will be subject to income tax, 0 otherwise
[P]	FitRate	= Policyowner's federal income tax rate
[P]	$7702Basis_0$	= Initial sum of premiums paid, 7702 basis
[P]	DBInd	= 1 for an increasing death benefit policy 0 for a level death benefit policy
[P]	E%	= Percent of premium expense charge
[P]	EF	= Amount per thousand of face amount deducted at

[1]In this chapter, "q" will be used to denote a rate of death. To avoid confusion between this "q" and a cost-of-insurance rate, in this chapter the cost of insurance rate will be denoted COI_t.

the beginning of each monthly cycle

[P] EP = Per policy charge deducted at the beginning of each monthly cycle

[P] $PlanPrem_t$ = Planned or billed-for premium for month t (This should reflect the premium payment mode.)

[P] $PolAnnInd_t$ = 1 if the first day of the month is a policy anniversary, 0 otherwise

[P] v = Present value discount factor, reflecting an after-tax interest rate available on investments

This development is intended to be sufficiently flexible to allow for federal income taxation of death proceeds.[2] For the sake of simplicity, it is assumed that the policyowner pays the premium and is someone other than the insured, so that gift and estate tax treatment is not an issue.[3]

The following notation is used to describe policy elements through time:

[S] P_t = Premium paid, beginning of month t

[S] W_t = Withdrawal taken, beginning of month t

[D] $W_t^{TAXABLE}$ = Portion of W_t that is taxable

[D] $W_t^{TAXFREE}$ = Portion of W_t that is tax-free

---

[2]While death proceeds generally escape federal income tax, there are exceptions, notably the "transfer-for-value rule" under which death proceeds lose their federal income tax exemption if transferred for a consideration.

Recently, some firms have offered to purchase, at a significant discount from the expected death benefit, the life insurance contracts of people who are diagnosed as terminally ill. Though ostensibly providing a service by offering a (possibly taxable) "living death benefit," these firms have also come under some criticism for what many consider to be their less than altruistic motives [2]. Such a purchase will come under the transfer for value rule. This model provides a method for maximizing the return on such a purchase.

[3]Modelling estate tax treatment would require some care. If the insured maintains incidents of ownership in the policy, so that the death benefit is includable in the gross estate, taxation of death proceeds may be modelled if the marginal estate tax rate can be estimated.

In addition, the impact of cash that may optionally be held either within or outside of the insurance contract should be modelled. For example, presumably money paid as premium no longer will be includable in the estate and thus escapes estate taxes (except to the extent it may have an impact on the universal life contract's death benefit, in the case of increasing death benefit contracts or contracts in the cash value corridor). Conversely, some or all of the money withdrawn from the contract could end up being subject to income tax, and then be includable in the estate as well.

[D]	CV_t	= Cash value, end of month t
[P]	SC_t	= Surrender charge, end of month t
[S]	F_t	= Month t face amount
[D]	DB_t	= Month t death benefit
[P]	$Corr_t$	= Month t IRS Section 7702 corridor factor
[P]	COI_t	= Month t annual cost-of-insurance rate, per dollar net-amount-at-risk
[D]	ΣGAP_t	= Sum of guideline annual premiums, month t
[D]	GAP_t	= Guideline annual premium, month t
[P]	$GAPAdj000_t$	= Time t guideline annual premium adjustment required for each thousand dollar change in face (Needed for level death benefit policies only)
[D]	$TaxBasis_t$	= Policy's tax basis at time t
[D]	$7702Basis_t$	= Sum of premiums paid, 7702 basis

As an objective, it is desired to maximize the actuarial present value of future cash flows:[4]

MAXIMIZE:

$$\sum_{t=1}^{n} {}_{t-1}p\ v^{\frac{t-1}{12}} \left(W_t^{TAXFREE} + (1-FitRate)\, W^{TAXABLE} - P_t\right)$$
$$+\sum_{t=1}^{n} {}_{t-1}q\ v^{\frac{t}{12}} \left(DB_t - FitRate \cdot TaxDBInd(DB_t - TaxBasis_t)\right)$$

This maximization is done subject to the following constraints:

(I1) It is necessary to define the policy's death benefit, including IRS section 7702 corridor considerations. The following constraint equations will account for the mechanics of the corridor:

For t = 1 to n:

$$CorrExcPol_t - PolExcCorr_t =$$
$$(CORR_t - DBInd)\ (CV_{t-1} + (1-E\%)P_t - EP - EF \cdot F_t/1000 - W_t) - F_t$$

---

[4]It is implicitly assumed that since the insured is in ill-health, it is desirable to keep the coverage in force at all cost.

and

$$DB_t = F_t + DBInd\ (CV_{t-1} + (1-E\%)P_t - EP - EF \cdot F_t/1000 - W_t) + CorrExcPol_t$$

where:

$CorrExcPol_t$ = Amount by which the death benefit defined by the corridor exceeds the "normal" policy death benefit, month t

$PolExcCorr_t$ = Amount by which the "normal" policy death benefit exceeds the death benefit defined by the corridor, month t

The effect of these constraints is to define a variable, $CorrExcPol_t$, that is the increase in death benefit required under IRS section 7702 when the policy is in the corridor, and to add it to the death benefit that would be in effect in the absence of cash value corridor requirements.

(I2) The following constraints relate each month's cash value to the cash value of the previous month, per (2.1.1), (2.1.8) and (2.5.1):

For t = 1 to n:

$$CV_t = \left(\left(1+\frac{COI_t}{12}\right) \left(CV_{t-1} + (1-E\%)\,P_t - EP - EF\frac{F_t}{1000} - W_t\right) - \frac{COI_t}{12}\,\frac{DB_t}{(1+i_g)^{1/12}} \right) (1+i_c)^{1/12}$$

(I3) The following set of constraints ensures that the policy has sufficient cash value to avoid lapsing:

For t = 1 to n: $CV_t \geq SC_t$

(I4) The following constraints, required only for level death benefit policies, reduces the policy face amount each time there is a withdrawal at the beginning of the month:[5]

For t = 1 to n: $F_t = F_{t-1} - W_t$

(I5) The following constraints, required only for level death benefit policies, adjust the guideline annual premium for the reduction in face amount due to a

[5] Withdrawals of cash value when the policy is in the corridor would not reduce the face amount of the policy if the policy were still in the corridor after the withdrawal. No attempt has been made to distinguish in this set of constraints between withdrawals that occur outside the corridor and withdrawals that occur when the policy is in the corridor. It would appear unlikely that an optimal solution maximizing the return on the policy on an impaired life would involve withdrawing money when the policy was in the corridor, since this would result in a more than dollar-for-dollar reduction in the death benefit.

cash value withdrawal:

$$\text{For } t = 1 \text{ to } n: \quad GAP_t = GAP_{t-1} - GAPAdj000_t \cdot W_t/1000$$

(I6) The following constraints define the sum of the guideline annual premiums. Each anniversary, the sum of the guideline annual premiums is incremented by the current guideline annual premium, adjusted as necessary for level death benefit policies for the effects any withdrawals for the current month have on the face amount. On monthiversaries that are not policy anniversaries, withdrawals on level death benefit policies cause the sum of the guideline annual premiums for the current year to be adjusted downward to account for the reduction in face. As developed here, mid-year downward adjustments to the guideline annual premiums are treated effectively as if they occurred on the prior policy anniversary:

For $t = 1$ to n:

$$\text{If } PolAnnInd_t = 1, \ \Sigma GAP_t = \Sigma GAP_{t-1} + GAP_t$$

$$\text{If } PolAnnInd_t = 0, \ \Sigma GAP_t = \Sigma GAP_{t-1} - GAPAdj000_t \cdot W_t/1000$$

(I7) The following constraints relate W_t, $W_t^{TAXFREE}$, and $W_t^{TAXABLE}$:

For $t = 1$ to n:

$$W_t = W_t^{TAXFREE} + W_t^{TAXABLE}$$

$$W_t^{TAXFREE} \leq TaxBasis_{t-1}$$

As noted in developing the General Model, the optimal solution to a linear programming problem in which withdrawals up to the tax basis may be taken tax-free is to take any such withdrawals before taking taxable withdrawals. Thus, no additional constraints are required to ensure the proper treatment of the order of withdrawals.

(I7) The following constraints adjust the tax basis for the effects of any premium payments or withdrawals:

$$\text{For } t = 1 \text{ to } n: \quad TaxBasis_t = TaxBasis_{t-1} + P_t - W_t^{TAXFREE}$$

(I8) The following constraints adjust the 7702 basis for the effects of any premium payments or withdrawals and ensure that the 7702 basis does not exceed guideline premium limitations:

$$\text{For } t = 1 \text{ to } n: \quad 7702Basis_t = 7702Basis_{t-1} + P_t - W_t^{TAXFREE}$$

$$7702Basis_t \leq \Sigma GAP_t$$

(19) Constraints are required to account for a company's rules regarding planned premium changes and the payment of unscheduled premiums. The following contract language is typical:

> Planned annual premiums are shown on the first page of this contract. Payments can be annual, semi-annual or quarterly, or can be at any frequency agreed to by the Company. You may increase or decrease the amount of subsequent payments, subject to the Limits on Premiums below. Unscheduled payments may be made at any time prior to the maturity date, subject to the Limits on Premiums below.
>
> Limits on Premiums:
>
> * Total planned and unscheduled payments will be limited to the Company's published maximums.
>
> * Proof that the Insured is insurable will be required if an unscheduled payment increases the Death Benefit by more than it increases the cash value.

Many companies allow the planned premium to be adjusted to any "reasonable" level without evidence of insurability; unscheduled payments, however, are allowed without underwriting only when the payment does not bring the policy into the corridor to such an extent that the net-amount-at-risk increases. Interestingly enough, underwriting is generally not required if a person pays the largest premium such that the policy enters the corridor but the net-amount-at-risk remains the same.[6] Following such an unscheduled premium payment, the policyholder could resume his scheduled premium payments, and drive up the net-amount-at-risk with each payment. Under some circumstances, the policyholder will benefit by this strategy.

The limitations on premium payments contained in the contract language above may be modelled using the following constraint equations:

---

[6]Any premium payment on a level death benefit policy not in the cash value corridor will reduce the policy's net-amount-at-risk until the policy reaches the corridor. When the policy reaches the corridor, any further payment will result in an increase in the net-amount-at-risk. Thus, for a level death benefit policy not in the corridor, one may solve for the premium payment that brings the policy into the corridor and results in the net-amount-at-risk being equal to the net-amount-at-risk before the payment was made.

On an increasing death benefit policy, a premium payment will have no effect on the net-amount-at-risk until the policy reaches the corridor. Thus, for an increasing death benefit policy one may solve for the largest premium payment that keeps the net-amount-at-risk from increasing.

For $t = 1$ to n:

$$PlanPrem_t + CPExcPP_t - PPExcCP_t =$$

$$\{F_t + (EF \cdot F_t/1000 + EP - CV_{t-1})(Corr_t - DBInd)\}/\{(Corr_t - 1)(1 - E\%)\}$$

and $$P_t \leq PlanPrem_t + CPExcPP_t$$

where:

$CPExcPP_t$ = Amount by which the premium necessary to bring the policy into the corridor, keeping the net-amount-at-risk level, exceeds the month t planned premium

$PPExcCP_t$ = Amount by which the month t planned premium exceeds the premium necessary to bring the policy into the corridor, keeping the net-amount-at-risk level.

Assuming the policy is not already in the corridor, the right hand side of the first equation expresses the premium required to bring the policy into the corridor without changing the net-amount-at-risk.[7] If this amount exceeds the planned premium, the second equation adds the amount by which it exceeds the planned premium to the planned premium to get the total premium allowed for the month.

Several comments should be made regarding this model. First, in running the model, it is desirable to have the planned premium as high as possible, so that the option is there to pay generous premiums, should such a strategy be optimal. This suggests that at the time the model commences, a policyholder should request that the planned premium be increased to the maximum a company would allow for the contract under consideration. These maximums would then be reflected in the input factors $PlanPrem_t$. Second, under the premise that the insured is very ill when the model commences, it is unlikely that either withdrawals or premium needed to bring a policy not already into the corridor

[7] If the cash value of the policy is already sufficiently large that the death benefit is determined by the product of the cash value and the corridor factor, the expression on the right hand side is negative. In this case, $CPExcPP_t$ will be zero, PPExcCP will a meaningless positive number, and the premium for the month will be constrained to be not larger than the planned premium.

into the corridor will occur after the first month.[8] Assuming that both withdrawals and payments necessary to bring a policy not in the corridor into the corridor occur only in the first month significantly reduces the programming required for the model.

6.2 Examples

Tables 6.1 through 6.3 show the optimal strategy of a policyholder who owns a $ 100,000 level death benefit policy from Company A on an insured who was age 45 at issue and who, having just reached age 65, has fallen into ill health. It is assumed that the insured's death will occur within the next thirty-six months, with death being equally likely to occur in any of those months. The policyowner will recover the death benefit tax-free; present values of future cash flows are taken at 8%. The tables differ in that the premiums paid during the first twenty years are not identical; thus, the model commences with different starting cash values and different tax bases.

In Table 6.1, the policyholder paid $ 1,200 each year for five years, and then stopped paying premium. As a result, the policy has almost no cash value. Not surprisingly, the optimal strategy is to pay nothing until premium is required to keep the policy from lapsing, and then to pay only the minimum required each month to keep the policy in force. The resulting actuarial present value of future cash flows of $ 87,720 compares with an actuarial present value of $ 34,536[9] had the policyowner not reacted optimally, but rather had continued with his prior behavior of not paying premium. This comparison, of course, is misleading since a rational policyholder would not fail to react to a lapse

[8] As will be seen, withdrawals are sometimes desirable if by getting money out of a policy, the policyowner can cause future net-amounts-at-risk to be greater than what they would have been had the withdrawal not been taken. In these cases, it is desirable to take the withdrawal as early as possible, so that the impact on future net-amounts-at-risk can commence as early as possible.

Paying premium to put a policy into the corridor can be desirable if, by doing so, the policyowner can cause an increase in future net-amounts-at-risk, due to the operation of the corridor factor. Again, when this is desirable, it should be done as early as possible.

[9] This figure assumes a two month grace period, but ignores certain adjustments to the death benefit that would be made by the insurer upon death.

Table 6.1: Optimal Transactions for an Age 65 Level Death Benefit Policyholder

Month	Face Amount	Premium (BOM)	Withdrawals Tax-free (BOM)	Withdrawals Taxable (BOM)	Cash Value (EOM)	Death Benefit (EOM)	Tax and 7702 Basis	Maximum Guideline Premium
Start	100,000				1,547		6,000	36,876
1	100,000				1,420	100,000	6,000	36,876
2	100,000				1,291	100,000	6,000	36,876
3	100,000				1,162	100,000	6,000	36,876
4	100,000				1,031	100,000	6,000	36,876
5	100,000				899	100,000	6,000	36,876
6	100,000				766	100,000	6,000	36,876
7	100,000				631	100,000	6,000	36,876
8	100,000				495	100,000	6,000	36,876
9	100,000				358	100,000	6,000	36,876
10	100,000				220	100,000	6,000	36,876
11	100,000				80	100,000	6,000	36,876
12	100,000	64			0	100,000	6,064	36,876
13	100,000	165			0	100,000	6,230	38,632
14	100,000	165			0	100,000	6,395	38,632
15	100,000	165			0	100,000	6,561	38,632
16	100,000	165			0	100,000	6,726	38,632
17	100,000	165			0	100,000	6,892	38,632
18	100,000	165			0	100,000	7,057	38,632
19	100,000	165			0	100,000	7,223	38,632
20	100,000	165			0	100,000	7,388	38,632
21	100,000	165			0	100,000	7,554	38,632
22	100,000	165			0	100,000	7,719	38,632
23	100,000	165			0	100,000	7,885	38,632
24	100,000	165			0	100,000	8,050	38,632
25	100,000	183			0	100,000	8,233	40,388
26	100,000	183			0	100,000	8,416	40,388
27	100,000	183			0	100,000	8,599	40,388
28	100,000	183			0	100,000	8,782	40,388
29	100,000	183			0	100,000	8,965	40,388
30	100,000	183			0	100,000	9,148	40,388
31	100,000	183			0	100,000	9,330	40,388
32	100,000	183			0	100,000	9,513	40,388
33	100,000	183			0	100,000	9,696	40,388
34	100,000	183			0	100,000	9,879	40,388
35	100,000	183			0	100,000	10,062	40,388
36	100,000	183			0	100,000	10,245	40,388

Assumptions: Policy was issued at age 45; policyholder paid $1,200 per year for five years, and then stopped paying premium.

Death is assumed to occur during the next 36 months, and is equally likely in any month. Present values assume death occurs end-of-month.

BOM = Beginning-of-month, EOM = End-of-month

Actuarial present value of future cash flows @ 8% = $ 87,720.

Table 6.2: Optimal Transactions for an Age 65 Level Death Benefit Policyholder

Month	Face Amount	Premium (BOM)	Withdrawals Tax-free (BOM)	Withdrawals Taxable (BOM)	Cash Value (EOM)	Death Benefit (EOM)	Tax and 7702 Basis	Maximum Guideline Premium
Start	100,000				44,861		24,000	36,876
1	76,000		24,000		20,950	76,000	0	35,629
2	76,000				21,039	76,000	0	35,629
3	76,000				21,129	76,000	0	35,629
4	76,000				21,219	76,000	0	35,629
5	76,000				21,311	76,000	0	35,629
6	76,000				21,403	76,000	0	35,629
7	76,000				21,497	76,000	0	35,629
8	76,000				21,591	76,000	0	35,629
9	76,000				21,686	76,000	0	35,629
10	76,000				21,782	76,000	0	35,629
11	76,000				21,879	76,000	0	35,629
12	76,000				21,977	76,000	0	35,629
13	76,000				22,068	76,000	0	36,139
14	76,000				22,159	76,000	0	36,139
15	76,000				22,251	76,000	0	36,139
16	76,000				22,344	76,000	0	36,139
17	76,000				22,438	76,000	0	36,139
18	76,000				22,533	76,000	0	36,139
19	76,000				22,629	76,000	0	36,139
20	76,000				22,726	76,000	0	36,139
21	76,000				22,824	76,000	0	36,139
22	76,000				22,922	76,000	0	36,139
23	76,000				23,022	76,000	0	36,139
24	76,000				23,122	76,000	0	36,139
25	76,000				23,215	76,000	0	36,648
26	76,000				23,309	76,000	0	36,648
27	76,000				23,403	76,000	0	36,648
28	76,000				23,499	76,000	0	36,648
29	76,000				23,595	76,000	0	36,648
30	76,000				23,692	76,000	0	36,648
31	76,000				23,791	76,000	0	36,648
32	76,000				23,890	76,000	0	36,648
33	76,000				23,990	76,000	0	36,648
34	76,000				24,091	76,000	0	36,648
35	76,000				24,193	76,000	0	36,648
36	76,000				24,296	76,000	0	36,648

Assumptions: Policy was issued at age 45; policyholder paid $1,200 per year for twenty years.

Death is assumed to occur during the next 36 months, and is equally likely in any month. Present values assume death occurs end-of-month.

BOM = Beginning-of-month, EOM = End-of-month

Actuarial present value of future cash flows @ 8% = $ 91,647.

Table 6.3: Optimal Transactions for an Age 65 Level Death Benefit Policyholder

Month	Face Amount	Premium (BOM)	Withdrawals Tax-free (BOM)	Withdrawals Taxable (BOM)	Cash Value (EOM)	Death Benefit (EOM)	Tax and 7702 Basis	Maximum Guideline Premium
Start	100,000				81,738		16,000	36,876
1	100,000	10,186			92,015	109,575	26,186	36,876
2	100,000	146			92,862	110,583	26,332	36,876
3	100,000	146			93,715	111,599	26,479	36,876
4	100,000	146			94,575	112,623	26,625	36,876
5	100,000	146			95,441	113,655	26,771	36,876
6	100,000	146			96,314	114,694	26,918	36,876
7	100,000	146			97,194	115,742	27,064	36,876
8	100,000	146			98,080	116,798	27,210	36,876
9	100,000	146			98,974	117,862	27,357	36,876
10	100,000	146			99,874	118,934	27,503	36,876
11	100,000	146			100,781	120,014	27,649	36,876
12	100,000	146			101,695	121,102	27,796	36,876
13	100,000	146			102,615	121,181	27,942	38,632
14	100,000	146			103,542	122,276	28,088	38,632
15	100,000	146			104,476	123,378	28,235	38,632
16	100,000	146			105,417	124,490	28,381	38,632
17	100,000	146			106,365	125,610	28,527	38,632
18	100,000	146			107,321	126,738	28,674	38,632
19	100,000	146			108,284	127,875	28,820	38,632
20	100,000	146			109,254	129,021	28,966	38,632
21	100,000	146			110,232	130,176	29,113	38,632
22	100,000	146			111,217	131,339	29,259	38,632
23	100,000	146			112,210	132,512	29,405	38,632
24	100,000	146			113,210	133,693	29,552	38,632
25	100,000	146			114,217	133,751	29,698	40,388
26	100,000	146			115,231	134,938	29,844	40,388
27	100,000	146			116,253	136,135	29,991	40,388
28	100,000	146			117,283	137,341	30,137	40,388
29	100,000	146			118,321	138,557	30,283	40,388
30	100,000	146			119,367	139,781	30,430	40,388
31	100,000	146			120,421	141,015	30,576	40,388
32	100,000	146			121,483	142,259	30,722	40,388
33	100,000	146			122,553	143,512	30,869	40,388
34	100,000	146			123,631	144,775	31,015	40,388
35	100,000	146			124,718	146,047	31,161	40,388
36	100,000	146			125,813	147,329	31,308	40,388

Assumptions: Policy was issued at age 45; policyholder paid a single premium of $16,000 at issue.

Death is assumed to occur during the next 36 months, and is equally likely in any month. Present values assume death occurs end-of-month.

BOM = Beginning-of-month, EOM = End-of-month

Actuarial present value of future cash flows @ 8% = $ 100,451.

notice sent by Company A if the insured was ill-health. Rather, the Impaired Life Model confirms the behavior that one would expect of a rational policyholder in this situation, perhaps improving it slightly by guiding the policyholder to pay only the bare minimum required to keep the coverage in force.

In Table 6.2, the policyholder has paid $ 1,200 per year for twenty years and has a cash value of $ 44,861. The optimal strategy is to withdraw an amount that precisely equals the tax basis of $ 24,000, and then pay no more premium. By halting premium payments and taking the maximum tax-free withdrawal, the policyholder slows the growth of cash value and thus slows the erosion of the insurance element in the policy.[10] The actuarial present value of future cash flows of $ 91,647 compares with a present value of $ 86,726 had the policyholder continued his behavior of paying a $ 1,200 annual premium each year, and a present value of $ 89,009 had the policyholder ceased premium payments but not taken the withdrawal.

In Table 6.3, the policyholder paid a single premium of $ 16,000 at issue, and has paid no premium since. The resulting $ 81,738 cash value almost puts the policy into the corridor. Under the assumption that Company A would allow premium of $ 1,756 per year[11] even if the policy is in the corridor, the optimal strategy is to pay $ 10,186 in month one, thus bringing the policy into the corridor with neither an increase nor a decrease in net-amount-at-risk, and then to pay each succeeding month one-twelfth of the maximum allowable planned premium of $ 1,756. The resulting actuarial present value of future cash flows of $ 100,451 compares with a present value of $ 98,846 had the policyholder continued his prior behavior and paid no future premium.

Several variables play a key role in determining whether the optimal strategy for a particular policy involves funding the policy to the maximum

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[10]Had the policyholder taken a sufficiently larger withdrawal, the interest on the remaining cash value would be insufficient to cover the monthly deductions, and the net-amount-at-risk would, in fact, increase with time. In the current situation, however, this is less than optimal, due to taxation of any withdrawals above the tax basis: It is better to leave the money in the policy, and receive it tax-free as a death benefit upon the insured's death.

[11]This figure is the guideline annual premium for the policy at issue.

extent allowed, so as to take advantage of net-amount-at-risk manipulation due to the impact of the corridor on the death benefit. Among them are:

(1) The insured's attained-age. The corridor factor of 1.20 at attained-age 65 yields an increase of twenty cents in net-amount-at-risk for each dollar of corridor cash value. By comparison, at attained-age 40, the corridor factor of 2.50 yields an increase of one dollar and fifty cents for each dollar of corridor cash value. Thus, all other things being equal, the lower the attained-age, the greater the return on a dollar of corridor cash value, and the greater the potential for net-amount-at-risk manipulation.

(2) The policy's percent-of-premium load factor. Paying the premium required to get the policy into the corridor involves a cost. The lower the percent-of-premium load factor, the more likely this cost can be recouped.

(3) Limitations on premium payments, either due to guideline premium limitations or a company's maximum allowable planned premiums. The more generously a contract can be funded, the greater the opportunity to take advantage of the operation of the corridor.

(4) The policy's credited interest rate. A substantial portion of the growth in net-amount-at-risk due to the operation of the corridor comes from interest credits on these high cash value policies. The greater the interest rate, the more rapid the growth in cash value due to interest credits, and hence the greater the growth in net-amount-at-risk.

(5) The assumed future lifetime random variable of the insured. Very short expected future lifetimes allow only very limited funding in excess of the amount required to bring the policy into the corridor, and thus limit the opportunity to manipulate the net-amount-at-risk. Conversely, very long expected future lifetimes may result in significant insurance charges being incurred before death occurs and may bring the insured's attained-age at death up to a point at which the corridor factor adds little to the death benefit.

(6) The policy's starting cash value, compared with the cash value required to put the policy in the corridor. On low and moderate cash value policies, expenses involved in paying premium sufficient to bring the policy into the

Table 6.4: Optimal Transactions for an Age 65 Increasing Death Benefit Policyholder

Month	Face Amount	Premium (BOM)	Withdrawals Tax-free (BOM)	Withdrawals Taxable (BOM)	Cash Value (EOM)	Death Benefit (EOM)	Tax and 7702 Basis	Maximum Guideline Premium
Start	55,139				44,861		24,000	37,400
1	55,139				45,299	100,438	24,100	37,400
2	55,139				45,647	100,786	24,100	37,400
3	55,139				45,998	101,137	24,100	37,400
4	55,139				46,352	101,491	24,100	37,400
5	55,139				46,708	101,847	24,100	37,400
6	55,139				47,068	102,207	24,100	37,400
7	55,139				47,430	102,569	24,100	37,400
8	55,139				47,795	102,934	24,100	37,400
9	55,139				48,163	103,302	24,100	37,400
10	55,139				48,533	103,672	24,100	37,400
11	55,139				48,907	104,046	24,100	37,400
12	55,139				49,283	104,422	24,100	37,400
13	55,139				49,662	104,801	24,100	39,680
14	55,139				50,043	105,182	24,100	39,680
15	55,139				50,428	105,567	24,100	39,680
16	55,139				50,815	105,954	24,100	39,680
17	55,139				51,205	106,344	24,100	39,680
18	55,139				51,599	106,738	24,100	39,680
19	55,139				51,995	107,134	24,100	39,680
20	55,139				52,394	107,533	24,100	39,680
21	55,139				52,797	107,936	24,100	39,680
22	55,139				53,202	108,341	24,100	39,680
23	55,139				53,611	108,750	24,100	39,680
24	55,139				54,023	109,162	24,100	39,680
25	55,139				54,437	109,576	24,100	41,960
26	55,139				54,854	109,993	24,100	41,960
27	55,139				55,275	110,414	24,100	41,960
28	55,139				55,699	110,838	24,100	41,960
29	55,139				56,126	111,265	24,100	41,960
30	55,139				56,556	111,695	24,100	41,960
31	55,139				56,990	112,129	24,100	41,960
32	55,139				57,427	112,566	24,100	41,960
33	55,139				57,867	113,006	24,100	41,960
34	55,139				58,311	113,450	24,100	41,960
35	55,139				58,758	113,897	24,100	41,960
36	55,139				59,208	114,347	24,100	41,960

Assumptions: Policy was issued at age 45 as a level death benefit policy; policyholder paid $ 1,200 per year for twenty years. Death benefit option change was elected before the model was run.

Death is assumed to occur during the next 36 months, and is equally likely in any month. Present values assume death occurs end-of-month.

BOM - Beginning-of-month, EOM - End-of-month

Actuarial present value of future cash flows @ 8% - $ 95,081.

corridor may be too large to be able to be recouped. Also, guideline premium limits may not allow sufficient funding to bring the policy into the corridor.

Tables 6.1 through 6.3 illustrate the optimal strategy for a level death benefit policy. Most universal life contracts allow the policyholder to change the death benefit option without evidence of insurability. Although upon electing a change in death benefit option the face amount of the policy is adjusted to equate the net-amount-at-risk before and after the change, electing the option can have an impact on future net-amounts-at-risk. Thus, this option can sometimes be used to a policyholder's advantage.

Table 6.4 illustrates the optimal strategy on the policy illustrated in Table 6.2, after a change in death benefit option has been elected. The actuarial present value of future cash flows of $ 95,081 exceeds the present value of $ 91,647 that was obtained under the optimal level death benefit strategy. The increase is made up of two components. First, under the level death benefit solution, the $ 24,000 withdrawal slowed but did not halt the erosion in net-amount-at-risk due to interest credits; conversely, when an increasing death benefit option is elected, the net-amount-at-risk in the future remains constant. Second, under the level death benefit case, the $ 24,000 withdrawal contributes exactly $ 24,000 to the present value. Under the increasing death benefit option, that $ 24,000, which stays in the policy, contributes more than $ 24,000, because it accumulates at 10% as cash value, is received tax-free upon the death of the insured, and then is discounted at 8%. This example illustrates that in solving for an optimal solution, it would be advantageous to run the Impaired Life Model once under the policy's current death benefit option, and once assuming a death benefit option change has been elected.[12]

While the examples above illustrate common optimal strategies, other

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[12]Examples can be developed in which it is optimal to change from an increasing death benefit policy to a level death benefit policy.

Changes in death benefit option can have an impact on the maximum funding allowed for a policy. Due to an anomaly in the attained-age-decrement method specified for adjusting guideline premiums when an adjustment in benefits is elected, a change from a level death benefit to an increasing death benefit policy does not necessarily cause an increase in guideline premiums.

Table 6.4: Optimal Transactions for an Age 65 Increasing Death Benefit Policyholder

Month	Face Amount	Premium (BOM)	Withdrawals Tax-free (BOM)	Withdrawals Taxable (BOM)	Cash Value (EOM)	Death Benefit (EOM)	Tax and 7702 Basis	Maximum Guideline Premium
Start	55,139				44,861		24,000	37,400
1	55,139				45,299	100,438	24,100	37,400
2	55,139				45,647	100,786	24,100	37,400
3	55,139				45,998	101,137	24,100	37,400
4	55,139				46,352	101,491	24,100	37,400
5	55,139				46,708	101,847	24,100	37,400
6	55,139				47,068	102,207	24,100	37,400
7	55,139				47,430	102,569	24,100	37,400
8	55,139				47,795	102,934	24,100	37,400
9	55,139				48,163	103,302	24,100	37,400
10	55,139				48,533	103,672	24,100	37,400
11	55,139				48,907	104,046	24,100	37,400
12	55,139				49,283	104,422	24,100	37,400
13	55,139				49,662	104,801	24,100	39,680
14	55,139				50,043	105,182	24,100	39,680
15	55,139				50,428	105,567	24,100	39,680
16	55,139				50,815	105,954	24,100	39,680
17	55,139				51,205	106,344	24,100	39,680
18	55,139				51,599	106,738	24,100	39,680
19	55,139				51,995	107,134	24,100	39,680
20	55,139				52,394	107,533	24,100	39,680
21	55,139				52,797	107,936	24,100	39,680
22	55,139				53,202	108,341	24,100	39,680
23	55,139				53,611	108,750	24,100	39,680
24	55,139				54,023	109,162	24,100	39,680
25	55,139				54,437	109,576	24,100	41,960
26	55,139				54,854	109,993	24,100	41,960
27	55,139				55,275	110,414	24,100	41,960
28	55,139				55,699	110,838	24,100	41,960
29	55,139				56,126	111,265	24,100	41,960
30	55,139				56,556	111,695	24,100	41,960
31	55,139				56,990	112,129	24,100	41,960
32	55,139				57,427	112,566	24,100	41,960
33	55,139				57,867	113,006	24,100	41,960
34	55,139				58,311	113,450	24,100	41,960
35	55,139				58,758	113,897	24,100	41,960
36	55,139				59,208	114,347	24,100	41,960

Assumptions: Policy was issued at age 45 as a level death benefit policy; policyholder paid $ 1,200 per year for twenty years. Death benefit option change was elected before the model was run.

Death is assumed to occur during the next 36 months, and is equally likely in any month. Present values assume death occurs end-of-month.

BOM - Beginning-of-month, EOM - End-of-month patterns are possible. For example, even when the cash value corridor cannot be reached, the optimal strategy on an increasing death benefit policy may be to fund the contract to the maximum extent allowed for several months. This would be true if the credited interest rate exceeds the present value interest rate, the percent-of-premium load is small, and the expected future lifetime is long enough so that premium deposited into the contract in the early months can be expected to be recovered tax-free upon death, having accumulated to a greater amount than it would have had it been kept outside the contract. In such a case, the policy's tax-free death benefit status allows it to be used as a proxy for a tax-free investment, with the funds earning a higher rate than would typically be available on a tax-free investment. As a second example, on level death benefit policies in which the tail end of the future lifetime random variable is long, the optimal strategy could involve withdrawing money at the time the model commences, in order to draw down the cash value to such an extent that the net-amount-at-risk increases with time. Several years in the future, if the insured is still living, premium would then have to be repaid in order to keep the policy in force.

6.3 Other Options with Potential for Net-Amount-at-Risk Manipulation

As noted in the previous section, the ability to change death benefit options contributes to the potential for net-amount-at-risk manipulation within the typical universal life contract. Several other options exist that can also be used to manipulate the net-amount-at-risk.

First, companies will routinely allow a policyholder to change his planned premium billing mode. The table below illustrates the present value of future cash flows under the optimal solution to the Impaired Life Model for the examples in Tables 6.1 and 6.3, as the billing mode is varied:

	Monthly	Quarterly	Semi-Annual	Annual
Table 6.1	\$ 87,720	\$ 87,621	\$ 87,476	\$ 87,192
Table 6.3	\$ 100,451	\$ 100,431	\$ 100,402	\$ 100,348

As can be seen, the billing mode has some impact on the present values. For example, if a level death benefit policy has virtually no cash value and is to be funded so as to just barely keep the policy in force, the billing mode should be switched to monthly, so that premiums can be paid as slowly as possible, in order to avoiding paying any unnecessary premium.[13] If the optimal strategy involves funding a policy so as to take advantage of cash value corridor considerations, the billing mode should be switched to the mode that will allow cash values to build most rapidly.[14]

Second, on corridor policies in which it appears desirable to allow the cash value to grow as large as possible and as quickly as possible, any riders on the policy that appear unlikely to provide a benefit payout could be removed. By reducing the amount of each monthly deduction, this would increase each month's cash value, thus increasing the death benefit by more than a dollar-for-dollar basis due to the operation of the corridor.

Finally, holders of cash value rich policies who are unable to take advantage of corridor manipulation due to a lack of available funds may be able to take advantage of the policy loan option. Taking a policy loan and paying

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[13] On an increasing death benefit option policy, the death benefit will increase as premiums are paid, but paying for the policy as slowly as possible is still the most efficient approach since the percent of premium load would be lost from any unnecessary premium.

It is interesting to note that billing mode is generally irrelevant to the holder of a traditional contract, due to provisions for the return of unearned fractional premium upon death. The universal life counterpart, the return of the unearned portion of the month of death's cost-of-insurance deduction, does not have the effect of putting policyholders with different billing modes in an equal position.

[14] The mode selected will depend upon the month of the policy year that the model commences. For example, if the model commences at the start of a policy year, and premiums in excess of the planned premium are paid, no further funding that year would be allowed if the billing mode were annual. However, 11/12th of the planned premium would be allowed if the payment mode had been changed to monthly. If the model commences at the start of the sixth month and the billing mode is monthly, 6/12th of the planned premium can be paid in subsequent months. If the billing mode is changed to semi-annual, 6/12th of the premium can still be paid, but since it will be paid earlier in the year, the funds (plus interest earnings) will impact the death benefit earlier.

Actually, in the former example, the best strategy would be to commence with monthly mode, switch to quarterly mode at the beginning of month four, to semi-annual mode at the beginning of month seven, and to annual mode at the beginning of month thirteen.

premium with those funds will reduce future cash surrender values due to the deduction of the percent-of-premium load on the money loaned and redeposited and due to the generally lower interest rate credited on cash value held as collateral for the loan. The interest due on the loan will also add to the holding cost of the policy. However, the cash value of the policy will increase, resulting in a greater than dollar-for-dollar increase in the death benefit when the policy reaches the corridor that could exceed the cost of loaning and redepositing funds. Subject to limitations on the payment of premiums, this borrowing and redepositing could be repeated.

Chapter 7

Some Additional Considerations

7.1 Limitations of Linear Programming Cost Comparison Methods

While linear programming methods are of greater value than traditional cost comparison methods when comparing policies with flexibility, linear programming cost comparison methods nonetheless share several of the significant drawbacks of the traditional methods.

First, traditional cost comparison methods have been criticized because the interest rate used to discount or accumulate cash flows is often chosen arbitrarily, yet a ranking of policies can be affected by the rate chosen. Similarly, both the allocation of funds among investment alternatives under the General Model, and the ranking of policies when the General Model is used as a cost comparison method, are affected by the interest rates assumed to be available on the investments.

A related issue is that policies are usually compared using interest rates representing the "current" environment. This is a palatable assumption, and may be a necessary assumption when comparing contracts with non-guaranteed elements in which an interest assumption underlies the current rate scale, but in a period of volatile interest rates, it is probably not a "good" assumption. For universal life products, one could theoretically confront this problem by basing purchase decisions on the relative performance of policies under several future interest rate scenarios. However, there are practical difficulties with this approach. On portfolio rate products, it would be difficult to model the rates credited to a policy as the interest environment changes in the absence of rather specific information about a company's investment and interest crediting strategies. On new money products, the mechanics of following different generations of cash flow may be difficult to model using linear programming techniques. In either case, it would be necessary to make an assumption regarding how the relationship between the rates on the various products and investment alternatives changes as market rates vary. Would spreads remain constant? The multiple scenario approach could add significantly to the work without adding much to the final product.

Another drawback of all cost comparison methods is that on contracts with non-guaranteed elements, they are necessarily based upon the cost structure at issue. Faith is thus placed not only in the stability of underlying assumptions, but also in the integrity of the issuing company's illustrations. If the non-guaranteed elements are changed, a retrospective study may show that in fact a different contract would have been a better purchase.

While linear programming cannot undo a prior purchase decision that turns out, with the benefit of hindsight, to have been less than the optimal purchase, with only minor modification the General Model can be used to optimize the allocation of funds between an existing universal life contract and investment alternatives. Thus, the best can be made of the situation at hand. Since interest rates and policy cost structures will change over time, as will tax laws and other elements affecting investment decisions, it seems clear that any purchase decision based on linear programming should be followed up from time to time with an exercise to reoptimize the allocation of funds under then current conditions.

7.2 Computational Issues

Linear programming provides a convenient language to phrase policyholder optimization problems. Linear programming routines such as the Simplex Algorithm theoretically should solve these problems fairly directly, and the size of the problems involved are manageable within a mainframe computer environment. However, the clear trend in insurance sales and financial planning illustrations is towards the use of personal computers, especially laptops. Implementing a linear programming based illustration system for use on a personal computer may require the development of computational and memory management shortcuts.

Several streamlined versions of the Simplex Algorithm exist for solving linear programming problems in which the constraints meet specified conditions. For example, the Danzig-Wolfe decomposition algorithm [26, page 448] can be used to solve some linear programming problems in which the constraints can be grouped into k+1 subsets C_j, such that if A and B are distinct elements of $\{C_1, \ldots, C_k\}$, then there is no overlap in the variables used to express the constraints in A and B. (The constraints in C_{k+1}, called the *central constraints*, may involve any of the variables.) The constraints in the General Model may be decomposed in this manner by letting k be the number of investment alternatives (including the insurance policy) and letting C_j for $j=1,\ldots,k$ contain those constraints involving only investment j. C_{k+1} is left for the constraints that link the investments.

Similarly, some of the constraints in the models in this paper are upper bound constraints, that is, they are of the form

$$x_i \le u_i$$

where x_i is a variable and u_i is a constant. (For example, any annual limit on the amount of money contributed to any single investment is an upper bound constraint.) Techniques exist that significantly enhance the efficiency of the Simplex Algorithm when solving linear programming problems with upper bound constraints. (See, for example, [27, page 467], or [13, page 273].)

Using standard computer implementations of the Simplex Algorithm, the time required to solve linear programming problems typically increases somewhat less than linearly with the number of variables, and roughly with the cube of the number of constraints. This suggests that in setting up the linear programming models presented in this paper, it would be worthwhile checking for and eliminating any redundancies in the constraints, before commencing with the solution. For example, in the General Model there were constraints for the annual limit on amounts allocable to each investment. If these limits are sufficiently stringent to prevent the policy's guideline premium limitations being exceeded, the guideline constraints could be eliminated.

Investigating the utility of various streamlining techniques would be a reasonable avenue of further research. Alternatively, one might explore the feasibility of developing solution routines that are independent of standard linear programming methods such as the Simplex Algorithm. For example, consider the optimal solution under the rather simple Term Model. Under this model, money was applied to the insurance policy to the fullest extent possible, as soon as the mean rate of return over the period those funds were to remain in the contract exceeded the rate used in the discounting process. Logic would have yielded a solution, in the absence of formally setting up a linear programming problem. The applicability of this idea becomes significantly more clouded under the full version of the General Model, when several alternative investments are available, arbitrary patterns of future withdrawals are allowed, tax considerations are brought in, and so forth; however, even within this more complicated setting, an approach more direct than linear programming to the solution of optimal allocation of funds might be found. Even if this approach did not yield an algorithm independent of standard techniques for deriving an optimal solution, it might yield rules that could be used to set up an initial basic feasible solution that is fairly close to the optimal solution, thus reducing the number of iterations required to come up with the optimal solution.

This author has not attempted to program a full implementation of the General Model, as described in section 4.2. Such an undertaking would in itself constitute a rather substantial software development project. In programming subsets of the model while developing the examples presented in this paper, the author ran into several instances in which cumulative roundoff error materially impacted results derived by Simplex Algorithm techniques (using the code in [25]). In some instances, the algorithm derived an optimal solution that was, in fact, suboptimal; in others, the algorithm concluded that no feasible solution existed when in fact a feasible solution could be developed by inspection. Performing a transformation of variables so that the all the coefficients of the initial Simplex tableaux were of the same relative magnitude eliminated the cumulative roundoff error in many, but not all, cases. Any implementation of the models in this paper should be done keeping the potential for roundoff error in mind.

7.3 Modelling IRS Section 7702 and 7702A

In the General Model, compliance with the cash value corridor provision of the definition of life insurance was ensured by including constraints (G12) that prohibited the policy from entering the corridor.

When a policy is in the corridor, additional death protection is purchased. From the standpoint of an insurance and investment program in which the policy face amount fully meets the policyowner's need for insurance, the cost of the additional protection subtracts from the policy's investment element, and is therefore undesirable. However, if the insurance policy is a competitive investment vehicle relative to the other available options and if the cost of purchasing the additional death benefit is relatively small, it is possible that the optimal strategy would be to allow a policy to enter the corridor, in spite of the penalty.[1] ${}^{CORR}\Delta CVC_t$ from (2.5.2) can be used to calculate the "diluted" annual return on a dollar of corridor cash value; for example, the return on a corridor dollar at attained age 65 for a universal life policy from Company A is reduced from 10.00% to 9.64%.

Several methods can be used to develop forms of the General Model that permit policies to enter the corridor. First, a month-by-month approach in which cash values are explicitly calculated through iterative equations, in a manner similar to that used in the Impaired Life Model, could be used. While this approach would have the advantage of accounting exactly for the impact of the corridor, it has the disadvantage of greatly increasing both the number of variables and the number of constraints, thus increasing substantially the computer resources required for a solution.

Alternatively, one could avoid increasing the number of variables and constraints by approximating the buildup of cash value by an annual account mechanism. Under this approach, deductions for the year could be based upon an annual death benefit variable that is subject to two constraints: First, the death benefit must be greater than the face amount (in the case of a level death

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[1]Additionally, though not "needed," the extra death benefit purchased presumably has some economic value.

benefit policy), or the face amount plus the cash value (in the case of an increasing death benefit policy), and second, the death benefit must be greater than the prior end-of-year cash value times the corridor factor for the current year. Since the objective is to maximize cash accumulation, annual deductions will be based upon the smallest death benefit each year that meets these two constraints.[2] Of course, the savings in computation time is at a cost of lack of precision in the cash value calculation, and the resulting solution would only be approximately optimal.

The General Model includes constraints (G13) to ensure that the contract does not become a modified endowment.[3] In many circumstances, the tax consequences of becoming a modified endowment are not so onerous as to warrant excluding a contract from becoming one. For example, a policyholder who does not intend to take any pre-retirement withdrawals and whose objective is to maximize an age 65 after-tax cash surrender value will not be adversely affected by a policy becoming a modified endowment and thus should not be constrained to the premium limits required to avoid becoming one. From a linear programming standpoint, however, handling contracts that could become modified endowments in a single model that allows for intermediate withdrawals cannot be done, because the abrupt change in taxation that occurs when a policy crosses the modified endowment limits results in an objective function that is discontinuous.

One approach to handling this problem would be to build a second model (the

[2]This same technique can be used if a monthly accumulation mechanism is used in developing a General Model, and is more efficient than the technique used in the Impaired Life Model, in which the death benefit was split into two components, the face amount (level) or face amount plus cash value (increasing), plus any excess of the cash value times the corridor factor over the face amount or face amount plus cash value. This technique could not have been used in the Impaired Life Model, because a "solution" to the objective function in that model (of maximizing the actuarial present value of future cash flows) would then have involved attempting to maximize an unbounded death benefit.

[3]A policy becomes a modified endowment contract if during any of the first seven policy years, the cumulative premiums paid exceed those that would be paid on a seven-pay life plan, calculated on a basis prescribed in the regulation. The tax rules for modified endowments are very similar to the rules for premature withdrawals from annuities, i.e., withdrawals are treated first as taxable income and then as non-taxable return of investment, and taxable withdrawals before age 59 1/2 are subject to a ten percent penalty tax.

"MEC General Model") in which the constraints causing avoidance of modified endowment status are removed, and all withdrawals are taxed under the less favorable modified endowment rules. Then take as the optimal overall solution the better of the optimal solutions from the General Model and the MEC General Model.

The MEC General Model will allow solutions under which the contract does not, in fact, become a modified endowment. However, since the taxation of modified endowments is less favorable than the taxation of policies that avoid becoming modified endowments, the value of the objective function under the General Model will be at least as great as the value of the objective function under the MEC General Model for any point that is in the feasible region of both models. Thus, if the optimal solution under the MEC General Model in fact does not cause the contract to become a modified endowment, the solution under the General Model will be the more optimal of the two, and the solution under the MEC General Model would not be chosen.[4]

7.4 Modelling Traditional Plans

Partly in response to the development of universal life, recently forms of traditional participating insurance have been introduced that exhibit a significant degree of flexibility. These plans typically combine a base policy with the purchase of paid-up additions (both through dividends and through premiums paid on a paid-up additions rider) and a flexible death benefit yearly renewable term rider. This section outlines an approach to developing the insurance portion of the General Model in which such a flexible traditional plan

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[4]A contract becomes a modified endowment if it meets any one of the following seven conditions:

$$\begin{aligned} P_1 &\geq 7PayPrem \\ P_1 + P_2 &\geq 2 \cdot 7PayPrem \\ &\cdots \\ P_1 + P_2 + \ldots + P_7 &\geq 7 \cdot 7PayPrem \end{aligned}$$

It would not be possible to modify the General Model to meet this set of "or" conditions, thus keeping the solution within a region in which the modified endowment tax rules actually apply. Linear programming theory requires that the feasible region be convex; "or" conditions define a non-convex feasible region.

is substituted for a universal life plan.

This development is not intended to be complete; rather, it is meant to give the flavor of one possible approach, highlighting along the way similarities and differences between building a model for universal life plans and building one for traditional plans. For the sake of simplicity, the policy loan option is ignored, and the mechanics of policyholder taxation are not explored. In "real life" uses, it would be desirable to account for taxation; with sufficient effort, methods similar to those used in the General Model could be developed.

The key difference between modelling traditional contracts and universal life contracts is that in traditional contracts, the cash flow elements are inextricably tied to the amounts of each type of coverage (base policy, term, or paid-up additions) that are purchased. Any premium flexibility is obtained by varying the mix of these three types of coverage, subject to meeting the total coverage needs of the insured and any administrative rules imposed by the insurer. Since generally the base policy is not divisible, i.e., it cannot be surrendered in part,[5] any intermediate need for funds must be provided through the investment alternatives or through cash value on surrendered paid-up additions.

Define the following notation:

[P]	FMin_t	= Minimum total coverage required by insured, year t
[S]	F^{BASE}	= Base policy face amount, assumed fixed by duration
[S]	F_t^{TERM}	= Term face amount, year t, assumed variable
[S]	FBuy_t^{PUA}	= Paid-up additions face purchased, beginning of year t
[S]	FSurr_t^{PUA}	= Paid-up additions face surrendered, end of year t
[D]	F_t^{PUA}	= Paid-up additions face amount in force during year t
[P]	$\mathrm{PremPer000}_t^{*}$	= Premium per thousand, for $*\in\{BASE, TERM\}$
[P]	$\mathrm{PremPer000}_t^{PUA}$	= Paid-up additions purchase rate, beginning of year t

[5]Some companies will process a "partial surrender" of a traditional plan by effectively splitting the policy in two and surrendering one piece. This gives the policyholder the ability to reduce the face amount or to gain access to a portion of the cash value, without replacing the policy. When this is done, however, it is usually done extracontractually. It is doubtful that a company would be willing to process a series of annual policy surrenders in order to provide an income stream.

[P] PolFee = Policy fee, assumed fixed by duration

[P] $DivPer000_t^{*}$ = Dividend per thousand, end of year t, *∈{BASE,TERM,PUA}

[P] $CVPer000_t^{*}$ = Cash value, end of year t, *∈{BASE,PUA}

[D] $InsuranceCF_t$ = Insurance policy cash flow, beginning of year t

[P] Per = Period of time in years the policy is to stay in force

An objective function involving maximizing a future sum of insurance cash values and investment fund balances would be developed, subject to the following sets of constraints:

(TR1) The total coverage in force each year at least meets the insured's need for insurance:

$$\text{For } t = 1 \text{ to Per:} \quad F^{BASE} + F_t^{TERM} + F_t^{PUA} \geq FMin_t$$

(TR2) The total paid-up additions face amount in force each year is not less than zero:

For t = 1 to Per

$$F_t^{PUA} = \sum_{j=1}^{t} FBuy_j^{PUA} - \sum_{k=1}^{t-1} FSurr_k^{PUA} \geq 0$$

In the absence of constraints set (TR2), the optimal solution could involve surrendering nonexistent paid-up additions and "replacing" them with cheap term insurance.

(TR3) Each year's insurance plan cash flow, combined with the cash flow from investments, matches the policyholder's desired net (positive or negative) cash flow. In constructing these constraints, the cash flow into the insurance contract would be the sum of the premiums paid on the base policy, the yearly renewable term, and the current year purchase of paid-up additions, less any dividends payable on coverage in force through the previous year, less any cash value made available through the surrender of paid-up additions:

For t = 1 to Per:

$$\begin{aligned} 1000 \cdot InsuranceCF_t = \; & F^{BASE} \cdot PremPer000_t^{BASE} \\ & + F_t^{TERM} \cdot PremPer000_t^{TERM} \end{aligned}$$

$$+ \ FBuy_t^{PUA} \cdot PremPer000_t^{PUA}$$

$$+ \ PolFee$$

$$- \ F^{BASE} \cdot DivPer000_{t-1}^{BASE}$$

$$- \ F_{t-1}^{TERM} \cdot DivPer000_{t-1}^{TERM}$$

$$- \ F_{t-1}^{PUA} \cdot DivPer000_{t-1}^{PUA}$$

$$- \ FSurr_{t-1}^{PUA} \cdot CVPer000_{t-1}^{PUA}$$

Constraints imposed by the issuing company's administrative rules, of course, will vary from company to company, but (TR4) through (TR8) give examples that would be typical of this type of plan:

(TR4) The desired minimum total death benefit is level, or increases annually under some fixed schedule. Rather than defining constraints, these conditions could be met through appropriately entering the parameters $FMin_t$. In a solution maximizing an accumulation of cash values plus investment fund balances, term insurance would never be purchased if it would put the total death benefit above $FMin_t$, since such term insurance comes at a cost but is unnecessary. Paid-up additions might be purchased, however, if a company's administrative rules allow such a purchase (see (TR7)) and if the rate credited to additions dividends make them more attractive than other investments, in spite of the cost of the additional death benefit. This is analogous to the situation in the General Model for universal life in which it would be desirable to allow premium payments that would put the policy into the corridor, when the added cost of protection does not diminish the return on the universal life policy to a point that makes it less desirable than other investments.

(TR5) The base plan coverage is no less than a fixed percentage k' of the total plan coverage at issue:

$$F^{BASE} \geq k' \cdot FMin_1$$

(TR6) The total term coverage in any year cannot exceed a fixed percentage k" of the total plan coverage at issue:

$$\text{For } t = 1 \text{ to Per:} \quad F_t^{TERM} \leq k''(F^{BASE} + F_1^{TERM} + F_1^{PUA})$$

(TR7) A given year's purchase of paid-up additions is limited to the amount required to fill in the gap between the minimum planned or desired coverage,

$FMin_t$, and the existing base policy face plus paid-up additions face:[6]

$$\text{For } t = 1 \text{ to Per:} \quad FBuy_t^{PUA} \leq FMin_t - F_{t-1}^{BASE} - F_{t-1}^{PUA}$$

The intent of this set of constraints is to limit lump sum purchases of paid-up additions when such a purchase would result in total coverage that exceeded the amount for which the insured was originally underwritten.[7] A particular company's rules for the purchase of paid-up additions are likely to be more liberal than the rule stated above. For example, the prior year's dividend may be allowed to be applied to purchase paid-up additions, even if this raises the total death benefit above the anticipated total plan coverage; similarly, the insured may have the right to such purchases up to some pre-defined annual premium limit. Constraints (TR7) may be modified to account for one or both of these possibilities.

(TR8) The surrender of paid-up additions is allowed only to the extent that the total coverage in force for the previous year exceeds the desired minimum coverage for the coming year. The amount by which the previous year's coverage exceeds the coming year's minimum coverage may be obtained by adding the set of constraints:

$$\text{For } t = 1 \text{ to Per} - 1: \quad PreExcCom_t - ComExcPre_t \leq F^{BASE} + F_{t-1}^{TERM} + F_{t-1}^{PUA} - FMin_t$$

where $PreExcCom_t$ = Amount by which previous year's coverage in force exceeds the minimum coverage for the coming year.

$ComExcPre_t$ = Amount by which the minimum coverage for the coming year exceeds the previous year's coverage in force.

For each value of t, one or both of $PreExcCom_t$ and $ComExcPre_t$ will be zero. Then the desired restriction on the amount of paid-up additions that may be surrendered in any year is obtained by adding the set of constraints:

$$\text{For } t = 1 \text{ to Per} - 1: \quad FSurr_{t-1}^{PUA} \leq PreExcCom_t$$

The intent of the above constraints is to guard against paid-up additions being

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[6]A term for the reduction in face amount due to surrenders of paid-up additions is not necessary, since it would be sub-optimal to both surrender and purchase paid-up additions simultaneously.

[7]A company would likely allow such a purchase only with evidence of insurability.

surrendered and replaced with term insurance, which a company may wish to avoid since such a transaction will result in an increase in the net-amount-at-risk.

Policyholder taxation considerations also will need to be modelled. Generally, there will be no problem with meeting the definition of life insurance (each component part would be expected to meet the Net Single Premium test of IRS Section 7702); however, as with the General Model for universal life plans, there is the possibility that surrenders of paid-up additions within the first fifteen policy years will trigger a taxable event. Additionally, constraints to prohibit the contract from becoming a modified endowment contract, which results in an abrupt, non-linear change in tax status, would need to be developed.

7.5 Some Final Thoughts and Suggestions for Further Research

Making insurance purchase decisions based upon linear programming models ultimately will be worthwhile only if the resulting savings justify the effort required to obtain those savings. This paper has illustrated several instances in which linear programming models enhance policyholder value; however, a policyholder must have both the desire and the sophistication necessary to utilize these techniques for them to be worthwhile. Some further study of the potential of linear programming techniques in enhancing policyholder value, using policies actually available in the insurance marketplace, would be useful. For example, in chapter five of this paper, an example was given in which purchasing two universal life policies was shown to be superior to purchasing a single policy, when both the death benefits of the two policies and the premium streams were carefully constructed. It would be interesting to test the savings potential of this method by using combinations of policies that are actually available. It seems unlikely that the purchaser of a modest face amount would find it worth the inconvenience of keeping track of the varying premium payments required on two policies in order to obtain a small increase in cash accumulation. On the other hand, on large sales within sophisticated markets, the use of such techniques may not be outside the realm of possibility. The broker offering linear programming based insurance sales along with transparent administration of the multiple contracts (including re-optimization of the allocation of premium between contracts as companies adjust cost structures) could offer a better "product" than any single product on the market.

The Impaired Life Model illustrated the potential for net-amount-at-risk manipulation that exists in flexible premium universal life contracts. Again, it may be hard to visualize the average policyholder utilizing linear programming techniques to earn a few extra dollars on the policy of an insured who is near death. However, even at an unsophisticated level, it is likely that a material percentage of policyholders will recognize that premium payments can be skipped when it is clear that the cash value is sufficient to fund the policy until death, or that premiums can be resumed on low cash value policies that had, in effect, lapsed for extended term and were about to go out of force.[8] Furthermore, given the long term nature of insurance contracts, the sophistication of consumers and their access to information when currently issued contracts approach claims time could be substantial. The development of a secondary market for insurance contracts by firms willing to purchase contracts on impaired lives could also give rise to sophisticated techniques being used to utilize options in these contracts to their fullest.

The fact that the seemingly innocuous flexibility of the universal life contract may result in adverse mortality experience suggests several related areas for further research. First, although mortality studies generally use a contract's face amount as the measure of exposure, the considerations above suggest that both industry-wide and company mortality studies should be done separately for universal life products, using the net-amount-at-risk rather than the face amount. Second, until such actual data is available, some modelling to estimate future universal life mortality experience, as compared with experience

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[8]The option within the universal life contract to lapse for extended term and then to reinstate has been recognized and was an element of concern during the development of universal life valuation and nonforfeiture regulations. See [6], including the discussions.

Some anti-selection may also be present at issue, when the plan is set up. For example, those in less good health may be more likely to set the planned premium low, treating the contract more as term insurance; those in good health may be more likely to fund the contract with single premiums or generous annual premiums.

on standard ordinary lives, would be of value, along with the development of appropriate methodology to recognize any difference in pricing. Finally, if it appears universal life mortality experience can be expected to be significantly less favorable than traditional experience, valuation and nonforfeiture mortality tables distinct for universal life type contracts should be developed, as well as the regulatory apparatus necessary to permit their adoption.

It has been noted that "flexible premium universal life may be even more 'consumer-oriented' than companies realize."[9] This author agrees with that assessment of the universal life product. The linear programming models presented in this paper--the Term Model, the General Model and its associated cost comparison method, and the Impaired Life Model--provide a viable means to utilize this very consumer-oriented product to the consumer's best advantage.

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[9] See Thomas C. Kabele's discussion of [6].

Appendix

This appendix describes the cost structure of each of the three sample universal life plans used in the examples in this paper. The linear coefficients ${}^{L}\Delta CVC_t$, ${}^{L}\Delta CVC_t^{(12)}$, ${}^{L}\Delta CVF_t$, ${}^{L}\Delta CVL_t$, ${}^{I}\Delta CVC_t$, ${}^{I}\Delta CVC_t^{(12)}$, ${}^{I}\Delta CVF_t$, and ${}^{I}\Delta CVL_t$ developed in chapter two to express cash values as a linear function of prior transactions are also listed.

The Company A plan charges a 6% load on premiums, credits interest at 10%, and has no monthly per policy fee. The Company B plan charges a 6% load on premiums, credits interest at 9%, has no monthly per policy fee, and charges cost-of-insurance rates that are lower than those of Company A. The Company C plan charges a 2% load on premiums, credits interest at 9% during years 1-10 and 10% from years 11 on, and charges a monthly fee of $2.50; its cost-of-insurance charges fall somewhere between those of Company A and Company B. The plans were designed so that $ 100,00C level death benefit policies issued to a person aged 45 will develop identical 20th year cash values if $ 1,200 of annual premium is paid.

IRS section 7702/7702A compliance factors are also listed.

Appendix 1: Sample Product Data

Product Specifications and Linear Coefficients for Company A

Portfolio Rate, years 1 - 10	10.0%
Portfolio Rate, years 11 - 10	10.0%
Loans Charged at	8.0%
Loaned Fund Earns at	6.0%
Guaranteed Interest Rate	4.0%
Percent of Premium Load	6.0%
Per Month Charge	$0.00
Surrender Charge Per $1,000 Face (Issue Age 45)*	$7.50
Minimum Issue Face	$25,000

* Surrender charge shown is for year 1. Surrender charge grades down linearly in annual increments over 15 years.

Attained Age	Current Annual COI Rate Per $1,000	${}^{L}\Delta CVC_t$	${}^{L}\Delta CVC_t^{(12)}$	$1000\,{}^{L}\Delta CVF_t$	$1000\,{}^{L}\Delta CVL_t$	${}^{I}\Delta CVC_t$	${}^{I}\Delta CVC_t^{(12)}$	$1000\,{}^{I}\Delta CVF_t$	$1000\,{}^{I}\Delta CVL_t$
45	2.6563	1.102925490	1.054917009	-2.792414264	-40.049060836	1.100009534	1.053383069	-2.788969065	-40.000159970
46	2.8733	1.103164796	1.055042841	-3.020839471	-40.053071953	1.100010313	1.053383479	-3.016808139	-40.000173039
47	3.1046	1.103419924	1.055176985	-3.264368463	-40.057347931	1.100011144	1.053383915	-3.259661659	-40.000186968
48	3.3533	1.103694304	1.055321241	-3.526275993	-40.061946193	1.100012036	1.053384385	-3.520784506	-40.000201946
49	3.6336	1.104003623	1.055483853	-3.821534470	-40.067129478	1.100013042	1.053384915	-3.815086045	-40.000218826
50	3.9333	1.104334438	1.055657751	-4.137313596	-40.072672402	1.100014118	1.053385481	-4.129756931	-40.000236875
51	4.2816	1.104719013	1.055859888	-4.504410994	-40.079115341	1.100015368	1.053386139	-4.495455888	-40.000257851
52	4.6906	1.105170767	1.056097309	-4.935636475	-40.086682722	1.100016836	1.053386911	-4.924887544	-40.000282482
53	5.1466	1.105674633	1.056362084	-5.416609830	-40.095121738	1.100018473	1.053387773	-5.403667722	-40.000309944
54	5.6755	1.106259315	1.056669280	-5.974731734	-40.104912574	1.100020372	1.053388772	-5.958990563	-40.000341796
55	6.2602	1.106906013	1.057009004	-6.592056862	-40.115739722	1.100022470	1.053389876	-6.572902107	-40.000377009
56	6.9091	1.107624123	1.057386172	-7.277558137	-40.127759822	1.100024800	1.053391102	-7.254222353	-40.000416088
57	7.5984	1.108387410	1.057786988	-8.006192341	-40.140533047	1.100027274	1.053392404	-7.977962532	-40.000457600
58	8.3434	1.109212918	1.058220384	-8.794231428	-40.154343954	1.100029948	1.053393811	-8.760187412	-40.000502467
59	9.1843	1.110145366	1.058709808	-9.684368115	-40.169939559	1.100032966	1.053395400	-9.643106188	-40.000553109
60	10.1222	1.111186222	1.059255989	-10.678007386	-40.187342783	1.100036333	1.053397171	-10.627874605	-40.000609593
61	11.1648	1.112344321	1.059863511	-11.783588386	-40.206699453	1.100040075	1.053399141	-11.722578352	-40.000672382
62	12.3523	1.113664720	1.060555943	-13.044131043	-40.228760012	1.100044338	1.053401384	-12.969627680	-40.000743899
63	13.7078	1.115173676	1.061346956	-14.484712327	-40.253959413	1.100049203	1.053403945	-14.392678883	-40.000825532
64	15.2400	1.116881594	1.062241880	-16.115275632	-40.282466690	1.100054703	1.053406839	-16.001469051	-40.000917808
65	16.9334	1.118771987	1.063231942	-17.920095909	-40.314001505	1.100060782	1.053410038	-17.779525858	-40.001019793
66	18.7565	1.120810446	1.064298991	-19.866333262	-40.347984911	1.100067326	1.053413482	-19.693776894	-40.001129588
67	20.7331	1.123024385	1.065457240	-21.980173374	-40.384868670	1.100074421	1.053417215	-21.769214594	-40.001248629
68	22.8547	1.125405194	1.066702028	-24.253412209	-40.424503409	1.100082037	1.053421223	-23.996916747	-40.001376403
69	25.1299	1.127963510	1.068038747	-26.696221302	-40.467059756	1.100090204	1.053425521	-26.385916484	-40.001513428
70	27.7843	1.130954950	1.069600627	-29.552707720	-40.516777099	1.100099732	1.053430535	-29.173103746	-40.001673291
71	31.2288	1.134847653	1.071631228	-33.269974819	-40.581402922	1.100112097	1.053437042	-32.789949586	-40.001880739
72	34.1693	1.138180488	1.073368128	-36.452763792	-40.636670960	1.100122652	1.053442597	-35.877607736	-40.002057835
73	38.1024	1.142652388	1.075696271	-40.723562476	-40.710737155	1.100136771	1.053450027	-40.007584018	-40.002294713
74	42.5232	1.147698020	1.078319854	-45.542603070	-40.794181579	1.100152641	1.053458378	-44.649731735	-40.002560965

Appendix 1: Sample Product Data (Continued)

Product Specifications and Linear Coefficients for Company B

Portfolio Rate, years 1 - 10	9.0%
Portfolio Rate, years 11 - 10	9.0%
Loans Charged at	8.0%
Loaned Fund Earns at	6.0%
Guaranteed Interest Rate	4.0%
Percent of Premium Load	6.0%
Per Month Charge	$0.00
Surrender Charge Per $1,000 Face (Issue Age 45)*	$7.50
Minimum issue Face	$25,000

* Surrender charge shown is for year 1. Surrender charge grades down linearly in annual increments over 15 years.

Attained Age	Current Annual COI Rate Per $1,000	${}^{L}ACVC_t$	${}^{L}ACVC_t^{(12)}$	1000 ${}^{L}ACVF_t$	1000 ${}^{L}ACVL_t$	${}^{I}ACVC_t$	${}^{I}ACVC_t^{(12)}$	1000 ${}^{I}ACVF_t$	1000 ${}^{I}ACVL_t$
45	2.1248	1.092318289	1.049331219	-2.221950166	-30.029380601	1.090007557	1.048112224	-2.219760648	-30.000095814
46	2.2863	1.092494680	1.049426240	-2.391014043	-30.031615159	1.090008132	1.048112527	-2.388478922	-30.000103097
47	2.4573	1.092681476	1.049522743	-2.570050555	-30.033981381	1.090008740	1.048112848	-2.567121891	-30.000110808
48	2.6401	1.092881194	1.049628055	-2.761473146	-30.036511135	1.090009390	1.048113191	-2.758092372	-30.000119051
49	2.8455	1.093105643	1.049746400	-2.976600629	-30.039353956	1.090010121	1.048113576	-2.972673116	-30.000128313
50	3.0636	1.093344017	1.049872080	-3.205074589	-30.042372906	1.090010896	1.048113986	-3.200521657	-30.000138148
51	3.3170	1.093621032	1.050018122	-3.470585848	-30.045880941	1.090011798	1.048114461	-3.465248212	-30.000149574
52	3.6141	1.093945901	1.050189379	-3.781965502	-30.049994584	1.090012854	1.048115019	-3.775628343	-30.000162972
53	3.9439	1.094306630	1.050379522	-4.127717673	-30.054561787	1.090014027	1.048115638	-4.120170466	-30.000177843
54	4.3253	1.094723935	1.050599463	-4.527698037	-30.059844614	1.090015384	1.048116354	-4.518619549	-30.000195042
55	4.7446	1.095182875	1.050841319	-4.967588596	-30.065653695	1.090016875	1.048117141	-4.956663345	-30.000213950
56	5.2074	1.095689633	1.051108338	-5.453314614	-30.072067010	1.090018521	1.048118010	-5.440152351	-30.000234819
57	5.6950	1.096223779	1.051389749	-5.965295497	-30.078825786	1.090020256	1.048118925	-5.949550752	-30.000256807
58	6.2184	1.096797408	1.051691915	-6.515124991	-30.086082835	1.090022117	1.048119907	-6.496350449	-30.000280409
59	6.8065	1.097442274	1.052031550	-7.133241891	-30.094239490	1.090024209	1.048121011	-7.110743732	-30.000306928
60	7.4591	1.098158273	1.052408579	-7.819545195	-30.103293855	1.090026530	1.048122236	-7.792522023	-30.000336356
61	8.1805	1.098950253	1.052825531	-8.578687561	-30.113306603	1.090029096	1.048123590	-8.546177991	-30.000368887
62	8.9987	1.099849139	1.053298657	-9.440311117	-30.124667767	1.090032006	1.048125126	-9.400964178	-30.000405783
63	9.9287	1.100871668	1.053836721	-10.420464731	-30.137587625	1.090035314	1.048126871	-10.372552133	-30.000447720
64	10.9744	1.102022648	1.054442093	-11.523570943	-30.152122820	1.090039034	1.048128834	-11.465017265	-30.000494875
65	12.1920	1.103363793	1.055147473	-12.809366531	-30.169058090	1.090043364	1.048131119	-12.737074373	-30.000549781
66	13.5047	1.104811578	1.055908539	-14.197220297	-30.187328951	1.090048034	1.048133583	-14.108490095	-30.000608977
67	14.9278	1.106383092	1.056734307	-15.703709427	-30.207151516	1.090053095	1.048136254	-15.595250074	-30.000673150
68	16.4554	1.108072284	1.057621519	-17.323041454	-30.228447170	1.090058529	1.048139121	-17.191191835	-30.000742037
69	18.0935	1.109886291	1.058573837	-19.062065763	-30.251303403	1.090064355	1.048142196	-18.902585375	-30.000815906
70	20.0047	1.112006165	1.059686139	-21.094365183	-30.277996544	1.090071153	1.048145783	-20.899308996	-30.000902091
71	22.4847	1.114762477	1.061131632	-23.736891496	-30.312676327	1.090079975	1.048150438	-23.490302930	-30.001013926
72	24.6019	1.117120522	1.062367042	-25.997664332	-30.342320726	1.090087506	1.048154412	-25.702274971	-30.001109401
73	27.4337	1.120281603	1.064022220	-29.028454346	-30.382025431	1.090097578	1.048159728	-28.660856828	-30.001237102
74	30.6167	1.123844509	1.065886124	-32.444650352	-30.426729146	1.090108900	1.048165702	-31.986392558	-30.001380641

Product Specifications and Linear Coefficients for Company C

Portfolio Rate, years 1 - 10	9.0%
Portfolio Rate, years 11 - 10	10.0%
Loans Charged at	8.0%
Loaned Fund Earns at	6.0%
Guaranteed Interest Rate	4.0%
Percent of Premium Load	2.0%
Per Month Charge	$2.50
Surrender Charge Per $1,000 Face (Issue Age 45)*	$7.50
Minimum Issue Face	$25,000

* Surrender charge shown is for year 1. Surrender charge grades down linearly in annual increments over 15 years.

Attained Age	Current Annual COI Rate Per $1,000	${}^{L}\ddot{A}CVC_t$	${}^{L}\ddot{A}CVC_t^{(12)}$	$1000\,{}^{L}\ddot{A}CVP_t$	$1000\,{}^{L}\ddot{A}CVL_t$	${}^{I}\ddot{A}CVC_t$	${}^{I}\ddot{A}CVC_t^{(12)}$	$1000\,{}^{I}\ddot{A}CVP_t$	$1000\,{}^{I}\ddot{A}CVL_t$
45	2.7626	1.093015050	1.049698634	-2.889769641	-30.038206547	1.090009826	1.048113421	-2.886067727	-30.000124575
46	2.9451	1.093214496	1.049803792	-3.080932365	-30.040732578	1.090010475	1.048113763	-3.076724976	-30.000132804
47	3.1356	1.093422720	1.049913573	-3.280509463	-30.043369614	1.090011153	1.048114121	-3.275739915	-30.000141394
48	3.3365	1.093642352	1.050029362	-3.491020446	-30.046150917	1.090011867	1.048114498	-3.485619838	-30.000150454
49	3.5246	1.093848026	1.050137785	-3.688154761	-30.048755296	1.090012536	1.048114851	-3.682127735	-30.000158936
50	3.7760	1.094122971	1.050282716	-3.951683220	-30.052236535	1.090013430	1.048115323	-3.944765261	-30.000170272
51	4.0247	1.094395024	1.050426112	-4.212442092	-30.055680866	1.090014315	1.048115790	-4.204582294	-30.000181487
52	4.3623	1.094764425	1.050620802	-4.566508091	-30.060357166	1.090015516	1.048116423	-4.557273522	-30.000196711
53	4.7349	1.095172256	1.050835723	-4.957410318	-30.065519294	1.090016841	1.048117123	-4.946529723	-30.000213512
54	5.2215	1.095705075	1.051116475	-5.468116378	-30.072262429	1.090018572	1.048118036	-5.454882688	-30.000235455
55	5.8185	1.106417446	1.056752355	-6.125679784	-40.107560242	1.100020885	1.053389042	-6.109134725	-40.000350408
56	6.4255	1.107088903	1.057105069	-6.766641281	-40.118801295	1.100023064	1.053390188	-6.746460673	-40.000386964
57	7.0665	1.107798375	1.057477682	-7.443898778	-40.130676125	1.100025365	1.053391399	-7.419486552	-40.000425567
58	7.7594	1.108565761	1.057880631	-8.176447074	-40.143517200	1.100027852	1.053392708	-8.147006953	-40.000467296
59	8.5873	1.109483297	1.058362314	-9.052340552	-40.158866650	1.100030823	1.053394272	-9.016274573	-40.000517155
60	9.5149	1.110512155	1.058902295	-10.034516553	-40.176072976	1.100034153	1.053396024	-9.990226497	-40.000573019
61	10.6066	1.111724146	1.059538200	-11.191535299	-40.196334642	1.100038072	1.053398086	-11.136481975	-40.000638765
62	11.8582	1.113115148	1.060267771	-12.519469769	-40.219579188	1.100042564	1.053400451	-12.450632816	-40.000714142
63	13.2966	1.114715726	1.061106927	-14.047510171	-40.246312994	1.100047727	1.053403168	-13.960925482	-40.000800768
64	14.9352	1.116541647	1.062063786	-15.790722863	-40.276793807	1.100053609	1.053406263	-15.681432399	-40.000899452
65	16.5947	1.118393651	1.063033835	-17.558882848	-40.307691788	1.100059566	1.053409398	-17.423892763	-40.000999394
66	18.3814	1.120390759	1.064079350	-19.465628017	-40.340990056	1.100065979	1.053412773	-19.299921842	-40.001106998
67	20.3184	1.122559558	1.065214117	-21.536357338	-40.377126914	1.100072932	1.053416432	-21.333776944	-40.001223654
68	22.3976	1.124891857	1.066433700	-23.763262154	-40.415960118	1.100080396	1.053420360	-23.516955759	-40.001348874
69	24.6273	1.127397911	1.067743300	-26.156150785	-40.457654258	1.100088400	1.053424572	-25.858176283	-40.001483158
70	27.2286	1.130328089	1.069273435	-28.954116996	-40.506362658	1.100097737	1.053429486	-28.589602801	-40.001639823
71	30.6042	1.134140869	1.071262693	-32.595029205	-40.569674943	1.100109855	1.053435862	-32.134094860	-40.001843122
72	33.4859	1.137405106	1.072964175	-35.712277306	-40.623818063	1.100120199	1.053441306	-35.160004327	-40.002016676
73	37.3404	1.141784745	1.075244774	-39.894916979	-40.696374860	1.100134036	1.053448587	-39.207437287	-40.002248820
74	41.6727	1.146725728	1.077814557	-44.613949045	-40.778112059	1.100149588	1.053456772	-43.756642684	-40.002509742

IRS Section 7702/7702A Compliance Factors

		Companies A and B	Company C
Guideline Premiums Per Thousand:			
Issue Age 45	Single	$214.64	$205.88
	Annual Level	17.56	16.84
	Annual Increasing	40.72	39.05
Guideline Premiums Per Policy:			
Issue Age 45	Single	$0.00	$419.87
	Annual Level	0.00	29.02
	Annual Increasing	0.00	29.62
Seven Pay Limit Per Thousand:			
Issue Age 45		$46.86	$46.86

Cash Value Corridor Factors:

Age	Fac	Age	Fac	Age	Fac	Age	Fac	Age	Fac	Age	Fac
45	215%	50	185%	55	150%	60	130%	65	120%	70	115%
46	209%	51	178%	56	146%	61	128%	66	119%	71	113%
47	203%	52	171%	57	142%	62	126%	67	118%	72	111%
48	197%	53	164%	58	138%	63	124%	68	117%	73	109%
49	191%	54	157%	59	134%	64	122%	69	116%	74	107%

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