Diversification Benefits of the Variable Annuities and Equity-Indexed Annuities Mixture

Guanghua Cao[1]

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Abstract

A variety of equity-linked insurance contracts such as variable annuities (VAs) and equity-indexed annuities (EIAs) have gained their attractiveness in the recent decade because of the bullish equity market and low interest rates. Pricing and risk management of these products are quantitatively challenging and therefore have become sources of concern to many insurance companies. From a financial engineer's perspective, the options in VAs and those embedded in EIAs can be modeled as puts and calls respectively, whose values move in opposite directions in response to changes in the underlying equity value. Therefore, for insurers that offer both businesses, there are natural offsets or diversification benefits in terms of economic capital usage. In this paper, we consider two specific products: the guaranteed minimal account benefit (GMAB), and the point-to-point (PTP) EIA contract, which belong to the VA and EIA classes respectively. Taking into account mortality and dynamic lapse risk, we build a model that quantifies the natural hedging benefits based on risk-neutral option pricing theory and risk-adjusted performance measure (RAPM). Through a double-tier simulation framework, an optimum product mixture of those two contracts is achieved that provides the best RAPM and therefore deploys capital the most efficiently.

1. Introduction

The market for equity-linked insurance such as variable annuities (VAs) and equity-indexed annuities (EIAs) has grown tremendously over the recent past and has become a significant segment of our capital markets. This has been evidenced by the growing sales that have reached \$113 billion for VAs and \$13 billion for EIAs in 2003.[2] This is partly thanks to the bullish U.S. equity market along with relatively low interest rates over the past decade, which have led policyholders to be more aware of investment opportunities outside the traditional insurance sector so that they can enjoy the benefits from financial markets in conjunction with investment guarantees and tax advantages. Different from traditional insurance products, these equity-linked insurance contracts provide policyholders mortality or maturity protection, as well as the beneficial return based on the equity market's performance. The pricing and risk management of these products are quantitatively challenging and therefore have become sources of concern to both the regulator and many insurance companies. For instance, the limited capital of a life insurance company constrains the volume of its VA and EIA business; thus how to deploy the *economic capital* more efficiently turns out to be an urgent problem to frame.

It is important to stress that, from an option-pricing perspective, the options in VAs and those embedded in EIAs can be modeled as *puts* and *calls* respectively, which will be shown in detail later. The values of these embedded options move in opposite directions in response to underlying equity price changes. Suppose both products share the same underlying equity process, then these two types of options have payoffs which can partially offset each other, therefore natural diversification benefits exist in a portfolio that contains both VA and EIA products. This means that the economic capital that annuity writers need to hold decreases. From the insurance company's (risk management) point of view, it will be very useful to quantify these diversification benefits and derive an optimal business mix based on the most efficient way to deploy the capital. The framework of this paper, which differs from previous literatures, is based on this purpose.

Perhaps the best way to illustrate this intuition is through a simple numerical example. Table 1.1 provides the value at risk (VaR) of a European put, a European call and a 50/50 mixture of these two options (called a *straddle*) at time horizons of both one and two years. This example assumes both options are at-the-money, have maturity of four years, and are based on the same underlying asset price which follows a geometric

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[2] Source: National Association for Variable Annuities (NAVA).

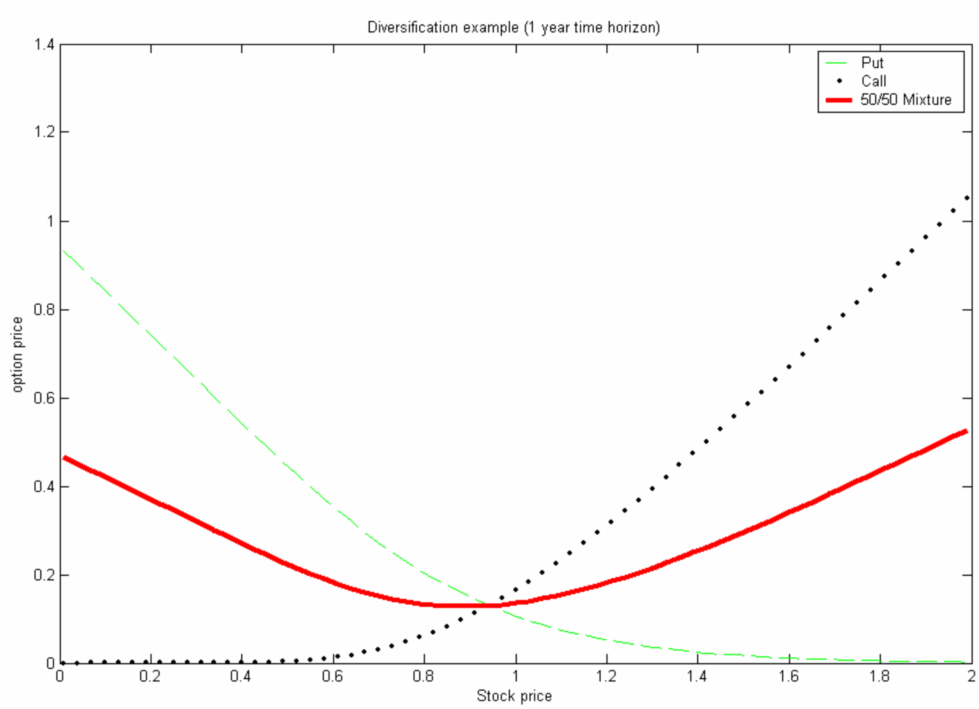
Brownian motion with drift $\mu = 8\%$, non-dividend-paying, volatility $\sigma = 0.2$, risk-free rate $r = 2\%$, initial price $S_0 = 1$.

TABLE 1.1
Diversification of a Put and Call

Year	Value at Risk (level 99%)		
	Put	Call	50/50 Mix
1	0.30	0.76	***0.38***
2	0.38	1.22	***0.61***

It is shown in Table 1.1 that the straddle portfolio has a much lower VaR than the average of these two options, which can be explained by Figure 1.1. The correlation between the prices of a put and a call is negative: when one option is in-the-money (therefore has a higher price), the other option is more likely to be out-of-the-money (with a lower price). This natural diversification lowers the VaR of that straddle portfolio (red line in Figure 1.1). It will be shown later that a similar diversification effect also exists in a portfolio that contains both VAs and EIAs.

Figure 1.1
Diversification of a Put and a Call (one-year time horizon)



There have been some previous literatures in this area. For research on VA, Brennan and Schwartz (1976) first introduced the famous Black-Scholes-Merton (1973) formula into this field. They assumed complete markets both for financial and mortality risk and derived risk-neutral price formulae. More recent work on equity-linked life insurance was done by Bacinello and Ortu (1993, 1996), Aase and Persson (1994) and Nielsen and Sandmann (1995). These authors allowed the risk-free interest rate to be stochastic. Follmer and Sonderman (1986) assumed an incomplete mortality market and introduced the concept of risk-minimizing strategies, which was extended by Moller (1998). Hardy (2003) offered risk-neutral pricing and dynamic hedging analyses on VAs. Milevsky analyzed VAs with mortality and lapse taken into account (Milevsky & Salisbury, 2002) and concluded that in today's market, the guaranteed minimum death benefit (GMDB) products were overpriced (Milevsky, 2001) and in contrast the guaranteed minimum withdrawal benefit (GMWB) products were underpriced (Milevsky, 2004).

In the field of EIA research, Tiong (2000) used Esscher transforms and derived closed form pricing formulae for several types of EIA products: point-to-point (PTP), cliquet and lookback, which were also covered by Hardy (2003). Lin and Tan (2003) extended the model to include stochastic interest rates.

The rest of this paper is organized as follows. We present the framework in Section 2. Analysis including risk-neutral pricing, VaR calculating and risk-adjusted performance measuring (RAPM) are implemented on two specific products: guaranteed minimum account benefit (GMAB) in Section 2.1, and the PTP EIA contract in Section 2.2, which belong to the VA and EIA classes respectively. In Section 2.3 we conduct the same analysis on the mixture of GMAB and PTP, and an optimal combination of these two products is achieved which provides the best RAPM. We conclude in Section 2.4 with closing remarks and summary. Numerical examples are listed at the end.

2. Formulation

2.1 GMAB Contract, Economic Capital and RAPM

2.1.1 Product Description

VAs are complex structured equity and interest rate products, but the basic idea behind them is an investment guarantee on a *separate* mutual fund account. The simplest VA product is the GMDB, which provides the beneficiary a minimal guarantee (which is put-like) in the event that the policyholder dies before the contract's maturity. The GMAB is similar to GMDB and its benefit is claimable on mortality or maturity, whichever one comes first. In this paper we focus on a GMAB account.

An example of a GMAB contract is as follows: at initiation, $t = 0$, the policyholder enters into a contract by paying the insurance company an initial amount P. The insurance company immediately invests the amount P into a mutual fund (such as an index fund), and there is no further payment from the policyholder. The insurance company guarantees a rate of return r_g up to the end of contract (which can be caused by mortality or maturity, but can NOT be caused by policyholder's lapse behavior), when the beneficiary will receive the greater of either the current mutual fund account value or the guaranteed amount. In exchange, the insurance company charges a certain percent of the account amount as the contract fees. If the policyholder decides to lapse the VA contract before maturity, he can get his mutual fund account value back after some penalty fees are charged, but the guarantee is not redeemable.

2.1.2 GMAB without Mortality and Lapse

Consider a GMAB contract with $1 initial account value and maturity time N (in years). Ignoring any mortality and lapse risk, the embedded option in GMAB turns to be a plain vanilla European put.

For the rest of this paper, the underlying equity price is assumed to satisfy a geometric Brownian motion; the interest rate is assumed to be constant; and continuous compounding will be used for simplicity. Given time horizon n prior to maturity, let G_n be the guaranteed amount

$$G_n = e^{r_g n} \cdot 1, \quad 0 \le n \le N$$

As we discussed before, G_n is going to be the strike price for its embedded option. Let $\{F_n\}$ be the account value process that satisfies

$$F_n = e^{-mn}\frac{S_n}{S_0}, \quad 0 \le n \le N$$

At any time n prior to N, suppose the underlying stock price is S_n. The embedded put option value in GMAB can then be calculated as

$$H_n = E_n^Q[e^{-r(N-n)}H_N]$$

here

$$\begin{aligned} H_N &= \left(G_N - F_N\right)^+ = \left(e^{r_g N} - e^{-mN}\frac{S_N}{S_0}\right)^+ \\ &= \frac{e^{-mN}}{S_0}\left(e^{(m+r_g)N}S_0 - S_N\right)^+ \end{aligned}$$

In the formula above, the H_N term, which is the final cash flow of the GMAB contract that happens at maturity N, is equivalent to the payoff of a vanilla European put option. Using notation $V_{put}(S_0, K, r, d, \sigma, t)$ as the price of a vanilla European put, then under the Black-Scholes-Merton (B-S-M) framework (Black & Scholes, 1973), the closed form of such an option value can be written as

$$\begin{aligned} H_n &= \frac{e^{-mN}}{S_0}\cdot V_{put}(S_n, e^{(m+r_g)N}S_0, r, d, \sigma, N-n) \\ &= e^{r_g N - r(N-n)}\Phi(-d_2) - \frac{e^{-mN}}{S_0}S_n e^{-d(N-n)}\Phi(-d_1) \end{aligned}$$

here

$$\begin{aligned} d_1 &= \frac{\log(S_n/S_0) - (m+r_g)N + (r - d + \frac{\sigma^2}{2})(N-n)}{\sigma\sqrt{N-n}} \\ d_2 &= d_1 - \sigma\sqrt{N-n} \end{aligned}$$

For a GMAB contract, the net value of adding the guarantee to the VA product at time n, noted by $NV_n(S_n)$, can be formulated as the difference between two parts: the embedded option (guarantee) value from time n to maturity N, and the present value of the benefit charge (noted as f_n) , as a portion of the total management fees charged to the policyholder's account. $NV_n(S_n)$ has the following form

$$NV_n(S_n) = H_n - f_n$$

and

$$f_n = E_n^Q[\sum_{t=n}^{N-1} e^{-r(t-n)} F_t \varepsilon \Delta t] \approx \int_n^N e^{-r(t-n)} E_n^Q[F_t]\varepsilon dt = \frac{1}{S_0} S_n \int_n^N e^{-mt} \varepsilon dt = \frac{\varepsilon}{mS_0} S_n [e^{-mn} - e^{-mN}]$$

The corresponding economic capital of GMAB is defined as the percentile risk measure of $NV(S_n)$.

$$\mathrm{P}[e^{-r\cdot n}(NV_n(S_n) - NV_0(S_0)) \ge EC_{GMAB}] < 1-\beta$$

Here β is the confidence level. Since $NV_n(S_n)$ is monotonic,[3] its analytical economic capital (or equivalent, VaR) can be directly calculated (Fong & Lin, 1999) in the following way:

$$Var[f(S)] = f(Var[S]) \text{ if } f(S) \text{ is monotonic.}$$

Suppose a 99 percent confidence level (notice this is under realistic measure) is applied, the economic capital under current framework is

$$\begin{aligned} EC_{\mathrm{GMAB}} &= e^{-r\cdot n}(NV_{n,99\%} - NV_0) = e^{-r\cdot n}[H_{n,99\%} - f_{n,99\%} - NV_0] \\ &= e^{-r\cdot n}[\frac{e^{-mN}}{S_0} \cdot V_{put}(S_{n,99\%}, e^{(m+r_g)N} S_0, r, d, \sigma, N-n) - f_{n,99\%} - NV_0] \\ &= e^{(r_g - r)N} \Phi(-d_2) - e^{-r\cdot n - mN - d(N-n)} e^{(\mu - d - \frac{\sigma^2}{2})n - 2.33\sigma\sqrt{n}} \Phi(-d_1) \\ &\quad - e^{-r\cdot n} f_n(S_0 e^{(\mu - d - \frac{\sigma^2}{2})n - 2.33\sigma\sqrt{n}}) - e^{-r\cdot n} NV_0 \end{aligned}$$

[3] Monotonicity of function $NV_n(S_n)$ is implied by the negativeness of its 1st derivative with respect to S_n.

here

$$d_1 = \frac{(\mu - d - \frac{\sigma^2}{2})n - 2.33\sigma\sqrt{n} - (m + r_g)N + (r - d + \frac{\sigma^2}{2})(N - n)}{\sigma\sqrt{N - n}}$$

$$d_2 = d_1 - \sigma\sqrt{N - n}$$

$$NV_0 = H_0 - f_0$$

2.1.3 GMAB with Mortality and Lapse

In the last section, mortality and lapse risk were totally ignored. In the real world, the involvement of mortality and lapse distinguishes GMAB from the normal financial instruments. Mortality leads the contract maturity time to be stochastic, and the lapse feature gives the policyholder an opportunity to abandon the contract. (Lapse happens when policyholders stop paying the management fee and exit their position with some certain amount of penalty charged.)

Let $\Psi(t)$ be the percentage of policyholders that survive and do not lapse before time t, $q(t)$ and $l(S_t, t)$ be the simultaneous mortality and lapse intensities (or equivalently, *hazard rates*) respectively. Independence between lapse risk and mortality risk is also assumed. Under an exponential model, $\Psi(t)$ has the following form:

$$\Psi(t) = e^{-\int_0^t [l(S_u, u) + q(u)]du}$$

Standard actuarial practice treats mortality risk as *diversifiable* or *non-systematic*, which means the mortality risk can be eliminated by issuing a large enough number of equivalent contracts. In this paper we adhere to this assumption. Then the benefits of a life insurance contract turn to be $\sum \Psi(t)q(t)\Delta t P(t)$, where $P(t)$ represents the payoff at time t.

However, since equity market performance has a huge impact on the policyholder's lapse behavior (Shumrak *et al.* 1999; Milevsky & Salisbury, 2002), lapse risk is not fully diversifiable and therefore is affected by the underlying equity price, which is implied in the form of $l(S_t, t)$. Therefore survival probability $\Psi(t)$ depends on the whole underlying equity price path $\{S_n\}$ prior to t.

Some researchers model the lapse behavior as a policyholder's fully rational decision, and treat it as an American-typed option. In this paper, we suggest that the lapse behavior of both VA and EIA policyholders can be rational or irrational just like other life insurance products and set up the model in a different way.[4]

We introduce the *dynamic lapse multiplier* in order to model dynamic lapse. At any time n, the instantaneous lapse rate can be modeled as

$$l(S_n, t) = f(R, t) \cdot l_B$$

here

$$R = \frac{F_n}{G_n} = \frac{1}{S_0} S_n e^{-(m+r_g)n}$$

The actual lapse rate l is the product of the base lapse rate l_B (Normally 2 percent for the GMAB product[5]) and the dynamic lapse multiplier $f(R,t)$. $f(R,t)$ depends on the ratio of guaranteed value to market value (GV / MV). The dynamic lapse multiplier is a non-decreasing function in variable S_n, which means a GMAB policyholder is more likely to lapse when the embedded option is more out-of-the-money (i.e., when the ratio of account value and guarantee is high).

Taking survival probability into account, the risk-neutral price of the embedded option is

$$\begin{aligned} H_n &= E_n^Q\left[\sum_{t=n}^{N-1} e^{-r(t-n)}\Psi(t)\cdot q(t)\Delta t\cdot(G_t - F_t)^+ + e^{-r(N-n)}\Psi(N)\cdot(G_N - F_N)^+\right] \\ &\approx E_n^Q\left[\int_n^N e^{-r(t-n)}\Psi(t)q(t)(G_t - F_t)^+ dt + e^{-r(N-n)}\Psi(N)\cdot(G_N - F_N)^+\right] \end{aligned} \quad (2.1)$$

and the PV of the fees

$$f_n = E_n^Q\left[\sum_{t=n}^{N}\Psi(t)\cdot e^{-r(t-n)}F_t\varepsilon\Delta t\right] \approx E_n^Q\left[\int_n^N \Psi(t)\cdot e^{-r(t-n)}F_t\varepsilon dt\right]$$

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4 This is because life insurance policyholders are neither financial professionals nor institutional investors, and sometimes lapse does happen for reasons unrelated to the equity performance. Liquidity problems and defaults can be examples.

5 Base lapse rate can be influenced by macro-economic factors such as the state of the domestic economy, federal rates, etc.

Let $NV_n(S_n)$ be the net value of adding the guarantee to the VA product, which is

$$NV_n(S_n) = H_n - f_n$$

Taking into account the mortality and lapse risk, economic capital of GMAB is defined in the same way as previously.

$$\mathrm{P}[e^{-r\cdot n}(NV_n(S_n) - NV_0(S_0)) \geq EC_{GMAB}] < 1 - \beta$$

An analytic form of EC_{GMAB} is difficult to achieve. In this paper, a double-tier simulation algorithm is implemented to calculate EC_{GMAB}.

In a simpler case, if lapse risks are assumed to be independent from the market (which means $l(t)$ is not dependent on S_n), a clearer form of the GMAB would be accessible. Let $\boldsymbol{BSP}(n,t)$ be at any time n, the value of the put option embedded in GMAB that matures at t, without taking lapse and mortality into account. From the last section we know

$$\begin{aligned}\boldsymbol{BSP}(n,t) &= \frac{e^{-mt}}{S_0} \cdot V_{put}(S_n, e^{(m+r_g)t}\eta S_0, r, d, \sigma, t-n) \\ &= e^{r_g t - r(t-n)}\Phi(-d_2) - \frac{e^{-mt}}{S_0} S_n e^{-d(t-n)}\Phi(-d_1)\end{aligned}$$

here

$$\begin{aligned}d_1 &= \frac{\log(S_n / S_0) - (m + r_g)t + (r - d + \frac{\sigma^2}{2})(t-n)}{\sigma\sqrt{t-n}} \\ d_2 &= d_1 - \sigma\sqrt{t-n}\end{aligned}$$

Unlike Formula (2.1), $\Psi(t)$ is no longer path-dependent and therefore can be factored out from the risk-neutral expectation. The embedded put option value in GMAB can be written as

$$\begin{aligned}H_n &= \sum_{t=n}^{N-1} \Psi(t) \cdot q(t)\Delta t \cdot \boldsymbol{BSP}(n,t) + \Psi(N) \cdot \boldsymbol{BSP}(n,N) \\ &\approx \int_n^N \Psi(t) q(t) \boldsymbol{BSP}(n,t) dt + \Psi(N) \cdot \boldsymbol{BSP}(n,N)\end{aligned}$$

The PV of the fees

$$f_n = E_n^Q[\sum_{t=n}^{N}\Psi(t)\cdot e^{-r(t-n)}F_t\varepsilon\Delta t] \approx \int_n^N e^{-\int_0^t[l(u)+q(u)]du}\cdot e^{-r(t-n)}E^Q[F_t]\varepsilon dt$$
$$= \frac{\varepsilon S_n}{S_0}\int_n^N e^{-\int_0^t[l(u)+q(u)]du}\cdot e^{-m(t-n)}dt$$

Proposition 1: In the case where both mortality and lapse risk are independent from the underlying equity prices, function $NV_n(S_n)$ is monotonically decreasing.

Proof: See the appendix.

Since $NV_n(S_n)$ is monotonic, its analytical economic capital (or equivalent, VaR) can be directly calculated in the same way as in the last section (Fong & Lin, 1999).

$$EC_{\text{GMAB}} = e^{-r\cdot n}(NV_{n,99\%} - NV_0) = e^{-r\cdot n}[H_{n,99\%} - f_{n,99\%} - NV_0]$$

here

$$H_{n,99\%} = \int_n^N \Psi(t)q(t)\boldsymbol{BSP_{99\%}}(n,t)dt + \Psi(N)\boldsymbol{BSP_{99\%}}(n,N)$$

$$f_{n,99\%} = e^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}}\varepsilon\int_n^N e^{-\int_0^t[l(u)+q(u)]du}\cdot e^{-m(t-n)}dt$$

$$\boldsymbol{BSP_{99\%}}(n,t) = \frac{e^{-mt}}{S_0}\cdot V_{put}(S_0e^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}}, e^{(m+r_g)t}\eta S_0, r, d, \sigma, t-n)$$
$$= e^{r_g t - r(t-n)}\Phi(-d_2) - e^{-mt}e^{(\mu-d-\frac{\sigma^2}{2})n-2.33\sigma\sqrt{n}}e^{-d(t-n)}\Phi(-d_1)$$

$$d_1 = \frac{(\mu-d-\frac{\sigma^2}{2})n - 2.33\sigma\sqrt{n} - (m+r_g)t + (r-d+\frac{\sigma^2}{2})(t-n)}{\sigma\sqrt{t-n}}$$

$$d_2 = d_1 - \sigma\sqrt{t-n}$$

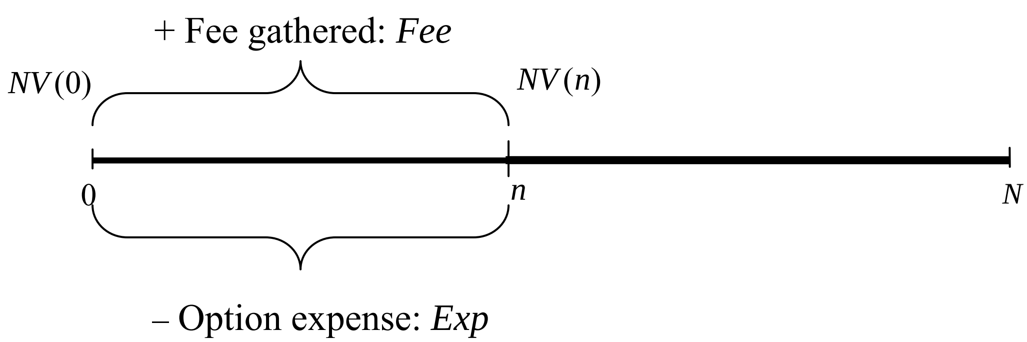
$$NV_0 = H_0 - f_0$$

2.1.4 RAPM of GMAB

Risk-adjusted performance measure (RAPM) is widely used in risk management work as a measure of returns in line with risks taken. Sitting at time horizon n, let $I(n)$ be the net income from a GMAB product during time period $[0,n]$ and let $EC(n)$ be the economic capital that is required to support such a contract. Then the *total return* of GMAB, noted by $TR(n)$, has the form of

$$TR(n) = \frac{I(n)}{EC(n)}$$

For a GMAB product that matures at N, the net income $I(n)$ is illustrated in the following graph.



There are three components that are contained in $I(n)$: fee gathered by insurer during $[0,n]$, option expense paid by insurer during $[0,n]$ and capital gain, which is the price difference between $NV_0(S_0)$ and $NV_n(S_n)$. Then $I(n)$ has the form as follows.

$$I(n) = Fee - Exp + NV_n(S_n) - NV_0(S_0)$$

RAPM of product is defined as the annualized return

$$RAPM = (1 + TR(n))^{\frac{1}{n}} - 1$$

Example 2.1.1. GMAB contract with maturity $N = 20$, guaranteed interest rate $r_g = 2\%$, underlying equity drift $\mu = 12\%$, volatility $\sigma = 0.2$,dividend yield $d = 0\%$, risk-free rate $r = 6\%$, benefit charge $\varepsilon = 1.5\%$, total fee charge $m = 3\%$, confidence level 99%. Taking into account mortality and lapse risk, the result in Table 2.1.1 is calculated through simulation.

TABLE 2.1.1
GMAB Contract: Economic Capital and RAPM

Year	Eco. Cap.	I(n)	TR(n)	RAPM
1	0.13	0.02	0.18	**18.4%**
2	0.17	0.05	0.30	**14.2%**
3	0.19	0.08	0.42	**12.4%**
4	0.21	0.11	0.53	**11.2%**
5	0.21	0.14	0.66	**10.7%**

2.2 Economic Capital of PTP

2.2.1 Product Description

Unlike VAs, EIAs are general account assets. EIA contracts vary between insurance companies, and the simplest EIA product is called point-to-point (PTP). This provides the beneficiary return on an index, but with a minimal guarantee (which is call-like) at the contract's maturity (usually death protection is included).

An example of a PTP contract is as follows: at the initiation, $t = 0$, the policyholder enters into a contract by paying the insurance company an initial amount P. The insurance company invests the amount P into the bond market, and there is no further payment from the policyholder. The insurance company guarantees a fixed rate of return r_g (with a pre-specified guaranteed proportion) up to the end of the contract (which can be caused by mortality, maturity or lapse decided by the policyholder), when the beneficiary will receive the greater of either the return on an index (with a pre-specified participation rate) or the guaranteed amount. If the policyholder lapses the EIA contract before maturity, he can get the guaranteed amount back after some penalty fees charged, but the return on that index is not redeemable.

2.2.2 PTP without Mortality and Lapse

Consider a simple PTP contract with $1 initial account value and maturity time N (in years) with fixed-interest rate r_g and guaranteed proportion η (95 percent or 100 percent is common). Also assume the underlying equity index price follows geometric Brownian motion with constant risk-free rate and volatility. Let

$$G_n = \eta \cdot e^{r_g n}, \quad 0 \le n \le N$$

be the amount of account value that is guaranteed. Similar to a GMAB contract, G_n is going to be the strike price for its embedded option. Let S_n represent the value at n of the equity index used. Given a participation rate α , the beneficiary of embedded call option payoff at maturity will be

$$H_N = (F_N - G_n)^+ = \left((1 + \alpha(\frac{S_N}{S_0} - 1)) - \eta \cdot e^{r_g N} \right)^+$$
$$= \frac{\alpha}{S_0}[S_N - \frac{S_0}{\alpha}(\eta e^{r_g N} - (1 - \alpha))]^+$$

with

$$F_N = (1 + \alpha(\frac{S_N}{S_0} - 1))$$

F_N is the available amount for participation. At any time n<N, the embedded call value on this contract can be formulated through risk-neutral pricing theory

$$H_n = E_n^Q[e^{-r(N-n)} H_N]$$

Let notation $V_{call}(S_0, K, r, d, \sigma, t)$ represent the price of a standard European call, under B-S-M framework (Black & Scholes, 1973), closed form of the embedded option value H_n can be written as

$$H_n = \frac{\alpha}{S_0} \cdot V_{call}(S_n, \frac{S_0}{\alpha}(\eta e^{r_g N} - (1 - \alpha)), r, d, \sigma, N - n)$$
$$= e^{-d(N-n)} \frac{\alpha S_n}{S_0} \Phi(d_1) - (\eta e^{r_g N} - (1 - \alpha)) e^{-r(N-n)} \Phi(d_2)$$

with

$$d_1 = \frac{\log(\alpha S_n / [S_0(\eta e^{r_g N} - (1 - \alpha))]) + (r - d + \frac{\sigma^2}{2})(N - n)}{\sigma\sqrt{N - n}}$$
$$d_2 = d_1 - \sigma\sqrt{N - n}$$

For the fees charged from PTP contracts at given time horizon, the following theorem is satisfied.

Proposition 2: Assume no mortality and lapse risk. For a PTP contract that is described in the previous section, for any $0 \le t_1 \le t_2 \le N$, let $I(t_1,t_2)$ represent the fee gathered during period $[t_1,t_2]$, compounded to time t_2. Then

$$I(t_1,t_2) = \eta[(e^{rt_2} - e^{r_g t_2}) - (e^{rt_1} - e^{r_g t_1})] + \frac{1-\eta}{1-e^{-rN}}(e^{rt_2} - e^{rt_1})$$

Proof: See the appendix.

Similar to a GMAB contract, the net value of adding the guarantee to the PTP product at time n, noted by $NV_n(S_n)$, can be formulated as the difference between two parts: the first part is the embedded option (guarantee) value, from time n to maturity N; the second part is the present value of the fee that is going to be charged in the future (noted as f_n). $NV_n(S_n)$ has the following form

$$NV_n(S_n) = H_n - f_n$$

and

$$f_n = e^{-r(N-n)} I(n,N) = e^{-r(N-n)}[\eta((e^{rN} - e^{r_g N}) - (e^{rn} - e^{r_g n})) + \frac{1-\eta}{1-e^{-rN}}(e^{rN} - e^{rn})]$$

The corresponding economic capital of the PTP is defined as the percentile risk measure of $NV(S_n)$.

$$\mathrm{P}[e^{-r\cdot n}(NV_n(S_n) - NV_0(S_0)) \ge EC_{PTP}] < 1-\beta$$

Here β is the confidence level. Since $NV_n(S_n)$ is again monotonic,[6] its analytical economic capital (or equivalent, VaR) is accessible (Fong & Lin, 1999). Suppose a 99 percent confidence level (notice this is under realistic measure) is applied, the economic capital under the current framework is

$$\begin{aligned} EC_{\mathrm{PTP}} &= e^{-r\cdot n}(NV_{n,99\%} - NV_0) = e^{-r\cdot n}[H_{n,99\%} - f_n - NV_0] \\ &= e^{-r\cdot n}[e^{-d(N-n)}\alpha e^{(\mu-d-\frac{\sigma^2}{2})n+2.33\sigma\sqrt{n}}\Phi(d_1) - (\eta e^{r_g N} - (1-\alpha))e^{-r(N-n)}\Phi(d_2) - f_n - NV_0] \end{aligned}$$

[6] Monotonicity of function $NV_n(S_n)$ is implied by the positiveness of H_n's 1st derivative with respect to S_n. Here f_n is not a function of S_n and therefore has no contribution to $\frac{d(NV_n(S_n))}{dS_n}$.

with

$$d_1 = \frac{(\mu - d - \frac{\sigma^2}{2})n + 2.33\sigma\sqrt{n} + \log(\alpha / [(\eta e^{r_g N} - (1-\alpha))]) + (r - d + \frac{\sigma^2}{2})(N-n)}{\sigma\sqrt{N-n}}$$

$$d_2 = d_1 - \sigma\sqrt{N-n}$$

$$NV_0 = H_0 - f_0$$

2.2.3 PTP with Mortality and Lapse

The effect of adding mortality into consideration in a PTP contract is similar to the GMAB case. By using the same terminology, let $\Psi(t)$ be the percentage of policyholders that survive and do not lapse before t, $q(t)$ and $l(S_t, t)$ be the mortality and lapse intensities (or equivalently, *hazard rates*) respectively. Independence between lapse risk and mortality risk is also assumed.

$$\Psi(t) = e^{-\int_0^t [l(S_u,u)+q(u)]du}$$

Similar to GMAB, lapse risk is not diversifiable and $\Psi(t)$ depends on the whole underlying equity price path $\{S_n\}$ prior to t. At any time n, the instantaneous lapse rate can be modeled as

$$l(S_n, t) = f(R,t) \cdot l_B$$

with

$$R = \frac{G_n}{F_n} = \frac{S_0}{S_n}\eta \cdot e^{r_g n}$$

The actual lapse rate l is the product of the base lapse rate l_B (normally 1 percent for the PTP product) and the dynamic lapse multiplier $f(R,t)$. $f(R,t)$ depends on the ratio of market value to guaranteed value (MV / GV, which is different from the GMAB). The dynamic lapse multiplier is again a non-decreasing function in variable S_n, which means a PTP policyholder is more likely to lapse when the embedded option is more out-of-the-money (i.e., when the ratio of account value and guarantee is high).

Taking survival probability into account, the risk-neutral price of the embedded option at time n is

$$H_n = E_n^Q\left[\sum_{t=n}^{N-1} e^{-r(t-n)}\Psi(t)\cdot q(t)\Delta t\cdot(F_t - G_t)^+ + e^{-r(N-n)}\Psi(N)\cdot(F_N - G_N)^+\right]$$
$$\approx E_n^Q\left[\int_n^N e^{-r(t-n)}\Psi(t)q(t)(F_t - G_t)^+ dt + e^{-r(N-n)}\Psi(N)\cdot(F_N - G_N)^+\right]$$

and the PV of the fees

$$f_n = e^{-r(N-n)}[E_n^Q\left[\sum_{t=n+1}^{N}\Psi(t)\cdot e^{r(N-t)}\eta[(e^{rt} - e^{r_g t}) - (e^{r(t-1)} - e^{r_g(t-1)})]\right] + \frac{1-\eta}{1-e^{-rN}}(e^{rN} - e^{rn})]$$
$$\approx e^{-r(N-n)}[E_n^Q\left[\eta(r - r_g)\int_n^N \Psi(t)e^{r(N-t)}dt\right] + \frac{1-\eta}{1-e^{-rN}}(e^{rN} - e^{rn})]$$

Let $NV_n(S_n)$ be the net value of adding the guarantee to the PTP product, which is

$$NV_n(S_n) = H_n - f_n$$

Taking into account the mortality and lapse risk, economic capital of PTP is defined as the percentile risk measure of $NV_n(S_n)$.

$$\mathrm{P}[e^{-r\cdot n}(NV_n(S_n) - NV_0(S_0)) \geq EC_{PTP}] < 1-\beta$$

An analytical form of EC_{PTP} is difficult to achieve. In this paper, a double-tier simulation algorithm is implemented to calculate EC_{PTP}.

In a simpler case, if lapse risks are assumed to be independent from the market (which means $l(t)$ is not depend on S_n), a clearer form of the PTP would be accessible. Let $\boldsymbol{BSC}(n,t)$ be at any time n, the value of the call option embedded in PTP that matures at t, without taking lapse and mortality into account. From the last section we know

$$\boldsymbol{BSC}(n,t) = \frac{\alpha}{S_0}\cdot V_{call}(S_n, \frac{S_0}{\alpha}(\eta e^{r_g N} - (1-\alpha)), r, d, \sigma, N-n)$$
$$= e^{-d(N-n)}\frac{\alpha S_n}{S_0}\Phi(d_1) - (\eta e^{r_g N} - (1-\alpha))e^{-r(N-n)}\Phi(d_2)$$

with

$$d_1 = \frac{\log(\alpha S_n / [S_0(\eta e^{r_g N} - (1-\alpha))]) + (r - d + \frac{\sigma^2}{2})(N-n)}{\sigma\sqrt{N-n}}$$

$$d_2 = d_1 - \sigma\sqrt{N-n}$$

$\Psi(t)$ is no longer path-dependent and therefore can be factored out from the risk-neutral expectation. The embedded call option value in PTP can be written as

$$\begin{aligned} H_n &= \sum_{t=n}^{N-1} \Psi(t) \cdot q(t)\Delta t \cdot \boldsymbol{BSC}(n,t) + \Psi(N) \cdot \boldsymbol{BSC}(n,N) \\ &\approx \int_n^N \Psi(t) q(t) \boldsymbol{BSC}(n,t) dt + \Psi(N) \cdot \boldsymbol{BSC}(n,N) \end{aligned}$$

PV of the fees

$$\begin{aligned} f_n &= e^{-r(N-n)} [\sum_{t=n+1}^{N} \Psi(t) \cdot e^{r(N-t)} \eta [(e^{rt} - e^{r_g t}) - (e^{r(t-1)} - e^{r_g (t-1)})] + \frac{1-\eta}{1-e^{-rN}} (e^{rN} - e^{rn})] \\ &\approx e^{-r(N-n)} [\eta (r - r_g) \int_n^N \Psi(t) e^{r(N-t)} dt + \frac{1-\eta}{1-e^{-rN}} (e^{rN} - e^{rn})] \end{aligned}$$

Here $NV_n(S_n)$ is again monotonic through similar steps to those in the proof of Theorem 1. The economic capital of PTP can be calculated through the same way as in the last section (Fong & Lin, 1999).

$$EC_{\text{PTP}} = e^{-r \cdot n} (NV_{n,99\%} - NV_0) = e^{-r \cdot n} [H_{n,99\%} - f_n - NV_0]$$

with

$$H_n = \int_n^N \Psi(t) q(t) \boldsymbol{BSC}_{\boldsymbol{99\%}}(n,t) dt + \Psi(N) \cdot \boldsymbol{BSC}_{\boldsymbol{99\%}}(n,N)$$

$$\begin{aligned} \boldsymbol{BSC}_{\boldsymbol{99\%}}(n,t) &= \frac{\alpha}{S_0} \cdot V_{call}(S_0 e^{(\mu - d - \frac{\sigma^2}{2})n + 2.33\sigma\sqrt{n}}, \frac{S_0}{\alpha}(\eta e^{r_g N} - (1-\alpha)), r, d, \sigma, N-n) \\ &= e^{-d(N-n)} \frac{\alpha S_n}{S_0} \Phi(d_1) - (\eta e^{r_g N} - (1-\alpha)) e^{-r(N-n)} \Phi(d_2) \end{aligned}$$

$$d_1 = \frac{(\mu - d - \frac{\sigma^2}{2})n + 2.33\sigma\sqrt{n} + \log(\alpha/[\eta e^{r_g N} - (1-\alpha)]) + (r - d + \frac{\sigma^2}{2})(N-n)}{\sigma\sqrt{N-n}}$$

$$d_2 = d_1 - \sigma\sqrt{N-n}$$

$$NV_0 = H_0 - f_0$$

2.2.4 RAPM of PTP

The definition or RAPM for PTP contract is similar to the previous GMAB contract and the author here will keep the same methodology.

$$I(n) = Fee - Exp + NV_n(S_n) - NV_0(S_0)$$

$$TR(n) = \frac{I(n)}{EC(n)}$$

$$RAPM = (1 + TR(n))^{\frac{1}{n}} - 1$$

Example 2.2.1. PTP contract with maturity $N = 10$, guaranteed interest rate $r_g = 2\%$, guaranteed amount $\eta = 100\%$, underlying equity index drift $\mu = 10\%$, volatility $\sigma = 0.2$,dividend yield $d = 0\%$, risk-free rate $r = 6\%$, participation rate $\alpha = 50\%$, confidence level 99%. Taking into account mortality and lapse risk, Table 2.2.1 is calculated through simulation.

TABLE 2.2.1
PTP Contract: Economic Capital and RAPM

Years	Eco. Cap.	I(n)	TR(n)	RAPM
1	0.27	0.01	0.04	**4.4%**
2	0.45	0.02	0.04	**1.9%**
3	0.66	0.03	0.04	**1.3%**
4	0.84	0.02	0.03	**0.7%**
5	1.03	0.02	0.02	**0.3%**

2.3 Economic Capital for VA and EIA Mixture

A natural diversification effect exists for a portfolio that contains both VA (which is put-like) and EIA (which is call-like) products. Suppose both products share the same underlying equity process, then such a portfolio can be modeled as a *straddle* (or *strangle*); that is, whenever either product is in-the-money, the other one is likely to be out-of-the-money. More specifically, when stock price is low and VA is in-the-money, the option value embedded in EIA drops and draws the portfolio value to remain regular; when the stock price is high and EIA is in-the-money, not only does the option value embedded in VA drop, but the policyholder's account is also charged by the insurance company with higher management fees. Both lower the total loss of the whole portfolio. Therefore the risk to the insurer that provides these products is reduced.

2.3.1 Simulation Framework

In this paper a double-tier Monte Carlo simulation framework is used to valuate both annuities' benefits. The simulation algorithm consists of the following steps:

1. Simulate the equity price process from time 0 to the horizon $[0,n]$: $\{S(t_i) \mid i=0,1,...,\hat{K}\}$. This is called the *outer* simulation path.

2. Based on a given mortality and lapse function, calculate the option expense and charged fee during this period $[0,n]$ through the above outer price path $\{S(t_i) \mid i=0,1,...,\hat{K}\}$ (both for VA and EIA).

3. Sitting at horizon time t, simulate the equity price process from time n to maturity $[n,N]$: $\{S(t_i) \mid i=\hat{K}+1,...,K\}$. This is called the *inner* simulation path.

4. Based on a given mortality and lapse function, calculate the option expense and charged fee during this period $[n,N]$ through the above inner price path $\{S(t_i) \mid i=\hat{K}+1,...,K\}$. One net value of the guarantee $NV_n(S_n)_1$ is calculated (both for VA and EIA).

5. Redo step 4 enough times and get series $\{NV_n(S_n)_1, NV_n(S_n)_2 ...\}$, take the mean as the estimator of $NV_n(S_n)$ (both for VA and EIA).

6. Return to step 1 and follow the steps enough times and get series of expenses, fees and net values; record the results to files.

7. In an Excel sheet, calculate the percentile value (VaR), net income and RAPM for VA and EIA, run Excel Solver to find the best products combination that provides the highest RAPM.

In practice, some variance reduction techniques can be used during the simulation. In the inner loops, the no mortality and lapse case have analytical solutions both for VA and EIA, thus these can be used as control variate to accelerate the convergence speed.

Example 2.3.1. Consider GMAB and PTP that share the same underlying equity process. Retain parameters from the previous examples. Double-tier simulation framework provides the results as following tables.

Table 2.3.1 provides the economic capital requirements for VA, EIA, 50/50 mixture and the optimal VA/EIA mixture based on different time horizons. Table 2.3.2 provides RAPM for VA, EIA, 50/50 mixture and the optimal mixture as well, and the last column is the weights of VA in the optimal portfolio. Graphical results are listed at the end of this section.

TABLE 2.3.1
Economic Capital Requirement for VA, EIA and Mixture

Economic Capital requirement				
Years	**VA**	**EIA**	**Mix(50/50)**	**Best Mix**
1	0.13	0.27	0.06	**0.04**
2	0.17	0.45	0.12	**0.07**
3	0.19	0.66	0.20	**0.09**
4	0.21	0.84	0.27	**0.12**
5	0.21	1.03	0.36	**0.13**

TABLE 2.3.2
RAPM for VA, EIA and Mixture

RAPM					
Years	**VA**	**EIA**	**Mix(50/50)**	**Best**	**Best Wgt of VA**
1	18.4%	4.4%	30.8%	**51.6%**	0.56
2	14.2%	1.9%	13.7%	**24.9%**	0.60
3	12.4%	1.3%	8.1%	**18.3%**	0.64
4	11.2%	0.7%	5.6%	**14.1%**	0.67
5	10.7%	0.3%	4.1%	**12.1%**	0.70

In this example, the natural hedging effect is significant. In the first five years, the optimal mixtures have an average of 52 percent smaller economic capital requirements than VA, and these are 86 percent smaller compare to EIA.

It is also observed that in Table 2.3.1, the economic capital of EIA goes up tremendously. This is because for shorting a call, there is no upper bound for the future loss. While in the mixture portfolio, the loss from EIA is balanced out by the moneyness of the option embedded in VA and the fees charged from the policyholder's account, as shown in the best mixture capital column. In Table 2.3.2, the weights of VA in the optimal mixture portfolio grow gradually. This is because the capital demand from EIA increases quickly and thus requires more weight in the VA in order to balance out.

2.4 Conclusion

This paper contributes to the literature in the area of analyzing natural diversification benefits between VA and EIA products. These benefits result from the phenomenon that the values of VA and EIA move in opposite directions in response to a change in the underlying equity value. The author models VA and EIA in the risk-neutral option pricing framework. Numerical examples show that natural hedging is feasible and the benefits are significant, which enables the insurance companies' capital to be deployed more efficiently.

Figure 2.3.1
Economic Capital Requirement for VA, EIA and the Best Mixture

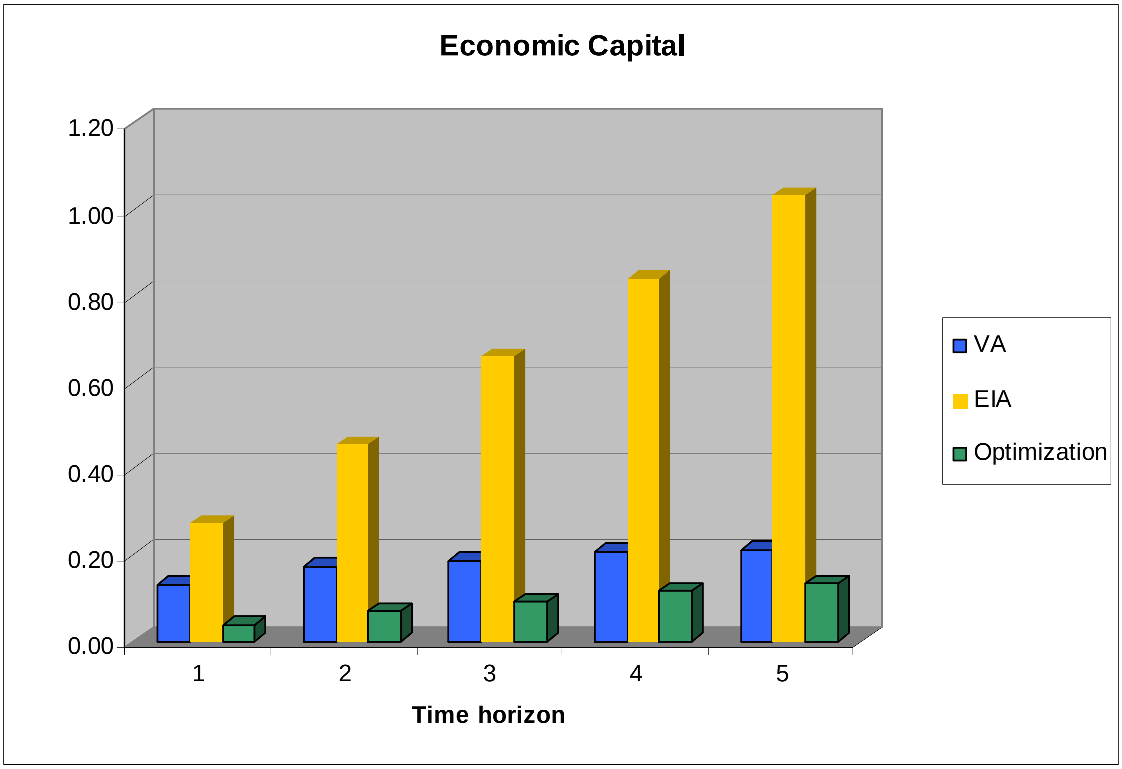
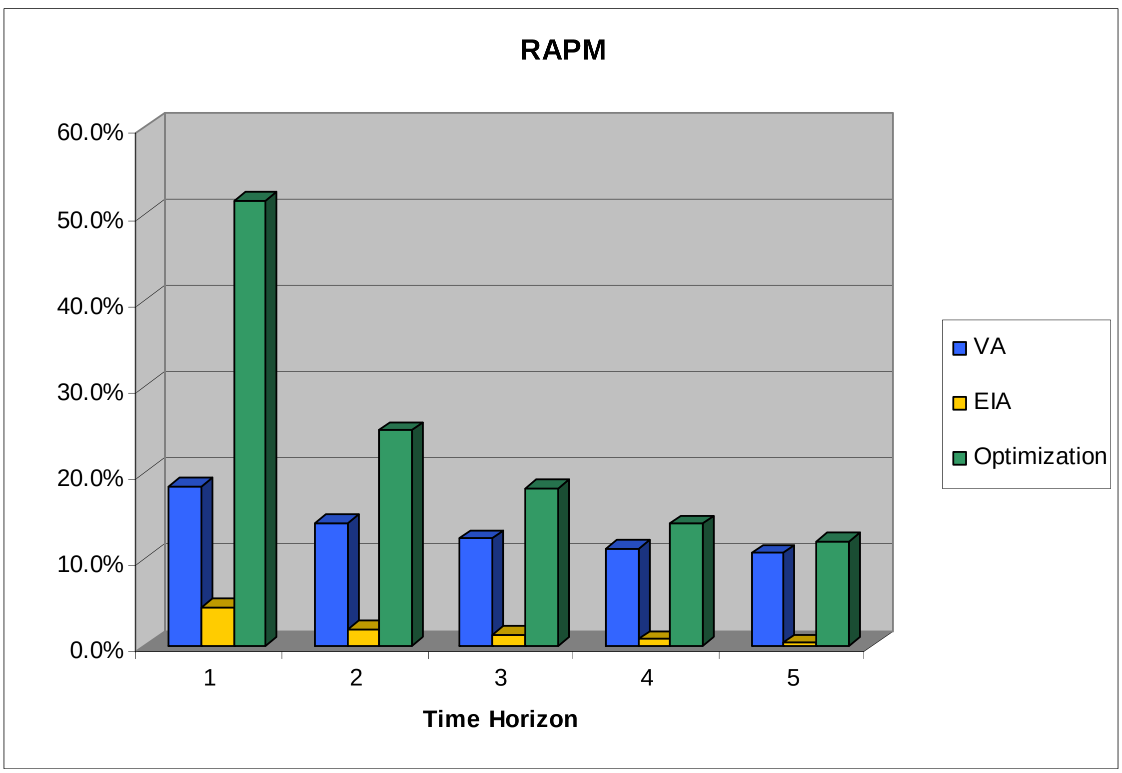


Figure 2.3.2
RAPM for VA, EIA and the Best Mixture



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Appendix

1.1 Notations

G_n	► guaranteed level
r_g	► guaranteed interest rate
N	► maturity of product
F_n	► account value at time n
H_n	► value of embedded option
S_0	► underlying equity price at time 0
m	► management fee of VA charged each year
ε	► benefit charge
η	► guaranteed amount of EIA
α	► participation rate
$V_{put}(S_0, K, r, d, \sigma, t)$	► price of a vanilla European put
$\Psi(t)$	► survival probability
$l(S_t, t)$	► instantaneous lapse rate
$q(t)$	► instantaneous mortality rate
NV_n	► net value of guarantee
f_n	► value of benefit charge
β	► confidence level
EC	► economic capital
$\boldsymbol{BSC}(n,t)$	► European call price at time n and maturities at t
$\boldsymbol{BSP}(n,t)$	► European put price at time n and maturities at t
$I(n)$	► net income during period $[0, n]$
$TR(n)$	► total return
$RAPM$	► risk-adjusted performance measure

1.2 Proof of Proposition 1

Proposition 1: In the case where both mortality and lapse risk are independent from the underlying equity price, the function $NV_n(S_n)$ is monotonically decreasing.

Proof:

If both risks are independent from underlying equity price S_n, $NV_n(S_n)$ has the following form

$$NV_n(S_n) = H_n - f_n$$

with

$$H_n = \int_n^N \Psi(t)q(t)\boldsymbol{BSP}(n,t)dt + \Psi(N)\cdot \boldsymbol{BSP}(n,N)$$

PV of the fees

$$f_n = \frac{\varepsilon S_n}{S_0}\int_n^N e^{-\int_0^t [l(u)+q(u)]du}\cdot e^{-m(t-n)}dt$$

Now take the 1st derivative to both H_n and f_n with respect to S_n.

$$\frac{dH_n}{dS_n} = \int_n^N \Psi(t)q(t)\frac{d(\boldsymbol{BSP}(n,t))}{dS_n}dt + \Psi(N)\cdot\frac{d(\boldsymbol{BSP}(n,N))}{dS_n}$$ [7]

Since

$$\frac{d(\boldsymbol{BSP}(n,t))}{dS_n} = -\frac{e^{-mt}}{S_0}e^{-d(t-n)}\Phi(-d_1) < 0$$

We know H_n is monotonically decreasing. For f_n,

$$\frac{df_n}{dS_n} = \frac{\varepsilon}{S_0}\int_n^N e^{-\int_0^t [l(u)+q(u)]du}\cdot e^{-m(t-n)}dt \geq 0$$

Which implies that f_n is monotonically increasing, therefore $NV_n(S_n)$ is monotonically decreasing.

---

[7] Here the author intentionally skipped rigorous mathematical proof of the interchange of derivative and integral. Precisely, formula (3.2.1) is valid only when the following technical conditions hold: 1. Both $\Psi(t)q(t)\boldsymbol{BSP}(n,t)$ and $\frac{d(\Psi(t)q(t)\boldsymbol{BSP}(n,t))}{dS_n}$ are continuous; 2. Both $\Psi(t)q(t)\boldsymbol{BSP}(n,t)$ and $\frac{d(\Psi(t)q(t)\boldsymbol{BSP}(n,t))}{dS_n}$ are bounded by a L^1 function. See Cheney (2001) for example.

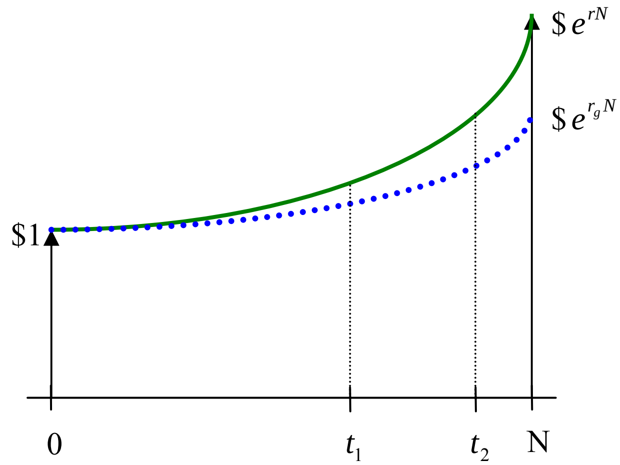
1.3 Proof of Proposition 2

Proposition 2: Assume no mortality and lapse risk. For a PTP contract that is described in the previous section, for any $0 \le t_1 \le t_2 \le N$, let $I(t_1,t_2)$ represent the fee gathered during period $[t_1,t_2]$, compounded to time t_2. Then

$$I(t_1,t_2)=\eta[(e^{rt_2}-e^{r_g t_2})-(e^{rt_1}-e^{r_g t_1})]+\frac{1-\eta}{1-e^{-rN}}(e^{rt_2}-e^{rt_1})$$

Proof:

First consider \$1 risk-free money account with guaranteed proportion 100% and guaranteed interest rate r_g. Risk-free rate is r. The account growth is shown in the following figure.



Fully ($\eta = 100\%$) guaranteed PTP income

The account value and guaranteed value increases with different rates, and the difference between them is the income. At any time t, the account value and guaranteed value will be e^{rt} and $e^{r_g t}$ respectively. Assuming no mortality and lapse risk, income from this particular account during period $[t_1,t_2]$, noted by $I_1(t_1,t_2)$, will be

$$I_1(t_1,t_2)=I_1(0,t_2)-I_1(0,t_1)=(e^{rt_2}-e^{r_g t_2})-(e^{rt_1}-e^{r_g t_1})$$

A PTP contract account can be modeled as two components: the guaranteed proportion η, and the pure profit proportion $1-\eta$. The corresponding income from the guaranteed proportion part, which is earned by holding policyholders' capital during period $[0,N]$, can be calculated by the above formula. This part of income can be affected by policyholders' behavior (death or lapse). On the other hand, the pure profit proportion $1-\eta$ is locked in at time 0, and therefore will not be affected by the policyholders' behavior. This proportion can be amortized to a continuous cash flow with a constant payout rate r_P through period $[0,N]$ as follows

$$1-\eta = \int_0^N e^{-rt} r_P dt$$

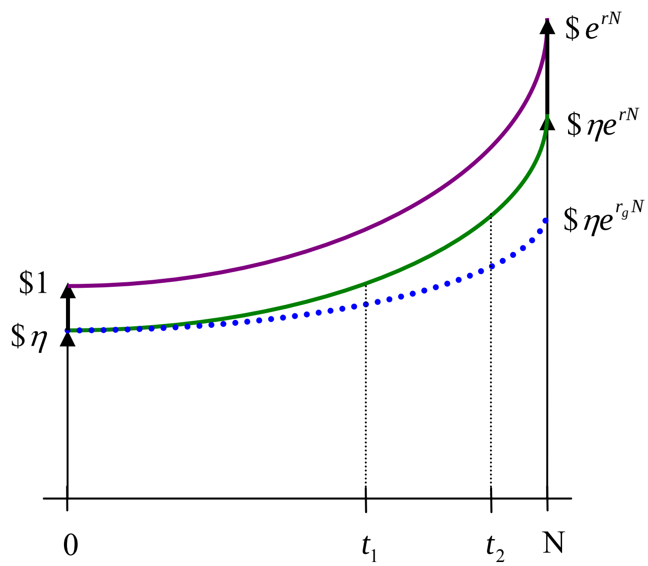
Solve for

$$r_P = \frac{(1-\eta)r}{1-e^{-rN}}$$

Income from this part can be modeled as

$$I_2(t_1,t_2) = I_2(0,t_2) - I_2(0,t_1) = \int_0^{t_2} r_P e^{r(t_2-s)} ds - \int_0^{t_1} r_P e^{r(t_1-s)} ds = \frac{r_P}{r}(e^{rt_2} - e^{rt_1})$$

An illustrative figure of the PTP contract account is in the following



Partly ($\eta < 100\%$) guaranteed PTP income

Total income of the PTP contract is the sum of the two parts, which is

$$\begin{aligned} I(t_1,t_2) &= \eta I_1(t_1,t_2) + I_2(t_1,t_2) \\ &= \eta[(e^{rt_2} - e^{r_g t_2}) - (e^{rt_1} - e^{r_g t_1})] + \frac{1-\eta}{1-e^{-rN}}(e^{rt_2} - e^{rt_1}) \end{aligned}$$