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The Valuation of Interest-Sensitive Cash Flows Using the Symbolic Method

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ABSTRACT

This paper introduces the symbolic valuation, a stochastic valuation which allows flexible interest rate and cash flow assumptions, and provides an illustration which values a simple SPDA product.

Overview

The fundamental problem of modern finance concerns how to analyze the relative worth of future obligations, with the most prominent measure of worth being the present value[1] of future cash flows. When the timing or amounts of the cash flows are uncertain, the value of future cash flows is measured as the *expected* present value with respect to a specific set of assumptions about future economic environments and the behavior of the cash flows with respect to these environments. In general, practitioners estimate this expected present value by averaging the present value over various scenarios. Unfortunately, this technique provides coarse estimates and implies rather unrealistic assumptions.[2] An alternate approach which more fully realizes realistic economic assumptions is needed. This paper presents an alternate approach for valuing cash flows, the symbolic method.

The symbolic method is intended to value cash flows whose amounts vary with interest rates. Interest-sensitive cash flows require a fundamentally different treatment than non-interest-sensitive flows. To illustrate, if the cash flows are fixed, as in the case of an annuity certain, the valuation follows almost immediately from the definition of present value. In this case, the valuation is often fine tuned by focusing on the exact timing of payments. (See Kellison [4].) When the cash flows are random variables independent of the discounting rate, as in the example of traditional life or health insurance, the important element on which to focus is the timing and size of the cash flows (claims). In this case, only the separate expected values of each cash flow and discount factor are needed. (See Bowers, *Et al.* [1].) In both of these cases, the interest rate is assumed to be level at some "reasonable" rate. However, when the amounts of the cash flows depend on rates, as with a Single Premium Deferred Annuity (SPDA), the valuation must account for the variability of interest rates as well as the behavior of the cash flows with respect to those variations. This additional consideration requires that these cash flows be valued with methods different from classical actuarial methods. While many practitioners employ Monte Carlo style scenario testing, the paper presents the symbolic method as an alternate technique.

Section 1 presents the general structure of a valuation, showing the calculations required to find the present value of a stream of cash flows through a specific scenario. In this case, all cash flows are fixed or known with certainty for the given scenario. Section 2 extends this general structure to encompass stochastic valuations in which the cash flows vary from scenario to scenario and the value is calculated as the expected present value over many scenarios. This section provides the theoretical framework from which we may value cash flows which vary according to the interest-rate scenario. Section 3 presents the symbolic method. The main elements are presented in sections 3.2 and 3.3, while section 3.4 presents a sample valuation. This example is deliberately simple in order to demonstrate the technique. Then, section 3.5 describes how the sample could

[1] Value, price, reserve, and book value are used throughout this paper as different names for a present value, each expressing a different use of the present value.

[2] For example, the binomial lattice assumes rate changes from one period to the next to be a binomial random variable.

be modified to accommodate varying assumptions and discusses concerns to be considered when employing the symbolic method. Section 3.5 also describes the benefits of the symbolic approach, which include: ease of changing assumptions, estimating duration, convexity and other economic statistics, and performing sensitivity analysis. Throughout the paper, the presentation focuses on the valuation of the SPDA, presenting all notation with the SPDA in mind.

1 General Structure of a Valuation

This section outlines a general structure in which financial instruments may be valued. The instrument to be valued has an underlying base value. For an SPDA this would be the account value. For a bond, it would be the outstanding par. For a mortgage, it would be the outstanding principle. This amount varies over time as the base accrues interest and cash payments retire a portion of the base.[3] The following assumptions apply to most valuations:

Cash flows occur at the end of each year[4].
Cash flows occur for a fixed number of years, say N.
The last cash flow is 100% of the outstanding base value.
The value is the present value of future cash flows.

These assumptions can be varied. Cash flows could occur in the middle of the year, or be distributed in some manner. The valuation rate could be fixed or could vary over time. In most cases, the general procedure is the same: calculate future base values and cash flows, then discount the flows to the present. The following notation is used:

$$\begin{aligned} A_n &:= \text{Base (Account) Value After Cashflow at time } n, \\ C_n &:= \text{Cashflow at time } n, \\ V_n &:= \text{Value at time } n \text{ of all cashflows after time } n, \\ {}_cr_n &:= \text{Credited Rate at time } n, \\ {}_mr_n &:= \text{Market Rate at time } n, \\ L_n &:= \text{Percent Decrement of Base Value at time } n. \end{aligned} \quad (1)$$

From these definitions and the above assumptions, the relationships of equation (2) follow.

$$\begin{aligned} C_n &= L_n\, A_{n-1}\, (1 + {}_cr) \\ A_n &= A_{n-1}\, (1 + {}_cr) - C_n \\ V_{n-1} &= (V_n + C_n)\, (1 + {}_mr)^{-1} \end{aligned} \quad (2)$$

In a deterministic model, these values are found iteratively. First, all account value and cash flow values are calculated progressing forward through time, first period to last. Then the market values are calculated backward through time, last period to first.

[3] A bullet bond which always pays only its interest coupon in cash until maturity is a special instrument whose outstanding base (par) remains constant over time.

[4] In this paper, we employ annual periods for valuation. In general, any time frame may be selected, i.e. quarterly, monthly, etc.

The forward pass is depicted in Diagram 1.

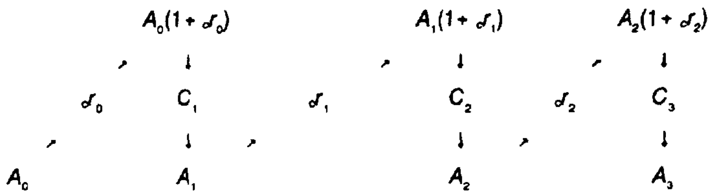

Diagram 1

In Diagram 1, the diagonal arrows indicate multiplication by $(1+{}_c r_0)$ and the vertical arrows indicate decreasing the account value (after interest) by the cash flow, C_n. The process could be extended to perform a valuation with a longer time period. Here, the credited rates are indexed to allow them to vary over time.

To calculate the present value of this block, the cash flows are discounted back to time zero, as depicted in Diagram 2. It is most efficient to perform this evaluation by discounting back period by period, starting with the last cash flow and proceeding to the first, at each step adding the next (earlier) cash flow to the market value.

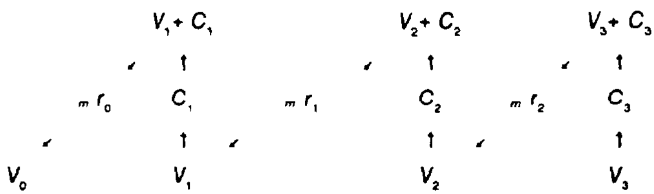

Diagram 2

In the Diagram 2, the vertical arrows indicate incrementing the discounted value by the current cash flow, C_n, and the diagonal arrows indicate discounting the market rate, ${}_m r_n$. At this point, it is clear why 100% lapses are a convenient assumption for the last period. If there were an outstanding account value in the last time period, it would be necessary to estimate the future value, V_N, at time N of cash flows past time N. The 100% lapse assumption at time N ensures $A_N=V_N=0$.

Combining the previous diagrams yields the following.

2 Stochastic Valuations

In a deterministic analysis, future variables such as market rates and lapse rates are assumed to be known with certainty for ease of computation. However, the future values of these variables are generally not known with certainty and can only be quantified probabilistically. The stochastic approach has traditionally been performed by repeating the deterministic approach many times for different scenarios and then combining the results, often by averaging the V_0's which arose from

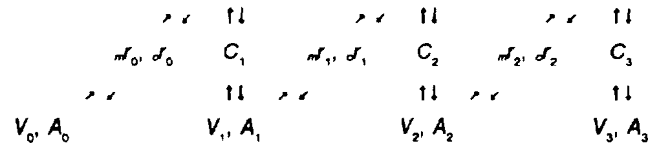

Diagram 3

each scenario. The following general definition of value applies: The **Present Value**, or **Price** of the future cash flows arising from a set of probabilistic assumptions is defined to be the expected value of the present value.

Any estimate of this expected value is a stochastic valuation. In contrast, the "discounted values [from deterministic valuations] are realizations of the random variable whose expected value is the ... price." [5] [6] For example, the present value of several different deterministic scenarios could be averaged together with equal weights to estimate the value of the stream of future cash flows. This is a stochastic valuation. This particular example is not satisfying because the weights are chosen rather arbitrarily.

When performing a stochastic valuation by averaging together several deterministic ones, the weights used for the scenarios greatly affect the final result. There is considerable debate about the proper weights to use, and different weights may be appropriate at different times. Indeed, a practitioner may perform different valuations under different assumptions to answer different questions, for example pricing versus establishing a reserve. It is not the purpose of this paper to define which assumptions should be used. Rather, our goal is to present a general frame work for performing a stochastic valuation given any assumptions.

In a fashion similar to the deterministic method, assume cash flows occur at distinct points in time, say annually.[7] Let the cash flow at time n be C_n and the applicable one-period market rate from time n to n+1 be $_{m}r_{n}$.[8] In the most general formulation, each market rate and cash flow has a conditional probability distribution contingent on all previous rates, and each cash flow has a distribution dependent on all previous and present interest rates. Most models display interdependence between rates from one year to the next. However, rarely are cash flows modeled with distributions conditional on rates; instead, cash flows are expressed as a function of previous and current rates. That is to say, if all rates before and during the cash flow are known, then the cash flow is assumed known with certainty as a function of this interest rate information.

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[5] Jacob, Lord and Tilley [3].

[6] Or, a deterministic valuation is the exact valuation arising from degenerate distributions, i.e. point distributions with one value having 100% certainty of occurrence.

[7] The assumption of annual time periods is not necessary for the valuation. It is simply convenient for our discussion.

[8] Rates varying by maturity are discussed in Section 3.5. See the discussion of Term *Structures and Time Series there.*

Thus, C_n is a function of the rates until that time, $C_n({}_m r_0, {}_m r_1, \dots, {}_m r_n)$, and ${}_m r_n$ has a conditional distribution $g({}_m r_n \mid {}_m r_0, \dots, {}_m r_{n-1})$. The definition of actuarial present value gives

$$\text{Value} = E\left(\sum_{n=1}^{N} \left[C_n \cdot \prod_{k=0}^{n-1} \frac{1}{(1 + {}_m r_k)} \right] \right) \tag{3}$$

where N is the maximum value of n for which C_n is not identically zero, and the expectation is taken over all possible combinations of market rates from time 1 to N.

Equation (3) is quite general and can be used in a variety of circumstances. Value has different interpretations according to the assumptions of the model employed. If the rate distribution is a market prediction then the value estimates a market price. If the rates are a fixed (deterministic) valuation rate and the cash flows are benefits and expense amounts, then the value is a reserve or a book value. If the flows are statutory profits, then the value is an estimate of the present value of future profits.

There are two methods for performing stochastic valuations: scenario testing and exact valuation. With scenario testing, individual cases are considered. In each case, the level of interest rates is decided at each point in the future and this entire movement is called the *path of interest rates*. Each scenario or path is valued deterministically and then the present values from all paths are weighted together.[9] The selection of paths and weights can be made in one of two ways: either the paths are selected by some method with the weights determined by the interest rate model, or the paths are selected by the interest rate generator[10] with the weights all equal. In either case, the practitioner attempts to find an unbiased estimator of the present value that would occur if all paths were sampled.

Unlike scenario testing, an exact valuation finds the value as if all paths were considered. Instead of estimating the value, we calculate it. Usually, testing all scenarios is impossible, so an approach equivalent to testing all scenarios must be employed. For example, we could integrate or sum the cash flow variables weighted by their relative possibilities. Exact valuations are obviously preferable if they are relatively easy to perform. The symbolic method presented in the next section is an exact valuation.

3 The Symbolic Method

3.1 Overview of the Symbolic Method

In section 2 we presented the definition of actuarial present value as the expected present value of future cash flows. When the cash flows are fixed (as in the case of an annuity certain) or random but independent of interest rates (as in the case of life insurance), the definition can be used directly by discounting the fixed flows or the expected value of the random flows by the appropriate expected discount factor. We find, however, that when the flows depend on interest rates the problem can be prohibitively complex. Until now, the standard technique to attack this problem was to employ Monte Carlo style scenario testing to approximate the true expected

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[9] Such performing of deterministic valuation is called Monte Carlo Sampling.

[10] The word generator is used in preference to the word model in this context to emphasize the rate model's role in selecting the rate path.

value. In this section, we present a method for computing the actuarial value which conforms to the above definition and is equivalent to testing all possible scenarios.

The difficulty of the interest-sensitive cash flows as discussed in section 2 arises from calculating each flow numerically. To circumvent this difficulty, we avoid this calculation. Instead, we leave each cash flow as functions of the most recent market credited rates, account values and lapse rates. Since the cash flows are only defined by current variables (i.e. C_k is a function of variables with subscripts K or K-1), we perform the valuation in a step by step manner, starting at the last flow and proceeding to the first flow. To illustrate the technique, we describe it using the general valuation described in section 1.

The new method performs one pass through Diagram 3 of section 1 from right to left. All variables at each point in time are calculated before proceeding to the next earlier time period. We do not calculate the numerical values of any variables. Instead, we express each algebraically in terms of other basic variables. For this reason, the new method is called the symbolic method.

The general method for performing a one-pass valuation for the deterministic model is as follows. At each time period n, the variables A_n, V_n, L_n and ${}_cr_{n-1}$ are written in terms of ${}_mr_{n-1}$. We proceed by reverse induction on n. So, from the variables at time n we proceed to the variables at time n-1, starting at the last time period in our horizon and iterating until deducing the expressions at time zero.

The rate variables ${}_mr_{n-2}$ and ${}_cr_{n-2}$ are introduced as they are needed to express new cash flow, account value, and market variables. At each time n-1, once all variables are expressed as functions of ${}_mr_{n-1}$, ${}_cr_{n-1}$, ${}_mr_{n-2}$ and ${}_cr_{n-2}$, the dependence upon ${}_mr_{n-1}$ and ${}_cr_{n-1}$ is eliminated to finish the progress from time n to n-1. This last step requires writing ${}_mr_{n-2}$ as a function of r_{n-1} and ${}_cr_{n-2}$ as a function of ${}_cr_{n-1}$. The former is derived from our belief of how interest rates move while the latter is derived from our crediting strategy.

We now describe the financial quantities which we will track throughout our sample. See Diagram 3.

A_n = Account Value at Time n
V_n = discounted value at time n of all cash flows past time n
C_n = Cash flow at time n
L_n = Decrement rate at time n **(4)**
${}_cr_n$ = Credited Rate for time period between n and $n+1$
${}_mr_n$ = Market rate for time period between n and $n+1$
$\hat{L}_n$ = Backward rate of Increment(describedbelow)

$$
\begin{aligned}
C_n &= A_{n-1}(1 + {}_c r_{n-1})L_n \\
A_n &= A_{n-1}(1 + {}_c r_{n-1}) - C_n \\
V_{n-1} &= \frac{1}{1 + {}_m r_{n-1}}(V_n + C_n).
\end{aligned}
\tag{5}
$$

In the deterministic case, we assume that the lapses are known for each time n. Thus equation (5) follows. Here, the first two equations above calculate A_n and C_n from L_n and quantities associated with time n-1. Since the symbolic method proceed backwards in time, it is more convenient to have quantities at time n-1 expressed in terms of those at time n. To facilitate such expression, we introduce $\hat{L}_n = L_n/(1-L_n)$. Thus, (5) becomes

$$
\begin{aligned}
C_n &= A_n \; \hat{L}_n \\
A_{n-1} &= \frac{1}{1 + {}_c r_{n-1}}(A_n + C_n) \\
V_{n-1} &= \frac{1}{1 + {}_m r_{n-1}}(V_n + C_n).
\end{aligned}
\tag{6}
$$

$\hat{L}_n$ is called the backward increment rate because it is the rate used to increment the account value while proceeding backwards through a valuation. It is equivalent to the decrement that would occur if the valuations were proceeding forward. Notice that C_n is now calculated from A_n not A_{n-1}, as in equation (5). A complication arises if L_n is 100%, since $\hat{L}_n$ is then undefined. The only time when 100% lapses are possible is at the end of our valuation. When all account values are assumed to cash out. Thus, to the above equations we add the conditions $V_N=0$, $A_N=0$, and $C_N=1$. This avoids any problem associated with 100% lapses.

Proceeding backwards from time N to time 0, the quantities V_n, A_n, and C_n are calculated for each n. After performing this calculation, the account value A_0 probably varies from the actual account value at time zero. So, all time zero variables must be scaled by an appropriate factor to have A_0 equal the actual initial account value. More directly, a practitioner may disregard the nominal amounts of V_0 and A_0 but only consider the ratio of the market value to the account value, V_0/A_0.

We now turn to our stochastic valuation with the symbolic method. Here, we assume the lapses are a function of the current environment, that is to say, the prevailing market rate. We assume the lapses at time n are a function of the rates at time n-1 to reflect that the decision to lapse occurs between time n-1 and n. Thus, $\hat{L}_n=\hat{L}_n({}_m r_{n-1} - {}_c r_{n-1})$. Next, we adjust equation (6) to reflect that the quantities V_n, A_n, and C_n are expected values given the rates of the previous time period. The result is equation (7).

$$
\begin{aligned}
C_n &= A_n \cdot \hat{L}_n \\
A_{n-1} &= E\left(\frac{1}{1+{}_m r_{n-1}}(A_n + C_n) \mid {}_m r_{n-1}, {}_c r_{n-1}\right) \\
V_{n-1} &= E\left(\frac{1}{1+{}_m r_{n-1}}(V_n + C_n) \mid {}_m r_{n-1}, {}_c r_{n-1}\right).
\end{aligned} \tag{7}
$$

Here, E( ) is the expectation over all time n variables conditional with respect to all time n-1 variables. Thus, A_{n-1} and V_{n-1} are functions of ${}_m r_{n-1}$ and ${}_c r_{n-1}$. As before, after the above calculations are performed, all account values, market values and cash flows are scaled to ensure A_0 is the actual initial account value.

That (7) yields the correct market value can be seen by considering the contribution to V_0 of a single cash flow C_n. The cash flow is properly discounted by the product of the one-year discount factors. Its amount arises as the product of the n^{th} year's lapse rate L_n with the previous rates of nondecrement, $1-L_j$ for $j=1,...,n-1$.

All that remains is to model the movement of market rates and credited rates to permit the calculation of expected values. In general, the market rates assumptions are the most difficult. For market rates, we specify the distribution of each interest rate ${}_m r_n$ and ensure that these rates vary from time period to time period with the desired interdependence. A method for modeling market rates is presented in the next two sections.

3.2 Distribution of Market Rates

In this section, we present a possible model of market rates. To ease our notation, for the remainder of the paper, we represent the n^{th} period market rate, ${}_m r_n$, by i_n.

Assume the distribution of interest rates to be continuous with a minimum value of 0% and a maximum value of i_{MAX}. Furthermore, assume the probability density function of i, f(i), to be piece-wise linear with $f(0)=f(i_{MAX})=0$. Let f have p+1 linear components with I_k for $k=1,...,p$ partitioning $[0,i_{MAX}]$ into the linear components of f. Let $a_k=f(I_k)$ for $k=1,...,p$. Thus, for p=19, the graph of f appears on the next page.

Since f is a density function, it must satisfy the equations in (8).

$$
\begin{aligned}
&0 < I_1 < I_2 < \cdots < I_p < i_{MAX}, \\
&a_k \geq 0 \quad \text{for all} \quad k=1,\ldots,p, \\
&\int_0^{i_{MAX}} f(i)\,di = 1.
\end{aligned} \tag{8}
$$

Piecewise Linear Distribution of Market Rates

(Sample with a Partition of 19 Points)

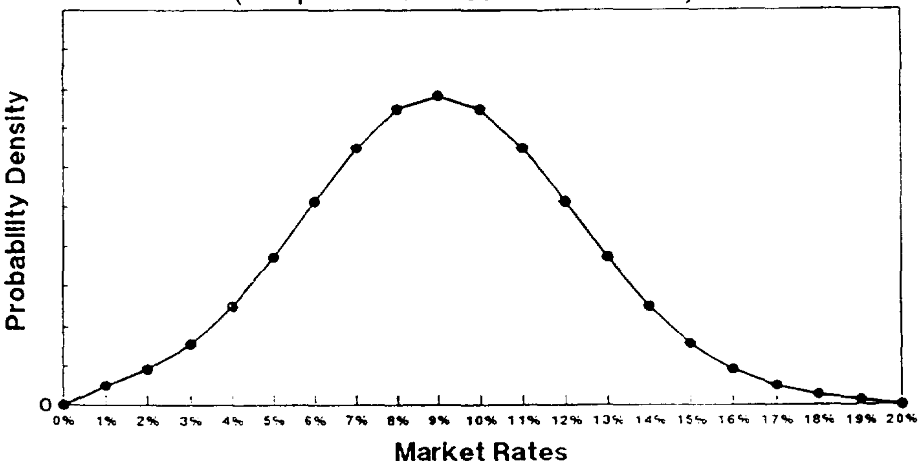


Diagram 3.1

Notice that the above integral requirement is equivalent to equation (9), since the distribution is piece-wise linear.

$$\frac{1}{2}l_1a_1 + l_2a_2 + l_3a_3 + \cdots + l_{p-1}a_{p-1} + \frac{1}{2}l_pa_p = 1. \qquad (9)$$

As illustrated above, the conditions prescribed for the distribution function give rise to polynomial expressions of the a_k's. In general, when all conditions of a distribution function may be depicted as polynomial equations, the distribution is easy to work with from a computational view point. It is interesting to note that if the distribution function is a cubic spline, not merely piece-wise linear, the above integral equations are still polynomials in the a_k's and l_k's. (See section 3.5)

While performing the valuation, we let the parameters vary to adapt the model as necessary. For example, if we wish the distribution to have mean μ and variance σ^2, we solve for the a_k's and I_k's such that
In general, it is necessary to add additional conditions such as the above to arrive at a unique distribution. In this manner, the practitioner may include more desired assumptions in to the model of interest rates. The important point is that, because of the piece-wise linear distribution, all integral requirements such as those in equation (10) are polynomials in I_k and a_k similar to equation (9).

The problem is simplified if we fix some of the parameters of f. To start, we fix the maximum possible rate, i_{MAX}. While the decision may be arbitrary, the alternative of allowing no maximum

$$\int_0^{i_{MAX}} f(i)di = 1$$
$$\int_0^{i_{MAX}} i\, f(i)di = \mu \tag{10}$$
$$\int_0^{i_{MAX}} i^2\, f(i)di = \sigma^2 + \mu^2$$

rate is equally arbitrary. Next, we either fix all of the I_k's or all of the a_k's. This greatly reduces the complexity of equations such as (10). Fixing the a_k's reduces (10) to a system of polynomial equations in the I_k's. However, fixing the I_k's in lieu of the a_k's reduces (10) to a system of *linear* equations in the a_k's. Clearly the latter is more desirable. This also has the added advantage of reducing the requirements to ensure f is a distribution function. We need only ensure that the a_k's are nonnegative.[11]

Sample distribution for section 3.4:
To illustrate, we choose $p=3$, $i_{MAX}=20\%$, $I_1=7.5\%$, $I_2=9.0\%$ and $I_3=10.5\%$. We will specify the expected value μ and variance σ^2 of interest rates and thus (10) is a nondegenerate linear system in a_1, a_2, and a_3 with a unique solution. We assume $\sigma^2=.0014924$, the historical variance of the one-year rate in the 1980's. What we assume for μ is the subject of the next section.

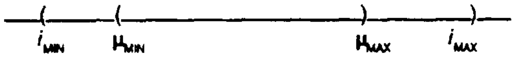
3.3 Expected Interest Rates and Mean Reversion

Continuing with the model of interest rates of the previous section, we focus on our requirements of μ. Given σ^2, there are values of μ which lead to negative solutions of (10). It follows that only a restricted set of values of μ produce valid distributions f(i). For example, in our sample with μ between 0 and 20%, a_1 is negative when μ is less than about 7.3% and a_3 is negative when μ is greater than about 11.2%. Thus, we require μ to stay between these two bounds to ensure that f(i) is always positive. This is sufficient to ensure f(i) is a distribution since a_2 is also positive on this range. To simplify some of our calculations, we restrict μ further to match the splines I_1 and I_3. Thus, we assume $7.5\% \le \mu \le 10.5\%$. This additional restriction is not necessary in general but is convenient for our calculations later.

In general, it is necessary to restrict the parameters of the rate's distribution to ensure that the derived function actually is a distribution. In our sample, we fix σ^2 and restrict μ to ensure that a_1, a_2, and a_3 are all positive. The requirement that μ be restricted to a subinterval of all the admissible values for the market rate will be a requirement for any model employing a distribution of market rates. To see why this is true, consider the effects when $\mu=0$. This requires that the *mean* of the distribution take on one of the boundary values of the market rates, and this is impossible. In our sample, the zero mean requirement leads to one of the values of a_1, a_2, or a_3 being negative. In other words, the mean of market rates must take on a value strictly between

[11] Assuming the a_k's to be fixed would require us to ensure that $0<I_1<I_2<...<I_p<i_{MAX}$ which is often harder.

the boundary values of market rates. This is true of any distribution of interest rates. The general picture for this is as follows.



Of course, any parameter of a function may have a limited range of values over which the function is a distribution. We focus on the limiting of the mean value because it directly affects our sample and because the limiting of the mean is a requirement of most models. This requirement is commonly referred to as mean reversion.

Before discussing mean reversion, let us see how this limiting of the mean's range fits into our analysis. We model future interest rates from period to period with i_t the rate at time t.[12] We require that the mean of i_t, μ_t, vary with i_{t-1}, and that low rates at time t-1 tend to be followed by low rates at time t. While it would be satisfying to let μ_t equal i_{t-1}, this would invalidate our t-period distribution function for some values of i_{t-1}, since i_{t-1} is itself a random variable with the same sample space as i_t. We must limit i_{t-1}'s impact on μ_t.

In our sample, i_{t-1} and i_t range between 0 and 20%. However, the admissible values for μ_t are between 7.5% and 10.5%. We therefore define,

$$\mu_t = .15\, i_{t-1} + .075. \tag{12}$$

Thus, all feasible values of i_{t-1} produce admissible values of μ_t.

This is an example of a more general financial modeling concept known as mean reversion. Mean reversion arises in an effort to maintain a model's integrity, in our case to ensure that our distribution assumes only nonnegative values. Usually mean reversion is used to force the expected value of interest rates to revert back to an overall mean. When rates exhibit such a propensity, they and their model are said to have a central tendency. As illustrated here, the mean reversion arises from a technical necessity of the model.

We note that the central tendency of our model is quite *harsh*, to the extent of being probably undesirable for modeling. Consider the case where the interest rate at time 2 is 1%. Our mean reversion given above would imply an expected rate at time 3 of 7.65%. It is unrealistic to expect rates to jump that much. They *could* move considerably from one period to the next but we would not *expect* this. Similarly, a rate of 18% at time 2 would produce the expected rate of 10.2% for time 3. For actual modeling, μ_t should be much closer to i_{t-1}. For example if i had a range of 0 to 20%, a desirable range for μ might be 1% to 19%, rather than 7.5% and 10.5%. In section 3.5, we note that this can be done by adding more partition points to the distribution

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[12] The reader should distinguish between I_k and i_t. I_k partitions the sample space of an arbitrary interest rate random variable for the purpose of defining the distribution function. i_t is the specific interest rate random variable at time t. In this paper, we do not vary the partition with time, so no ambiguity can arise.

function of rates. For example, the partition points could be $I_1=1\%$, $I_2=2\%$, ... $I_{19}=19\%$. This would expand the range of μ to at least 1% to 19% and allows the mild reversion $\mu_t=.9i_{t-1}+.01$.

This brings us to the question of how to judge mean reversions. A linear mean reversion, $\mu_t = a\ i_{t-1} + b$, has the property that $0<a<1$. Here, a is the *coefficient of the mean reversion*. The closer a is to 1, the milder the reversion since the expectation of the next period's rate varies only mildly from the current rate. The farther a gets from 1, the harsher the reversion. In our sample, squeezing a sample space which is .2 wide into an admissible band for the mean which is only .03 wide produced the very harsh coefficient of mean reversion of .15 (=.03/.2). The coefficient of reversion should be a concern of all practitioners. When mean reversion is motivated by technical concerns rather than philosophical ones, milder reversions are preferable.

One final note on mean reversion. The central rate towards which all rates revert is often defined to be that rate which implies an expected future rate equal to itself. That is to say, the rate i_{t-1} for which $\mu_t = i_{t-1}$. In our sample, the central rate is about 8.82%. Some practitioners believe the central rate should equal the historical average. Others believe it should equal the initial time zero rate, i_0. The importance or lack thereof of this rate is uncertain.[13]

We continue our valuation sample without reducing the harshness of our sample distribution's mean reversion, since the strong central tendency of this example's mean reversion does not detract from this illustration and making the reversion less harsh by increasing the partition of the distribution function would only cloud the presentation. However, in practice, a reversion much milder than our sample should be employed.

3.4 Sample Valuation

In this section, a sample valuation illustrates the concepts of the symbolic approach.

The previous two sections present our sample market rate assumptions. Two more assumptions are necessary, a crediting strategy and a lapse or cash flow function. A fixed credited rate of 8% is assumed for ease of presentation.

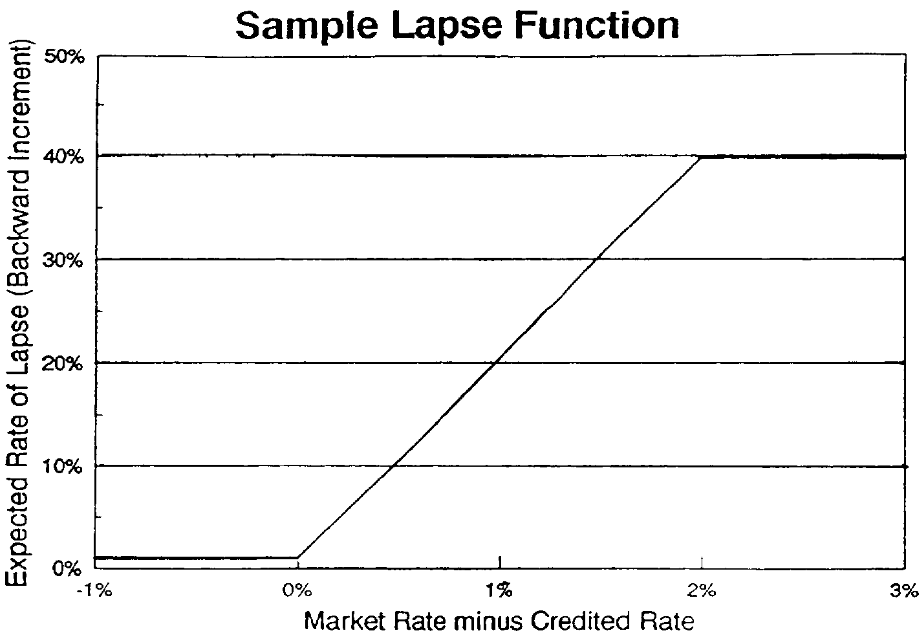
For a lapse function, let the rate of backward increment, $\hat{L}_n$, have a minimum of 1% and a maximum of 40% and be linear between these two values for differences in credited rate and market rate between 0% and 2%. For the reader' convenience, a graph of the sample lapse function is provided on the next page.

The functional form of the lapse function is given in equation (13).

$$\hat{L}_n(i, {}_cr) = \begin{cases} .01 & i \le {}_cr \\ 19.5(i - {}_cr) + .01 & {}_cr \le i \le {}_cr + .02 \\ .4 & i > {}_cr + .02 . \end{cases} \qquad (13)$$

The piece-wise linearity of the above lapse function is not necessary for the method, merely easy

[13] In the spirit of arbitrage-free pricing, the author believes there is no intrinsic answer to the question of the meaning of the reversion. The current feelings in the market place should be monitored, and pricing model should receive such feelings as inputs.



for the sample. In practice, cubic spline functions would work well.[14] Recall that $\hat{L}_n = L_n/(1-L_n)$ where $\hat{L}_n$ is the rate of backward decrement. The probability density function for market rates for the sample is given in section 3.2. Diagram 3.1 displays the density function. For the sample, I_1=7.5%, I_2=9%, I_3=10.5%, I_{MAX}=20%, and the density function is given in equations (14) and (15).

[14] The practitioner could use splines of higher degree than three. Cubic splines are recommended since they are best for estimating functions and commonly employed in practice.

$$P_\mu(i) = \begin{cases} \frac{40}{3} a\, i & 0 \le i \le .075 \\ 6a - 5b + \frac{200}{3}(b-a)\, i & .075 \le i \le .09 \\ 7b - 6c + \frac{200}{3}(c-b)\, i & .09 \le i \le .105 \\ \frac{40c - 200ci}{19} & .105 \le i \le .2, \end{cases} \tag{14}$$

$$\begin{aligned} a &= \frac{145059 - 2350000\,\mu + 10^7\mu^2}{1575} \\ b &= \frac{-3907714 + 871000000\,\mu - 46\cdot10^7\mu^2}{13125} \\ c &= \frac{165177 - 4050000\,\mu + 3\cdot10^7\mu^2}{6875}, \end{aligned} \tag{15}$$

Here, μ is the desired mean, which must be between 7.5% and 10.5% as discussed earlier. We assume that the variance of this distribution is $\sigma^2=.0014934$, the historical average of the one-year rate for the 1980's. Notice that $P(0)=P(.2)=0$, $p(.075)=a$, $P(.09)=b$, $P(.105)=c$ and P is linear between these points. For the reader's convenience, The following table is provided to illustrate the density function for various desired means.

μ	a	b	c
7.5%	15.9	2.8	4.4
8.0%	13.4	8.9	4.8
8.5%	11.2	13.2	5.5
9.0%	9.2	15.6	6.4
9.5%	7.7	16.4	7.5
10.0%	6.4	15.4	8.8
10.5	5.4	12.7	10.3

Table of Sample Distributions for Various Assumed Means

Recall from section 3.3 that the sample assumes market rates from one period to the next follow the mean reversion $\mu_t = .15\, i_{t-1} + .075$.

The time under consideration for the sample is 3 years, N=3. The valuation proceeds from the last time period to the first, and the diagram in section 1 is modified to produce the representation of our calculations in Diagram 4.

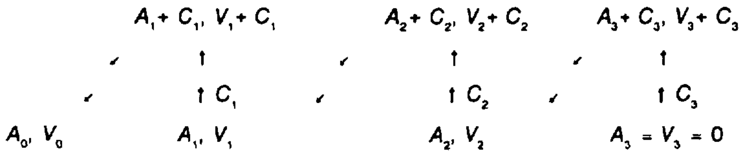


Diagram 4

In Diagram 4, the diagonal arrows represent discounting by the market or credited rates and the vertical arrows represent incrementing by the current cash flow.

The valuation follows from equation (7) and the assumption that ${}_c r$=8%. We assume C_3=1.

At Time 2:

$E(A_3 \mid i_2) = E(V_3 \mid i_2) = 0$. Therefore,

$$A_2 = \frac{1}{1+{}_c r} \qquad V_2 = \frac{1}{1+i_2} \tag{17}$$

At Time 1:

From equation (7),

$$\begin{aligned} A_1 &= \frac{1}{(1+{}_c r)^2}\left(1 + \hat{L}_2\,(i_1 - {}_c r)\right) \\ &= .857339\left(1 + \hat{L}_2\,(i_1 - {}_c r)\right) \\ V_1 &= \frac{1}{1+i_1}\left[E(V_2 \mid i_2) + \hat{L}_1\,(i_1 - {}_c r)\cdot E(A_2 \mid i_1)\right] \\ &= \frac{1}{1+i_1}\left[E(V_2 \mid i_1) + \hat{L}_1\,(i_1 - {}_c r)\cdot \frac{1}{1+{}_c r}\right] \end{aligned} \tag{18}$$

Now,

$$E(V_2 \mid i_1) = \int_0^{.2} \frac{1}{1+i_2}\, P_{\mu_2}(i_2)\, di_2. \tag{19}$$

Substituting equation (14) and equation (15) and the mean reversion into equation (19) yields,

$$E(V_2 \mid i_1) = .931405 - .130021\ i_1 + .0172431\ i_1^2. \tag{20}$$

Combining equations (18) and (20) yields,

$$\begin{aligned} A_1 &= .857339\ (1 + \hat{L}_2(i_1 - {}_c r)) \\ V_1 &= \frac{1}{1 + i_1} (.925926\ \hat{L}_2(i_1 - {}_c r) - .164507\ (1 + i_1) + 1.078670 \\ &\quad + .017243\ (1 + i_1)^2) \end{aligned} \tag{21}$$

The expressions are made in terms of $1+i_1$ rather than in terms of i_1, in order to facilitate calculations later. That is to say, since there exists a $(1+i_1)^{-1}$ factor, expressing functions as powers of $1+i_1$ reduces the complexity of later integrals. Specifically, terms like $(1+i_1)^n$ are easy to integrate while terms like $(1+i_1)^n i_1^2$ are troublesome.[15]

At Time 0:

From equation (7),

$$\begin{aligned} A_0 &= \frac{1}{1 + {}_c r} E(A_1 \mid i_0)(1 + \hat{L}_1(i_0 - {}_c r)) \\ V_0 &= \frac{1}{1 + i_0} [E(V_1 \mid i_0) + \hat{L}_1(i_0 - {}_c r) E(A_1 \mid i_0)] \end{aligned} \tag{22}$$

Now, using equation (18), it follows that,

$$E(A_1 \mid i_0) = \int_0^{.2} \frac{1}{(1 + {}_c r)^2} (1 + \hat{L}_2(i_1 - {}_c r))\ P_{\mu,l}(i_1)\ di_1. \tag{23}$$

As before, substituting the mean reversion into the above, integrating and then substituting the result into equation (22) produces,

$$A_0 = (1 + \hat{L}_1(i_0 - {}_c r))(.864546 + .260366\ i_0 + 7.972261\ i_0^2 - 22.548135\ i_0^3). \tag{24}$$

Similarly, it can be shown that,

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[15] Note to the reader who may use the software Mathematica from Wolfram Inc: Mathematica 1.0 does not integrate the latter term directly. It is necessary to specifically rewrite the functions in terms of $(1+i_1)$.

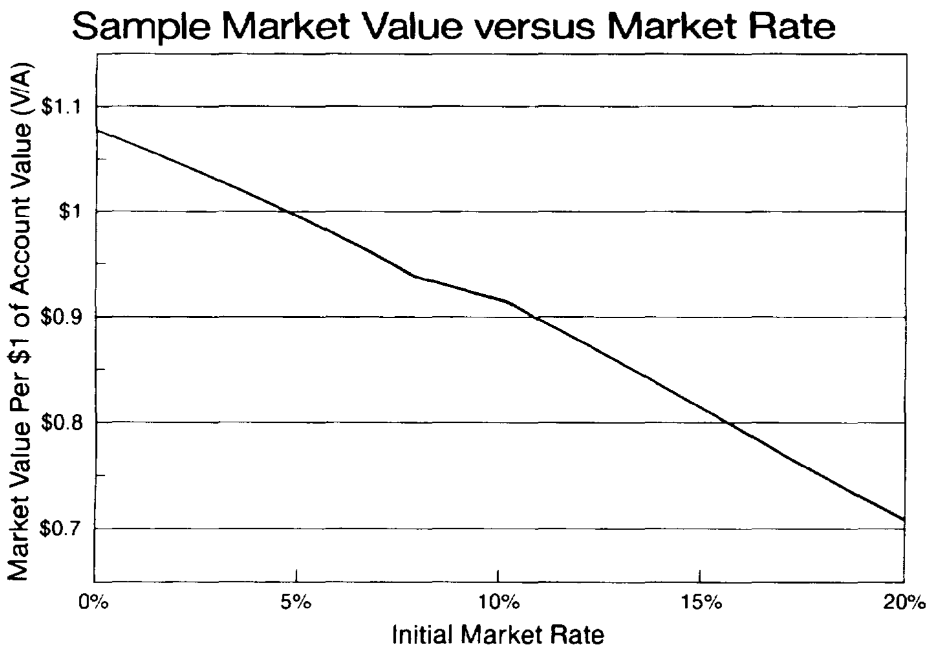
$$E(V_1 \mid i_0) = .931599 + .133443\, i_0 + 8.483157\, i_0^2 - 24.126775\, i_0^3, \tag{25}$$

and thus,

$$V_0 = \frac{1}{1+i_0}\Big(.931599 + .133443\, i_0 + 8.483157\, i_0^2 - 24.126775\, i_0^3 + \hat{L}_1(i_0 - {}_c r)(.933710 + .281195\, i_0 + 8.610042\, i_2^2 - 24.351985\, i_0^3)\Big) \tag{26}$$

The values of V_0 and A_0 in equations (24) and (26) may be scaled up and down by the same factor to ensure A_0 equals the actual account value.

On the next page, the graph of V_0/A_0 illustrates how the market value at time zero varies with the current market rate, i_0. Notice that having an explicit form for V_0/A_0 facilitates graphing the effects of interest rate shifts or calculating the duration and convexity. (In the sample, the duration for initial rates between 8 and 10% is in the range 1.02 to 1.11. Outside of this range, the duration varies more but is generally above 2.1 for initial rates close to 8 and 10%.)



In this sample graph, the curve has a noticeably different slope when the initial interest rate is in the range 7.5% to 10.5% than when it is outside of this range. This is due to our interest rate model which treats this range differently from the remaining space of possible interest rates. Using more partition points in our rate distribution would produce a smoother graph and this would be desirable in practice. However, this graph dramatically illustrates the effects of the interest rate assumption on a valuation and emphasizes the need to be able to incorporate realistic interest rate assumptions.

3.5 Points of Interest

This section highlights some advantages and concerns to be considered if the symbolic method is to be developed and used in practice. Some of these ideas indicate methods to improve the assumptions of the valuations. Others expand the possible analysis. All of these points are areas for future research and are only presented summarily here.

Improved distribution of market rates

The modeling of interest rates with a piece-wise linear density function with a partition of three points, as done in our sample valuation, is not suitable for practice. Great improvements occur by simply increasing the number of partition points, and reducing the space between partition points. A piece-wise linear distribution may still be employed, or a practitioner may move to a cubic spline function. Either way, increasing the number of partition points will improve the interest-rate assumptions and permit milder mean reversions. In addition, rates could be made more or less volatile in different rate environments by specifying that the variance of rates at time t is a function of the rate at time t-1. No matter what assumptions are used, the method requires integrable functions at each step and thus the distribution function should be fairly simple; a practitioner should stay with piece-wise polynomial functions.

Duration and Convexity

As shown at the end of the last section, the model produces the ratio of the market value to account value in functional form, from which the duration and convexity follow as direct calculations. It is also possible to calculate statistics needing high order derivatives. One may also calculate derivatives with respect to interest rates other than those at time zero. (See Reitano [5] and [6].)

Term structures and Time Series

The model presented above assumes only one market rate. In reality, there are many different rates at any given time and the collection of rates of varying maturities (for a particular market sector) is called the term structure. Some instruments demand that a term structure be employed. For example, any instrument whose cash flow amounts vary with more than one maturity rate would require a rate model which provides a rate for the corresponding maturities. Also, modeling a term structure is a necessary prerequisite for ensuring that a model is arbitrage-free. It is easy to add more rate parameters at each time in the model. In this case, the iterative step in the valuation would consist of integrating with respect to each rate parameter.

One may also employ a time series model of interest rates. This expresses each rate as a function of previous rates, usually more than just the immediately preceding rate. The symbolic model is flexible enough to permit such modeling. To do so, at each time n introduce any rates for earlier periods which are necessary to complete the time series model. The integration would take place over the "error" term in the time series model.[16]

Regardless of which interest rate model used, it is easiest to perform the integration in the model if each function is a product of powers of (1+rate) rather than linear combinations of rates. For example, if the one-period rate at time t, r_t, was a linear combination of two previous rates, $Ar_{t-1}+Br_{t-1}$, then discounting with r_t produces terms like $(1 + Ar_{t-1} + Br_{t-2})^{-1}$ which must be integrated over r_{t-1} and r_{t-2}. While such integration is possible, it becomes complicated. It is easier if the model can be expressed multiplicatively like

$$(1 + r_t) = (1 + r_{t-1})^A \ (1 + r_{t-1})^B \tag{27}$$

Terms such as the above can be integrated easily because the variable of integration are separated into different factors. Notice that in this case, discounting by r_t leads to terms which are still of the same form.

Arbitrage-Free Criterion
Once a model for the term structure of rates is selected, a practitioner can attack the problem of implementing the arbitrage-free criterion. In general, one must ensure that the rates at each time in the model are consistent with the expected price of the corresponding instrument. For example, the 4-year rate at time 3 must price a fixed 4-year obligation at time 3 exactly as the model does by back-valuing from time 7 to time 3. The exact method for ensuring this consistency is highly dependent on the rate model employed and is an area for future research.

Yields
A yield can be calculated by discounting the market values, v_n, with an arbitrary parameter, y, rather than the market rate. The final market-to-account value ratio so produced is a function of y, a polynomial in $(1+y)^{-1}$ to be precise. The roots of this function are a type of yield for the underlying instrument. The practitioner should be careful, however, since this is not the average yield over all scenarios; rather, it is the yield for one scenario in which the expected cash flow occurs at each time.[17]

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[16] Particular care must be taken with the error term and its relationships to the finite interest rate boundaries and mean reversions. For example, with a finite boundary, a normal error term could not be assumed. Note that the error term should have a piece-wise polynomial distribution for easy integration.

[17] The ranking of instruments by yields is done in practice but enjoys little theoretical foundation. To this author's knowledge, which type of "average yield" is best to use is unknown.

Crediting strategy
The example in the last section assumed a fixed credited rate, but this is not a requirement of the model. The symbolic model can incorporate crediting strategies in algebraic form. For example, if rates were always a constant spread below market and reset each period, the decrement function could be altered to account for this. Indeed, any polynomial of market rates could be employed as a reset strategy. Alternately, rates could vary less often than the valuation time period. This would require indexing the market rate at the reset period for all subsequent periods until the next reset. For example, if rates were reset every 4^{th} period, then the 4^{th} period market rate would be used to calculate the credited rate in the 4^{th}, 5^{th}, 6^{th}, and 7^{th} periods. Since the model proceeds symbolically backwards, there is no difficulty with introducing the 4^{th} period rate at time 7 and carrying it over to the 4th period. Similarly, averages of previous market rates (perhaps to simulate portfolio rates) or of the current market rate and previous credited rate can be used as a crediting strategy. As mentioned above, calculations proceed more expediently when relationships are expressed as functions of (1+rate) rather than of just the rate.

Problems can arise with crediting strategies not expressible in algebraic form. For example, the "sticky down" strategy which always resets rates at the lower of the current market or previous credited rates can cause difficulty. This strategy calls for a comparison of two rates which are kept in the model symbolically, without numeric values. Presently, the model only handles crediting strategies which do not require such comparisons.

Sensitivity analysis
The symbolic approach facilitates sensitivity analysis. For instance, consider the SPDA example of the previous section and suppose that the practitioner wished to know the sensitivity of the valuation to the lapse assumption. The valuation could be run with a free sensitivity parameter, S, as a factor of the lapse function. That is to say, S $\mathfrak{L}_n$ is used rather than $\mathfrak{L}_n$. The sensitivity parameter is carried throughout the valuation to yield the ratio of market-to-account value as a function of S. Evaluating this function at S=1 provides the valuation with normal lapses. Evaluating this function at S=2 provides the valuation if the lapses were more or less double.[18] Thus, the practitioner could observe the sensitivity. Similarly, the sensitivity of the instrument's duration to the lapse rate could be tested. One could also perform analysis to measure the sensitivity to the market volatility assumption. In practice, only one sensitivity analysis can be performed at a time, since the time and memory required for the calculation rises quickly with the number of free variables.

Monthly, Quarterly, and Other Time Periods
The symbolic method may employ any time period, not just annual ones. Suppose quarterly periods were employed. The lapse function would change and the discount factor would have an exponent of -1/4 but neither of these complicates the process significantly. Similarly, any time frame can be chosen.

Modeling other instruments
The example of the previous section models an SPDA, but it is possible to model other instruments. For example, a pass-through mortgage-backed security could be modeled similarly by replacing the lapse function with a prepayment function. Similarly, callable bonds can be modeled by using a discontinuous lapse (Call) function of 0% up to a certain call threshold and then 100%

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[18] The "more or less" arises from $\mathfrak{L}$ not being the lapse function L but rather the function of backward increment L/(1-L).

thereafter. Options could be valued similarly by introducing an exercise function.

The symbolic method needs further research before it can be employed to value certain instruments. Specifically, instruments whose cash flows depend heavily on the level of the underlying base value (par, account value, etc.) may not be valued as easily as the SPDA because the "account value" and market value may not be scaled up and down arbitrarily. The most important examples would be collateralized mortgage obligations and banded rate annuities. The cash flows for these instruments are amount dependent in the sense that they vary with the amount of the underlying base value: the CMO tranche's cash flow varies with the outstanding principle of the entire pool and the banded annuity's rate varies with the outstanding account value.

4 Conclusion

This paper presents the symbolic method, a general technique for valuing interest-sensitive instruments as the expected present value of all future cash flows. The method requires that the distribution of future one-period rates be specified with integrable functions and that the cash flows be expressed as function of these interest rates. The paper gives an example of piece-wise linear distributions to value an SPDA product with a simple lapse function. The method could easily be expanded to incorporate close approximations (say cubic spline functions) of other desired lapse and interest-rate distribution functions. In this way, the method may incorporate almost any interest rate assumption.

The main advantage of this approach is that it does not require sampling of specific paths. In essence, all possible future paths are sampled simultaneously. This approach also simplifies duration calculations and sensitivity analysis.

The symbolic method still requires future research to be refined. The main area for future research are: 1) the handling of amount-dependent cash flows (see Modeling Other Instruments in the previous section), 2) determination of a desired model of interest rates which can be sufficiently approximated with cubic spline functions, and 3) ensuring the arbitrage-free criteria for this model of interest rates.

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