DIFFERENTIAL EQUATION MODEL

FOR YIELD CURVES

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Abstract: This paper examines a differential equation model,
whose solutions have yield curve shapes. The paper also examines
the relevance of the model with historical monthly U. S. Treasury
nominal rates.

In this paper m will represent the maturity of a yield curve
and i(m) will denote a yield curve, of interest rates i.

In the study of yield curve behavior, yield curve models
should have at least the the following list of properities:

(1)   Short term interest rates should be more volatile than
long term rates. See [11]. That is, the short term rates
have little influence on the later maturity yield values.

(2)   Yield curves should converge to finite (and
reasonable) values at infinity and zero. See [17].
Reasonable here will mean interest rates that range from
0.5% to 30%.

(3) Yield curves should be continuous over all positive maturities.

(4) Yield curves should be bounded over all positive maturities, and again be reasonable as defined in (2).

(5) A specific interest rate at a specific maturity should be influenced by adjoining rates. This influence increases as the maturity m increases. This property is modeled as a feedback component.

(6) A model of yield curves should replicate the model yield curves used by Jetton and Pedersen et al. See [8] and [12].

(7) A model of yield curves should correspond with historical yield curves.

(8) A model of yield curves over time should have high autocorrelation at lag one between the interest rates at infinity (and zero). Otherwise stated, this period's interest rate should be closely related to last period's rate and to next period's rate.

(9) Ninety day interest rates should be highly correlated with the zero maturity interest rate.

Properties (2), (3), (4) will be restated as:

(4') Yield curves should be continuous and bounded over all nonnegative maturities (including infinity), and the bound should be reasonable.

Several models of yield curves have been studied. An incomplete list of them follows:

I   Polynomial fit of yield curves

The benefit of this type of model is that you will have a model that is reproducing, but it will not have a finite value at infinity. See [17].

II   Siegel's truncated model of yield curves

Here the yield curve is approximated by a form $b + c/m$, Where b is the interest rate at infinity and c is a constant. See [17]. This model does satisfy property (4') except at zero, and it satisfies property (1). This model even replicates the yield curves in [8]; however, this model does not replicate the special humped yield curve mentioned in Figure 16 of [12].

III   Power series in (1/m), and Siegel general model

The Siegel general model approximates yield curves by

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b + c/m + f(m), where f(m) tends to zero faster than 1/m. See
[17]. It appears that power series in (1/m) form a subset of
Siegel general model. If f(m) is not continuous or bounded at a
positive maturity other than infinity, the Siegel general model
will not satisfy (4'). Also f(m) must be carefully chosen to
replicate the humped yield curve of [12].

IV Rational approximation of yield curves

Here a yield curve is represented as a quotient of two
polynomials in m, say $P(m)/Q(m)$. Prior research using an N point
Pade approximate method called the Thiele Reciprocal Difference
Method (see [3]) generated a collection of Pade approximates of
the monthly U.S. Treasury nominal yield curves. These
approximations were reproducing, however one was not able to
guarantee that the result was continuous and bounded over the
nonnegative reals. This was especially a problem on yield curves
that were almost flat. Possibly there are other algorithms other
than the Thiele algorithm that will allow control over the
denominator $Q(m)$.

Also, the rational approximation converged to a finite value
at infinity only when $P(m)$ and $Q(m)$ had the same degree. This
occurs only when there are exactly an even number of known
points. See [3]. When there are an odd number of points, one
can guarantee a finite value at infinity by an introducing an
additional element obtained by interpolation.

V Differential Equation model of yield curves

The remaining part of the paper will examine the family of
coupled linear differential equations of the following form

$$\frac{di}{du} + \epsilon i = \epsilon \left(\frac{bm^2 + cm + d}{m^2 + fm + g} \right) \quad (0.1a)$$

$$\frac{dm}{du} = a + h(\ln(m + q)) \quad (0.1b)$$

where m represents the maturity, $i(m)$ represents a yield
curve and u is a parameter. This model is not necessarily the
simplest model, and may very well be a subset of the Siegel
general form. It is not even reproducing as in the polynomial or
rational approximation methods, but it does attempt to address
the yield curve properties (1) through (9). Also note the right
hand side of (0.1a) can written as

$$\epsilon' \left(\frac{(m-a_1)(m-a_2)}{(m-a_3)(m-a_4)} \right)$$

so the six parameters can be reduced to five. However, we will

continue to use the six parameter model, because of the
interpretation of the b coefficent that follows.

Differential equations (0.1a) and (0.1b) were obtained by
using various formulas and numerically solving the equations.
These solutions were then graphed and examined. This approach is
called geometric modeling. See [2] for further discussion and
references on geometric modeling.

One can reduce (0.1a) and (0.1b) to the following form

$$\frac{di}{dm} + \epsilon i \; r(m) = \epsilon(\frac{bm^2+cm+d}{m^2+fm+g})r(m) \qquad (0.1)$$

where $r(m) = 1/(a + h(ln(m+q)))$.

Since (0.2) is a linear differential equation of the form

$$\frac{di}{dm} + P(m)i = Q(m) \qquad (0.2)$$

the general solutions are of the form

$$i(m) = \frac{\int Q(m) \; exp(\int P(m)dm) \; dm}{exp(\int P(m)dm)} + C \qquad (0.3)$$

See [13]. When attempting to solve (0.1), the integration of the
numerator in (0.3) is prohibitive. To simplify a Pade [1/1]
rational approximation to the function r(m) is used. This
resulted in the approximation $r(m) = (jm+k)/(m+p)$, where j, k, p
are constants. See [4] for details to make this rational
approximation. With these restictions and approximations (0.1)
becomes:

$$\frac{di}{dm} + \epsilon i \frac{(jm+k)}{(m+p)} = \epsilon(\frac{bm^2+cm+d}{m^2+fm+g}) \frac{(jm+k)}{(m+p)} \qquad (0.4)$$

In Appendix I, an analytic solution of a restriction of
(0.4) will be derived with the use of (0.3). This derivation
shows that the b coefficent above in (0.4) corresponds to the
interest rate at infinity.

Now if

$$\frac{bm^2+cm+d}{m^2+fm+g}$$ (0.6)

is expressed as a power series in (1/m) by an argument similar to
Appendix I, i(m) will be a power series in (1/m). But by
expressing the model as a rational expression of two second
degree polynomials, one can easily control nonsingularity in the
differential equation. By controlling noningularity one can
assure the continuity of the solutions over the nonnegative
reals. This is because if a derivative of a function exists at a
point the function is continous at that point. However this does
not allow one to assume that the values are bounded or reasonable
as stated in property (4) above.

To control nonsingularity, chose a value p that is positive
and r(m) will be well defined on all nonnegative m. To assure
(0.6) will be nonsingular observe the following argument:

First assume $f > 0$, and $0 < g < f^2/4$. So

$$0 < \quad 4g \quad < f^2$$

$$0 \quad < \quad f^2 - 4g < \quad f^2$$

$$0 \quad < \sqrt{f^2 - 4g} < f$$

$$-f + \sqrt{f^2 - 4g} < 0$$

$$\frac{-f + \sqrt{f^2 - 4g}}{2} < 0$$

This shows the right most root of $m^2 + fm + g$ is less than zero.

Next assume that $f \leq 0$, and $0 \leq f^2/4 < g$. So

$$0 \quad \leq \quad f^2 \quad < 4g$$

$$f^2 - 4g < 0$$

$$\sqrt{f^2 - 4g} \quad \text{is not real.}$$

This shows that $m^2 + fm + g$ has no real roots. So by placing
either of these restrictions on f and g, and assuming p is
strictly positive, we can guarantee nonsingularity in (0.4).

To guarantee that the solutions to (0.4) are bounded, one
must use the union of two arguments. By expressing (0.6) as a
power series in (1/m), one can show that the solution is finite
and bounded by an identical argument in Appendix I. However, to
show the solutions are bounded at and near zero will require one

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to rewrite (0.4) as

$$\frac{di}{dm} = - \epsilon i \frac{(jm+k)}{(m+p)} + \epsilon (\frac{bm^2+cm+d}{m^2+fm+g}) \frac{(jm+k)}{(m+p)} \qquad (0.7)$$

Note if the choice of p, f, and g have been chosen as above, to control nonsingularity, this will guarantee that the partial derivative of the right hand side of (0.7) with respect to i is continuous in a neighborhood of zero. This allows one to conclude by the existence theorem of order one differential equations, that the solution is bounded. See [14].

Also, one is be able to obtain the humped yield curve, by forcing the derivative of i with respect to m to be zero at a certain point, possibly allowing one to generate a humped yield curve. For instance, if the interest rate at maturity m is a local maximum, you can force the value of d to equal

$$i(m)(m^2+fm+g) - bm^2 - cm \qquad (0.8)$$

and this will assure i'(m) is zero.

The concept of feedback is modeled as a system that takes a current value of the solution and multiplies it by a constant and places that value back into the system. This is usually written as a differential equation of the form:

$$\frac{dy}{dx} + cy = f(x,y)$$

As one can see (0.4) has this form. So property (5) is satisfied.

If one multiplies (0.4) through by (m + p), the following is obtained

$$(m+p) \frac{di}{dm} + \epsilon i (jm+k) = \epsilon (\frac{bm^2+cm+d}{m^2+fm+g}) (jm+k) \qquad (0.9)$$

Note, that for maturities near -p, the effect of the derivative diminishes, and this allows for a greater volatility of a curve's values near that point. So if p is a positive number near zero, property (1) is satisfied.

The coefficent ϵ, has two major influences on the solutions of (0.4). First, observe what occurs when $\epsilon = 0$. This produces the simple differential equation

$$\frac{di}{dm} = 0 \qquad\qquad (0.10)$$

which has constant solutions $i(m) = c$. This corresponds to the flat yield curve. Secondly, ϵ is used to guarantee that solutions of (0.4) are constrained. Here ϵ will be used in the example that follows to guarantee the reasonableness of the solutions.

The other coefficents c, d, f, and g do not have an interpretation, within the model.

The differential equation method has no means to guarantee that property (8) or (9) holds. All that one can hope is that these properties will hold after modeling. This will be addressed in the historical study in section II.

The following section will discuss the method of solution that is taken to examine the historical yield curves.

I Boundary Value Problem and Multistage Monte Carlo Algorithm

Solving for exact solutions to (0.4) is called boundary value problems. However boundary value problems for linear differential equations, require matching the solution to at most two points on the curve. For the historical model one desires an exact fit with every point on a given yield curve, i.e. reproducing. Since the analytical solution is a complex infinite series in (1/m), it is prohibitive to attempt this approach. So, the approach was to check for historical validity by minimizing

$$\sum_{m \in I} | i_m - \hat{i}_m | \qquad\qquad (1.1)$$

where the index set I is the set of all known maturities for the historical yield curve i_m. Here $\hat{i}_m$ would be the corresponding values generated by solutions to (0.4). The sum of absolute differences, instead of least squares was used, to reduce the influence of outliers. See the discussion in [10] about this issue.

The equation (0.4) is arbitarily restricted to the following model for the historical study:

$$\frac{di}{dm} + \epsilon i \frac{(2m+2)}{(3m+2)} = \epsilon \left(\frac{bm^2+cm+d}{m^2+fm+g} \right) \frac{(2m+2)}{(3m+2)} \qquad\qquad (1.2)$$

This assures that (1.2) is nonsingular at zero. This also allowed the problem to determine only six coefficents instead of

nine. So this assumes that there is an universal r(m) for a
specific family of yield curves.

The approach taken to provide coefficents to (0.4) was to
use the Multistage Monte Carlo Optimization algorithm (see [6]
and [7]) to select the coefficents, making sure that the choice
of f and g controls nonsingularity. Also, when the interest rate
at one year exceeds the surrounding two rates, the coefficent d
is constrained by equation (0.8) to model the historical humped
yield curves.

Once the coefficents are determined, the four point Runge-
Kutta algorithm is used to generate the numerical solution to the
specific differential equation. See [5]. The starting point for
the numerical solution is at the highest maturity known for the
historical curve which was 20 for the curves from January 1, 1970
to Febuary 1, 1977, and 30 for all subsequent curves. The step
size is set at 0.25, so the solutions pass through the maturities
of the indexed set I.

For each yield curve, there is between 15000 to 75000
random samples taken to generate the optimal solution.

See the Appendix II for a copy of the Quick Basic program
that was used for the study.

II Historical Results

The historical yield curves are the nominal monthly Treasury
curves from January 1970 through December 1990. See [18] and
[19]. There were 252 yield curves modeled. Table 1 indicates
which rates are available. This table also describes the indexed
set I.

Date	90 day	1 yr	2 yr	3 yr	5 yr	7 yr	10 yr	20 yr	30 yr
1/1/70 - 2/1/77	x	x	NA	x	x	NA	x	x	NA
3/1/77 - 3/1/84	x	x	NA	x	x	NA	x	x	x
4/1/84 - 12/1/86	x	x	x	x	x	x	x	x	x
1/1/87 - current	x	x	x	x	x	x	x	NA	x

TABLE 1

The error term (1.1) for the yield curves has a mean of
0.00211, and a standard deviation of 0.00113.   Skewness is
0.406, and kurtosis is 2.849.   The lowest value for (1.1) is
0.0002, and the highest is 0.00601.   A frequency distribution of
the error is in Figure 1.   Assuming that the error term fits a
gamma distribution, using the mean and variance above one obtains
an estimate for α and β.   These are 3.441 and 1634.4
respectively.   A gamma distibution with these parameter values
for α and β has 5% of its area above the value of .00425.   This
model has 3.57% above this value.   So assuming a gamma
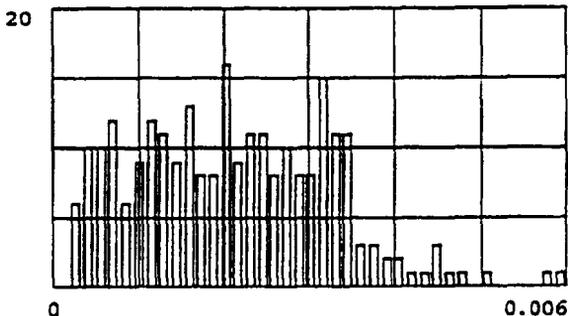distribution as the hypothesis, the hypothesis is accepted at 5%.

Figure 1

Treating the sum

$$\sum_{m\in I} i_m - i_m \qquad\qquad (4.1)$$

as another error term, it has mean -0.00016, standard deviation
of 0.00139, skewness of -0.264 and kurtosis of 3.648.   The
minimum value of (4.1) is -0.04727, and the maximum is .003749.
A frequency distribution of the error is in Figure 2.   This error
term appears to fit a normal distribution with the above sample
statistics.   A hypothesis that this error is normal is rejected
at the 5% level because there are 15 out of 252 terms outside the
second deviation from the mean.   So 6% of the sample are
outliers.   Now the hypothesis can be accepted at the 1% level
because 0.8% of the sample are outliers from the third deviation.

Let z be the interest rate at maturity zero, and q be the
interest rate at 90 days.   The statistics of the coefficents are
in table 2.

Note the high autocorrelation for the z.   This demonstrates
property (8) for the zero maturity.   However for the interest
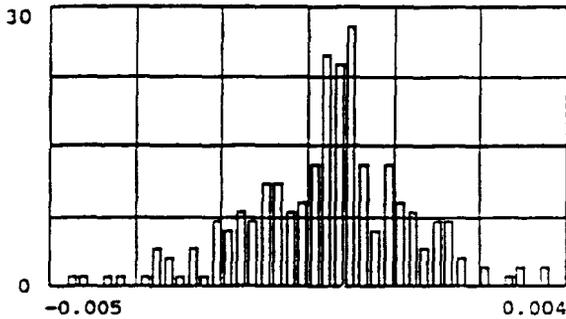
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Figure 2

	mean	standard deviation	skewness	kurtosis	min	max	lag one auto-correlation coefficent
b	0.0951	0.0289	1.462	8.753	0.030	0.262	0.3692
c	0.1203	0.4540	-2.495	18.071	-3.33	1.321	0.1875
d	-0.0236	1.0283	-0.673	8.39	-4.94	3.470	-0.0661
f	2.0060	2.998	0.108	1.684	-3.51	7.824	0.5515
g	2.5250	1.788	0.608	2.366	0.031	7.182	0.3488
ϵ	-0.6575	0.8265	-1.231	3.531	-3.20	0.184	0.0983
q	0.0758	0.0266	1.105	4.104	0.032	0.163	0.9658
z	0.0693	0.0260	1.181	4.694	0.020	0.162	0.8696

TABLE 2

rate b at infinity, the autocorrelation coefficent is disappointing. This may be due the fact that the approximation to r(m) is not optimal. Also, it may be improved by increased computer runs to reduce the randomness. This may also be solved by modifying the differential equation (0.4) to clamp the values at infinity.

The correlation between q and z above is found to be 0.882. So, property (9) is satisfied.

The joint distribution of these coefficents through time will be the content of a future paper.

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III Further Research and Closing Remarks

As one can see, the use of differential equations can aid one in the process of fitting data. Even though the process is somewhat prohibitive in the fact that random Monte Carlo techniques must be used to solve for various coefficents in the differential equation, there is an increased control over the process, where one can restrain the solutions to take on behavior that best satisfies the data. This paper showed that properties (1) through (9) were satisfied with the model (except for property (8) at infinity). Hopefully some of the concepts of this research may find uses in other topics such as graduation and other statistical fitting problems.

Other topics for future study are as follows:

(1) Find the optimal choice of the linear rational term $r(m) = (jm+k)/(m+p)$. Possibly this will clear up the disappointing result concerning property (8) at infinity, and improve other results.

(2) Determine the marginal and joint distributions of the coefficents b, c, d, f, g, ϵ, and z through time. If this is completed, one may be able to use (0.4) in scenario generation as in [9]. The use of these distributions, may also lead to futher analysis of (0.4) as a random differential equation. There is the possibility that this model may have influence on multidimensional immunization as referred in [15] and [16].

(3) Improve the historical accuracy of this model, by examining the behavior of daily yield information.

(4) Determine a method to test for a coefficent's relevance in a differential equation by some statistical measure.

(5) Examine the model (0.4) for bifurcation as in [2] and for chaotic behavior. The author is especially interested in chaotic behavior of (0.4) possibly being exhibited historically in the early 1980's. This study may result in chaotic indicators to help prepare the industry for extreme swings in the value of short term rates.

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Following is the proof that the differential equation

$$\frac{di}{dm} + \epsilon i(jm+k)\frac{}{m+p} = \epsilon(b)\frac{(jm+k)}{m+p}. \tag{A.1}$$

has a solution of the form $b + O(1/m)$.

The proof of (A.1) will allow us to demonstrate, that as m approaches infinity, the solutions will approach a constant function, as referred to in [17].

Proof. Now using equation (0.3),

$$P(m) = \frac{\epsilon(jm+k)}{m+p} \tag{A.2}$$

and

$$\exp\left(\int P(m)\,dm\right) = \exp(j\epsilon m)(m+p)^{(k-jp)\epsilon} \tag{A.3}$$

Now $Q(m)$ corresponds to

$$\frac{\epsilon b(jm+k)}{m+p} \tag{A.4}$$

So

$$\int Q(m) \ \exp\left(\int P(m)\,dm\right) \ dm \tag{A.5}$$

$$= \int \frac{b\epsilon(jm+k)\exp(j\epsilon m) \ (m+p)^{(k-jp)\epsilon}}{m+p} \ dm$$

$$= jb\epsilon \int \frac{(m+k)\exp(j\epsilon m)(m+p)^{(k-jp)\epsilon-1}}{j} \ dm \tag{A.6}$$

$$= jb\epsilon \int (m)\exp(j\epsilon m)(m+p)^{(k-jp)\epsilon-1} \ dm \tag{A.7}$$

$$+ \quad kb\epsilon \int \exp(j\epsilon m)(m+p)^{(k-jp)\epsilon-1} \ dm$$

Discard the second term, because it will have lesser degree than the first term. Now expand $(m+p)^{((k-jp)\epsilon-1)}$ by the binomial expansion theorem one obtains the following formula

$$jb\epsilon \int m \ \exp(j\epsilon m)\left[(m)^{(k-jp)\epsilon-1}+((k-jp)\epsilon-1)p(m)^{(k-jp)\epsilon-2} \right.$$

$$\left. + \ \dots \ \right] \ dm \tag{A.8}$$

Discarding the $((k-jp)\epsilon-2)$th and higher terms of m, we obtain

$$jb\epsilon \int m \ \exp(j\epsilon m)\left[(m)^{(k-jp)\epsilon-1}\right] \ dm$$

$$= \quad jb\epsilon \int m^{(k-jp)\epsilon} \ \exp(j\epsilon m) \ dm \tag{A.9}$$

Using integration by parts, (A.9) reduces to

$$
jb\epsilon \left[\frac{m^{(k-jp)\epsilon}}{(j\epsilon)} \exp(j\epsilon m) \right.
$$

$$
- \frac{((k-jp)\epsilon) \, m^{(k-jp)\epsilon-1}}{(j\epsilon)^2} \exp(j\epsilon m) \tag{A.10}
$$

$$
\left. + \quad \ldots \right]
$$

$$
= jb\epsilon \exp(j\epsilon m) \left[\frac{1}{j\epsilon} m^{(k-jp)\epsilon} \right. \tag{A.11}
$$

$$
- \frac{(k-jp)\epsilon}{(j\epsilon)^2} m^{(k-jp)\epsilon-1}
$$

$$
\left. + \quad \ldots \right]
$$

Expanding (A.3) by the binomial theorem we obtain

$$
\exp(j\epsilon m) \left[m^{(k-jp)\epsilon} + ((k-jp)\epsilon)pm^{(k-jp)\epsilon-1} \right.
$$

$$
\tag{A.3b}
$$

$$
+ \quad ((k-jp)\epsilon)((k-jp)\epsilon-1)p^2 m^{(k-jp)\epsilon-2}
$$

$$
\left. + \quad \ldots \right]
$$

Dividing (A.3b) into (A.11) we obtain

$$jb\epsilon \left[\frac{1}{j\epsilon} + O(1/m) \right] = b + O(1/m) \qquad \text{(A.12)}$$

Since in each yield curve ϵ is fixed. The linear term above dominates when m approaches infinity. By a similar argument if $Q(m)$ was replaced by

$$\epsilon(b+c/m) \ \frac{(jm+k)}{m+p} \qquad \text{(A.4b)}$$

the solution would be of the form

$$b + \frac{c'}{m} + O(1/m^2) \qquad \text{(A.12b)}$$

In fact, if a power series of $(1/m)$ is used one will obtain a power series of $(1/m)$ with the leading constant term the same.

```
DECLARE SUB RK4 (W!(), N!, H!, X!, A!, B!, C!, D!, F!, G!, E!, Q!, R!, S!, L!())
DECLARE FUNCTION XDEF2! (X!, Y!, A!, B!, C!, D!, F!, G!, E!)
DECLARE FUNCTION XDEF1! (X!)

DIM W(2)

SCREEN 2
' THE VARIABLE XX IS THE TREAS.DAT FOR A SPECIFIC YIELD CURVE
DIM XX(9, 2)
' THE VARIABLE TABLE GIVES THE FIELDS THAT PARSE  AS IN THE TREAS.DAT
DIM TABLE(9, 2)

TABLE(1, 1) = 7: TABLE(1, 2) = 12
TABLE(2, 1) = 13: TABLE(2, 2) = 18
TABLE(3, 1) = 19: TABLE(3, 2) = 24
TABLE(4, 1) = 25: TABLE(4, 2) = 30
TABLE(5, 1) = 31: TABLE(5, 2) = 36
TABLE(6, 1) = 37: TABLE(6, 2) = 42
TABLE(7, 1) = 43: TABLE(7, 2) = 48
TABLE(8, 1) = 49: TABLE(8, 2) = 54
TABLE(9, 1) = 55: TABLE(9, 2) = 60
'VARIBLES Q,R,S ARE USED TO PROPERLY PLOT ON THE SCREEN A SOLUTION OF DE
Q = 20
R = 100
S = 100

OPEN COMMAND$ FOR INPUT AS #1
TOP:
'GET A YIELD CURVE
GOSUB TREAS
DD$ = LEFT$(A$, 6)

'SET UP SIEVE FOR RUNGE KUTTA BASED ON WHETHER 30 YEAR OR 20 YEAR IS THE
'LAST YIELD CURVE VALUE.  THE SIEVE MUST BE DESIGNED SO TO PASS THROUGH
'KNOW YIELD CURVE MATURITIES, TO TEST THE FIT OF THE SOLUTION TO THE
'ACTUAL YIELD CURVE.
IF XX(9, 2) = 0 THEN
        N = 20 * 4
        INTERVAL = 20
ELSE
        N = 30 * 4
        INTERVAL = 30
END IF
'RUNGE KUTTA SOLUTION IS PLACED INTO THE VARIABLE L
DIM L(N, 2)
DIM PL(N, 2), QX(5)



DIM A(7), N(7), X(7), B(7)
DIM LL(7), AU(7)
KOR = INSTR(COMMAND$," ")
IF KOR = 0 THEN KOR=LEN(COMMAND$)
XRB$=LEFT$(COMMAND$,KOR)
```

```
SCRIT$=LTRIM$(RIGHT$(COMMAND$,LEN(COMMAND$)-KQR))
OPEN "R" + XRB$ FOR APPEND AS #2
PRINT #2, "NEWTEST6 " + XRB$
CLOSE #2

XXXRRB:

MOLD = .19
CEILING = 25
X(1)=0
IF QX(2) = 0 THEN CEILING = 10
IF QX(1) = 0 THEN CEILING = 5

FOR K8 = 1 TO CEILING
CLS
RANDOMIZE TIMER
IF SCRIT$="" THEN PRINT A$; "NEWTEST6 " + XRB$
M = .19
MERR = 0

IF XX(9, 2) = 0   THEN
B(1) =   0  *  QX(2)
A(1) =   0
N(1) =   0 *  QX(2)
B(2) =   .015
A(2) = XX(8,2)
M(2) =   .40
B(3) = -8  * QX(3)
A(3) =   0
N(3) =   8 * QX(3)
B(4) =   -8 * QX(4)
A(4) =   0
N(4) =   8 * QX(4)
B(5) =   -8* QX(4)
A(5) =   0   * QX(4)
M(5) =   8 * QX(4)
B(6) =   0 * QX(5)
A(6) =   4 * QX(5)
N(6) =   8 * QX(5)
B(7) =   -4
A(7) =   0
N(7) =   4
END IF
IF XX(8, 2) = 0   THEN
B(1) =    0 * QX(2)
A(1) =   0
N(1) =   0 * QX(2)
B(2) =   .015
A(2) =   XX(9,2)
N(2) =   .40
B(3) =   -8  * QX(3)
A(3) =   0
N(3) =   8 * QX(3)
B(4) =   -8 * QX(4)
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A(4) = 0
N(4) = 8 * QX(4)
B(5) = -8* QX(4)
A(5) = 0  * QX(4)
N(5) = 8 * QX(4)
B(6) = 0 * QX(5)
A(6) = 4 * QX(5)
N(6) = 8 * QX(5)
B(7) = -8
A(7) = 0
N(7) = 8
END IF
IF XX(8, 2) <> 0 AND XX(9, 2) <> 0 THEN
B(1) = 0  * QX(2)
A(1) = 0
N(1) = 0 * QX(2)
B(2) = .015
A(2) = XX(9,2)
N(2) = .40
B(3) = -8  * QX(3)
A(3) = 0
N(3) = 8 * QX(3)
B(4) = -8 * QX(4)
A(4) = 0
M(4) = 8 * QX(4)
B(5) = -8* QX(4)
A(5) = 0  * QX(4)
N(5) = 8 * QX(4)
B(6) = 0 * QX(5)
A(6) = 4 * QX(5)
N(6) = 8 * QX(5)
B(7) = -4
A(7) = 0
N(7) = 4
END IF
MTRS:


FOR J = 1 TO 15
Z = 40
IF M < .003 THEN Z = 100
IF M > .01 AND J > 13 THEN Z = 10
FOR I = 1 TO Z
TRIES = 0
RETRY:
TRIES = TRIES+1
FOR K = 2 TO 7
IF (A(K) - N(K) / (2)   J < B(K)) THEN
    LL(K) = B(K)
ELSE
    LL(K) = A(K) - N(K) / 2   J
END IF
    IF (A(K) + N(K) / 2   J > N(K)) THEN
        AU(K) = N(K) - LL(K)
    ELSE
```

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          AU(K) = A(K) + W(K) / 2    J - LL(K)
      END IF
      U = RND
      X(K) = LL(K) + U * AU(K)
NEXT K
      IF TRIES > 150 THEN GOTO ROUTE
      IF X(5) < 0 AND X(6)<=X(5) 2/4 THEN GOTO RETRY
      IF X(5) >= 0 and X(6)>= X(5) 2/4 THEN GOTO RETRY
      IF XX(3,2) <> 0 THEN
         IF XX(2,2) > XX(1,2) AND XX(2,2) > XX(3,2) THEN
         X(4) = XX(2,2)*(1+X(5)+X(6)) - X(2) - X(3)
         END IF
      ELSE
         IF XX(2,2) > XX(1,2) AND XX(2,2) > XX(4,2) THEN
         X(4) = XX(2,2)*(1+X(5)+X(6)) - X(2) - X(3)
         END IF
      END IF
'START
'H DETERMINES THE STEP SIZE OF THE RK ALGORITHM
H = -(INTERVAL / N
'LET THE STARTING VALUE FOR THE RK ALGORITHM TO BE THE YIELD CURVE
'VALUE FOR THE 1 YEAR MATURITY.
IF XX(9,2)=0 THEN
XXX = 20
W(2) = XX(8, 2)
ELSE
XXX = 30
W(2) = XX(9, 2)
END IF
L(0, 1) = XXX: L(0, 2) = W(2)
IF SCRITS ="" THEN
LOCATE 2, 5
PRINT USING " ##.####"; X(1); X(2); X(3); X(4); X(5); X(6); X(7)
END IF
CALL RK4(W(), N, H, XXX, X(1), X(2), X(3), X(4), X(5), X(6), X(7), Q, R, S, L())

'ACCUMULATE THE VARIABLE SSUM TO TEST TO SEE HOW FAR THE ACTUAL
'YIELD CURVE IS AWAY FROM THE DE SOLUTION.
'NOTE:  USED ABS INSTEAD OF LEAST SQUARE.

  SSUM = 0
  CERR = 0
IF SCRITS="" THEN
LOCATE 5, 1
PRINT USING " ### ### ###  #.#### #.#### #.####"; KB; J; I; M; HOLD; HERR
END IF
   FOR IX = 0 TO N
'THIS SELECT CASE IS SET UP TO PROPERLY OBTAIN THE SSUM VARIABLE
      SELECT CASE L(IX, 1)
         CASE .25
           IF XX(1, 2) <> 0 THEN
             SSUM = SSUM + ABS(L(IX, 2) - XX(1, 2))
             CERR = CERR + (L(IX, 2) - XX(1, 2))
           END IF
         CASE 1
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              IF XX(2, 2) <> 0 THEN
                SSUM = SSUM + ABS(L(IX, 2) - XX(2, 2))
                CERR = CERR + L(IX, 2) - XX(2, 2)
              END IF
            CASE 2
              IF XX(3, 2) <> 0 THEN
              SSUM = SSUM + ABS(L(IX, 2) - XX(3, 2))
              CERR = CERR + (L(IX, 2) - XX(3, 2))
              END IF
            CASE 3
            IF XX(4, 2) <> 0 THEN
            SSUM = SSUM + ABS(L(IX, 2) - XX(4, 2))
            CERR = CERR + (L(IX, 2) - XX(4, 2))
            END IF
            CASE 5
            IF XX(5, 2) <> 0 THEN
            SSUM = SSUM + ABS(L(IX, 2) - XX(5, 2))
            CERR = CERR + (L(IX, 2) - XX(5, 2))
            END IF
            CASE 7
            IF XX(6, 2) <> 0 THEN
            SSUM = SSUM + ABS(L(IX, 2) - XX(6, 2))
            CERR = CERR + (L(IX, 2) - XX(6, 2))
            END IF
            CASE 10
            IF XX(7, 2) <> 0 THEN
            SSUM = SSUM + ABS(L(IX, 2) - XX(7, 2))
            CERR = CERR + (L(IX, 2) - XX(7, 2))
            END IF
            CASE 20
            IF XX(8, 2) <> 0 THEN
            SSUM = SSUM + ABS(L(IX, 2) - XX(8, 2))
            CERR = CERR + (L(IX, 2) - XX(8, 2))
            END IF
            CASE 30
            IF XX(9, 2) <> 0 THEN
            SSUM = SSUM + ABS(L(IX, 2) - XX(9, 2))
            CERR = CERR + (L(IX, 2) - XX(9, 2))
            END IF
            CASE ELSE
          END SELECT

        NEXT IX


PF = SSUM
IF PF >= M THEN GOTO LOC3
M = PF
MERR = CERR
FOR N1 = 1 TO 7
A(N1) = X(N1)

NEXT N1
ZED = L(N,2)
IF MOLD > M THEN MOLD = M
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LOC3: NEXT I

NEXT J
OPEN "R" + XRBS FOR APPEND AS #2
f1S="\       \ #.###### #.###### ##.###### ##.###### ##.###### ##.###### ##.###### ##.###### #.#####"
PRINT #2, USING f1S; DDS; N; NERR; XX(1,2); A(2); A(3); A(4); A(5); A(6); A(7); ZED


CLOSE #2
IF MOLD < .0005 THEN GOTO OUTOFHERE


ROUTE:NEXT K8
IF MOLD > .002 THEN
            IF QX(1) = 0 THEN
                  QX(1) = 1
                  GOTO XXXRRB
            END IF
            IF QX(2) = 0 THEN
                  QX(2) = 1
                  GOTO XXXRRB
            END IF
            IF QX(3) = 0 THEN
                  QX(3) = 1
                  GOTO XXXRRB
            END IF
            IF QX(4) = 0 THEN
                  QX(4) = 1
                  GOTO XXXRRB
            END IF

END IF

OUTOFHERE: GOSUB TREAS
DDS = LEFTS(AS, 6)
GOTO XXXRRB


TREAS:
QX(1) = 1
QX(2) = 1
QX(3) = 1
QX(4) = 1
QX(5) = 1
IF EOF(1) THEN GOTO ENDUP
LINE INPUT #1, AS
FOR I = 1 TO 9

YS = LTRIMS(RTRIMS(MIDS(AS, TABLE(I, 1), TABLE(I, 2))))

IF YS = "NA" THEN XX(I, 2) = 0 ELSE XX(I, 2) = VAL(YS)
   SELECT CASE I
      CASE 1
      XX(I, 1) = .25
      CASE 2
      XX(I, 1) = 1
      CASE 3
```

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        XX(I, 1) = 2
        CASE 4
        XX(I, 1) = 3
        CASE 5
        XX(I, 1) = 5
        CASE 6
        XX(I, 1) = 7
        CASE 7
        XX(I, 1) = 10
        CASE 8
        XX(I, 1) = 20
        CASE 9
        XX(I, 1) = 30

    END SELECT

NEXT I
RETURN

ENDUP: CLOSE #1
      END

SUB RK4 (W(), N, H, X, A, B, C, D, F, G, E, O, R, S, L())
  DIM K(4, 2)
START = X
FOR I = 1 TO N
    K(1, 2) = H * XDEF2(X, W(2), A, B, C, D, F, G, E)
    K(2, 2) = H * XDEF2(X + 1 / 2 * H, W(2) + 1 / 2 * K(1, 2), A, B, C, D, F, G, E)
    K(3, 2) = H * XDEF2(X + 1 / 2 * H, W(2) + 1 / 2 * K(2, 2), A, B, C, D, F, G, E)
    K(4, 2) = H * XDEF2(X + H, W(2) + K(3, 2), A, B, C, D, F, G, E)
    W(2) = W(2) + (K(1, 2) + 2 * K(2, 2) + 2 * K(3, 2) + K(4, 2)) / 6
X = START + I * H
L(I, 1) = X: L(I, 2) = W(2)

NEXT I

END SUB

FUNCTION XDEF1 (X)
XDEF1 = .0001 + LOG(X)
END FUNCTION

FUNCTION XDEF2 (X, Y, A, B, C, D, F, G, E)
QRS = X * X + F * X + G
IF QRS = 0 THEN QRS = .0001
XDEF2 = -E * (Y - ( B*X*X +  C * X + D) / QRS) * (2*X+2) / (3 * X+2)

END FUNCTION
```

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