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AN INVESTIGATION OF THE GOMPERTZ LAW OF MORTALITY

Jacques F. Carriere
Warren Centre for Actuarial Studies & Research
The University of Manitoba, Winnipeg, Canada

ABSTRACT

This article investigates the properties of the Gompertz distribution function. Explicit formulas for continuous life insurances and annuities are given in terms of the left-truncated gamma function. Moreover, approximations for the mean, variance, skewness and kurtosis of the Gompertz distribution are also given. Finally, the article shows that using a Gompertz assumption at fractional ages produces better a approximation than the usual UDD assumption.

KEYWORDS

Left-Truncated Gamma Function; Gumbel Distribution; Complete Expectation of Life

1. INTRODUCTION

Consider the law of human mortality described by Gompertz (1825) where the force of mortality has the form $\mu_x = Bc^x$ with $x \geq 0$, $B > 0$, $c \geq 1$. It is well known that this model is an excellent description of the pattern of mortality at the adult ages. Nevertheless, actuaries rarely use it to calculate the value of annuities and insurances, as evidenced by the textbooks written by Jordan (1967), Wolff (1970) and Bowers, *et al* (1986). The probable reason for this is the intractability of the mathematical expectations that emerge from the analysis.

The purpose of this article is to elevate the Gompertz law and compile a set of facts that are accessible to the practicing actuary. We find that many of the actuarial functions can be expressed in terms of the left-truncated gamma function, as Mereu (1962) demonstrates. We hope that future actuaries will adopt the model for calculating the net single premiums of life insurances and annuities.

First, the paper gives a few definitions and background facts about the Gompertz law. We present a parametrization of the law that is statistically informative. Next, we show that this law readily explains the pattern of mortality for a valuation mortality table and we estimate the Gompertz parameters with the valuation mortality rates. Next, explicit expressions are derived for continuous life insurances, annuities, net level premiums and reserves by using the left-truncated gamma function which can easily be approximated. Using the properties of the Gumbel distribution, we also present approximations for the mean, variance, skewness and kurtosis of the Gompertz distribution. Finally, the article recommends the Gompertz law over the classical assumptions of UDD and Balducci for calculating probabilities at fractional ages.

2. DEFINITIONS AND BASIC RESULTS

In this section, we present some notation and basic results. We start by giving an informative representation of the Gompertz law. Carriere (1992) expresses the force as

$$\mu_x = \frac{1}{\sigma}\exp\{(x-m)/\sigma\},\ \sigma > 0,\ m \in \Re. \tag{2.1}$$

Note that $c = e^{1/\sigma}$ and $B = \sigma^{-1}e^{-m/\sigma}$. This representation is informative because m is approximately equal to the mean and σ is proportional to the standard deviation, as we show later. In human populations we usually find that $m > \sigma > 0$. For example, for the United States (US) population, Carriere (1992) found that these parameters were $m = 82.3$ and $\sigma = 11.4$. In section 3, we find that using valuation mortality rates yields estimates of m and σ that are very similar to the US population.

Using the relation $s(x) = \exp\left\{-\int_0^x \mu_t\, dt\right\}$, we find that the survival function for the Gompertz law is equal to

$$s(x) = \exp\left\{e^{-m/\sigma} - e^{(x-m)/\sigma}\right\}. \tag{2.2}$$

The new parameter σ can be interpreted as a dispersion parameter because if $m > 0$, then

$$\lim_{\sigma \to 0} \{s(m-\epsilon) - s(m+\epsilon)\} = 1, \ \forall \epsilon > 0. \tag{2.3}$$

This limit suggests that all the mass concentrates about m when σ is small and so m can also be interpreted as a location parameter when $m > 0$. Using the relation $f(x) = -\frac{d}{dx} s(x) = \mu_x\, s(x)$ we find that the density is equal to

$$f(x) = \exp\left\{e^{-m/\sigma} - e^{(x-m)/\sigma} + (x-m)/\sigma\right\}, \ x \geq 0. \tag{2.4}$$

It is easy to verify that the mode of the density is equal to 0 when $m \leq 0$ and that the mode is m when $m > 0$.

Pollard and Valkovics (1993) defined a Gompertz law for all $x \in \Re$ and found explicit formulas for the moments. Using our notation, they analysed the distribution function

$$G(x) \equiv 1 - \exp\left\{-e^{(x-m)/\sigma}\right\}, \ x \in \Re. \tag{2.5}$$

Pollard and Valkovics fail to mention that this extended Gompertz distribution is simply the Gumbel distribution for minima, an extreme-value distribution. For an extensive discussion about extreme-value distributions, consult Johnson and Kotz (1970). For a specific discussion about the Gumbel distribution, see Kotz and Johnson (1983). Appendix A summarizes certain facts about the standardized ($m = 0$, $\sigma = 1$) Gumbel distribution for minima. Appendix B presents the Inverse-Gompertz distribution and the corresponding Gumbel distribution for maxima.

3. ESTIMATION OF THE GOMPERTZ PARAMETERS

In this section, we estimate the parameters of the Gompertz law using the male and female ultimate mortality rates from the 1975-80 Basic Tables of the Society of Actuaries. These crude rates were prepared by the Committee on Ordinary Insurance and Annuities (1982). The Gompertz law is only applicable at the older ages and so we focus our analysis on the rates $\hat{q}_x$, for the ages $x = 40, \ldots, 100$.

A. Estimation using the crude rates.

Using the relation $q_x = 1 - \frac{s(x+1)}{s(x)}$, we find that the Gompertz law yields the identity

$$q_x(m,\sigma) \equiv 1 - \exp\left\{e^{(x-m)/\sigma}(1 - e^{1/\sigma})\right\}. \tag{3.1}$$

Let D_x denote the total amount of death claims associated with the *crude* rate $\hat{q}_x$. Carriere (1994) suggests that a good way of estimating m and σ is to minimize the robust loss function

$$L(m,\sigma) = \sum_{x=40}^{100} \sqrt{D_x}\left|1 - \frac{q_x(m,\sigma)}{\hat{q}_x}\right|. \tag{3.2}$$

The function $L(m,\sigma)$ was minimized using the NONLIN module of the statistical computer software SYSTAT. We found that the NONLIN *simplex* or *polytope* algorithm was very successful at minimizing the non-differentiable loss function $L(m,\sigma)$. Using the male data we found that the parameters $\hat{m} = 82.153$ and $\hat{\sigma} = 10.304$ minimized (3.2). The female data yielded the parameter estimates $\hat{m} = 87.281$ and $\hat{\sigma} = 10.478$. Figure 1 shows that the Gompertz model fit the data well. This graph and the others in the paper were produced with the computer package GAUSS.

B. Estimation using the graduated rates.

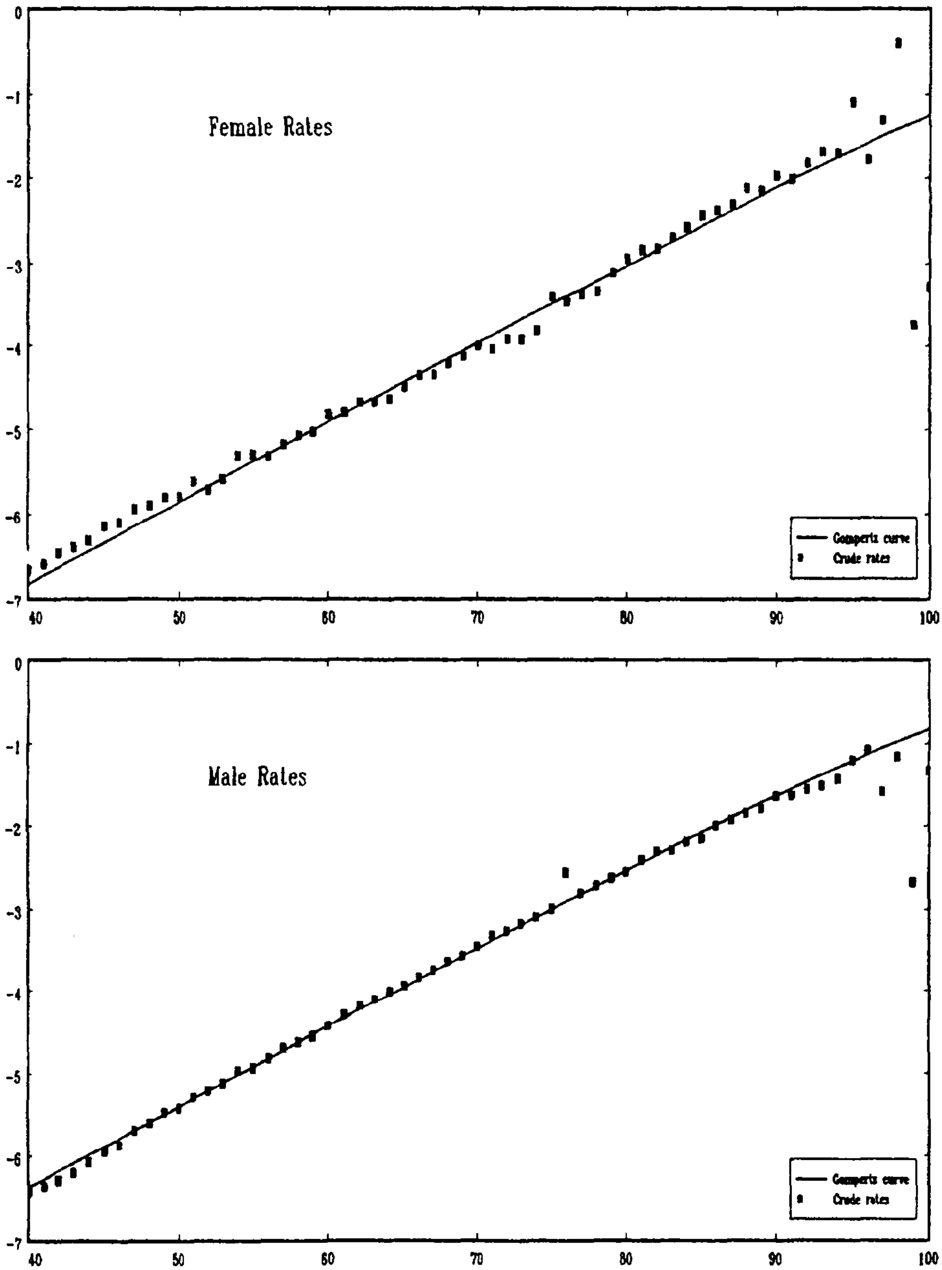
Using the relation ${}_{t|}q_0 = s(x) - s(x+1)$, we find that the Gompertz law yields the identity

$${}_{t|}q_0(m,\sigma) \equiv \exp\left\{e^{-m/\sigma}\right\} \times \left[\exp\{-\sigma\mu_x\} - \exp\{-\sigma\mu_{x+1}\}\right\} \tag{3.3}$$

FIGURE 1

A Comparison of the Best-Fitting Gompertz Model with the Crude Ultimate Rates

This is a plot of $\log_e\{\hat{q}_x\}$ and $\log_e\{q_x(\hat{m},\hat{\sigma})\}$ versus the age x.



Let ${}_{t\,|}\hat{q}_0$ denote a *graduated* probability from the Basic Tables. Another way of estimating m and σ is to minimize the function

$$L(m,\sigma) = \sum_{t=40}^{99} \left({}_{t\,|}q_0(m,\sigma) - {}_{t\,|}\hat{q}_0 \right)^2 . \tag{3.4}$$

Using the male data we found that the parameters $\hat{m} = 81.022$ and $\hat{\sigma} = 10.379$ minimized (3.4). The female data yielded the parameter estimates $\hat{m} = 86.235$ and $\hat{\sigma} = 9.593$. Note that these parameter estimates are similar to the ones given in section A. Figure 2 is a plot of ${}_{t\,|}\hat{q}_0$ and the best-fitting Gompertz model, ${}_{t\,|}q_0(\hat{m},\hat{\sigma})\}$. The male model fit the data better than the female model but the male rates were based on ten times the experience. In any case, the Gompertz model fit the data well.

4. CONTINUOUS INSURANCES AND ANNUITIES

In this section, we give explicit expressions for continuous life insurances, annuities, net level premiums and reserves by using the left-truncated gamma function. Some of these expressions may be found in Mereu (1962). Let $T(x) \geq 0$ denote the time-at-death of a life aged $x \geq 0$ and let ${}_tp_x \equiv s(x+t)/s(x)$ denote the probability of surviving to time $t \geq 0$. For the Gompertz law, we can write

$${}_tp_x = \exp\left\{ e^{(x-m)/\sigma}(1 - e^{t/\sigma}) \right\} = \exp\left\{ \sigma\mu_x(1 - e^{t/\sigma}) \right\}. \tag{4.1}$$

Let $M_x(u) \equiv E(\exp\{u\,T(x)\})$ denote the moment generating function of $T(x)$. Remember that the density of $T(x)$ is ${}_tp_x\,\mu_{x+t}$ and so $M_x(u)$ is equal to

$$M_x(u) = \int_0^\infty e^{ut}\,{}_tp_x\,\mu_{x+t}\,dt = \int_0^\infty e^{ut}\exp\left\{ \sigma\mu_x(1 - e^{t/\sigma}) \right\}\mu_x e^{t/\sigma}\,dt. \tag{4.2}$$

Applying the transformation $z = \sigma\mu_x e^{t/\sigma}$ yields

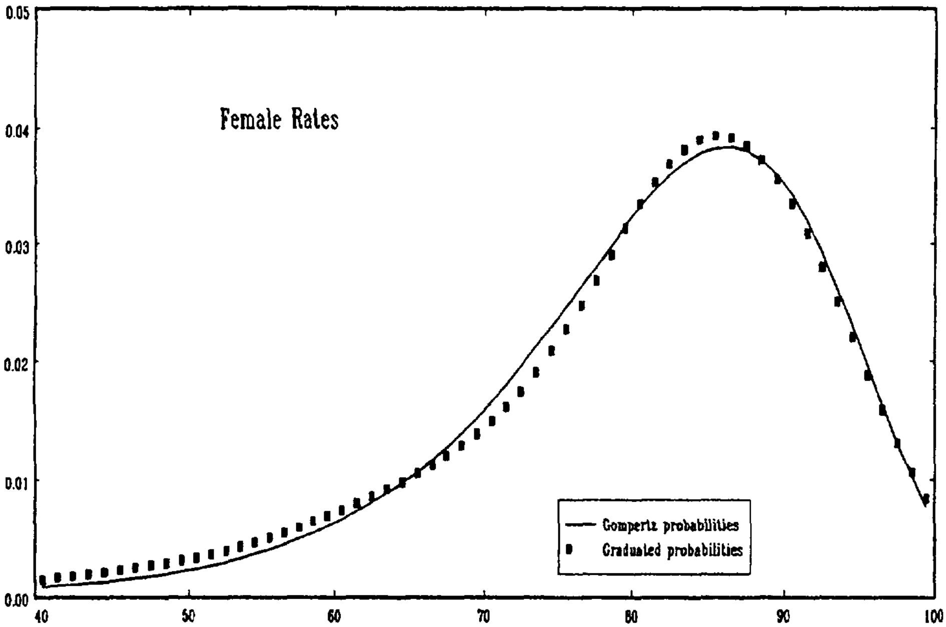
$$M_x(u) = \exp\{\sigma\mu_x - u(x-m)\} \int_{\sigma\mu_x}^\infty z^{u\sigma} e^{-z}\,dz. \tag{4.3}$$

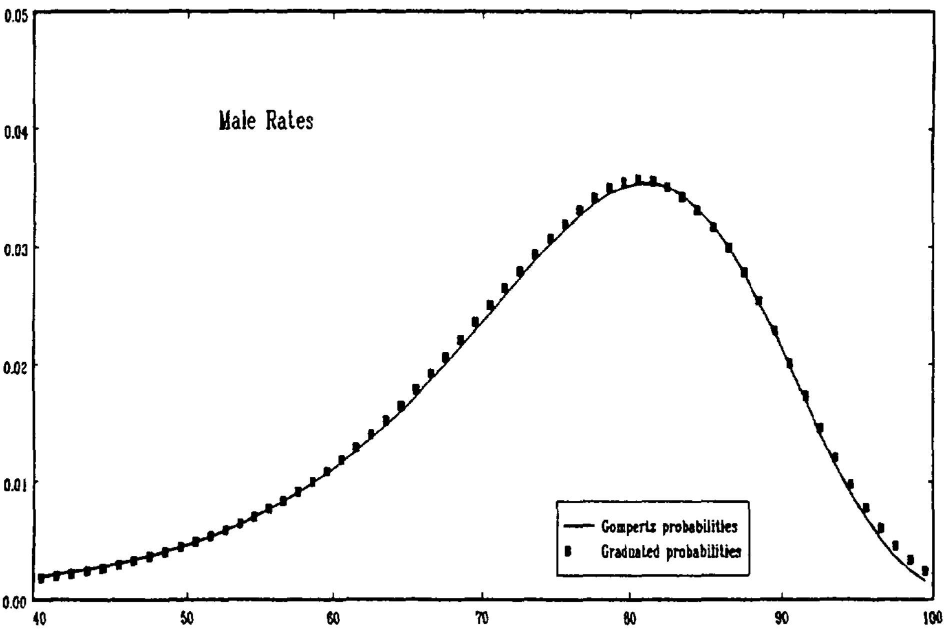
It is fairly easy to show that $M_x(u) < \infty$ for all $u \in \Re$. Let us denote the integral in (4.3) as $\Gamma(\sigma\mu_x, 1 + u\sigma)$ and formally introduce the left-truncated gamma function. This is defined as:

FIGURE 2

A Comparison of the Gompertz Model with the Graduated Probabilities

This is a plot of $_{t|}\hat{q}_0$ and $_{t|}q_0(\hat{m},\hat{\sigma})$ versus the time t.





$$\Gamma(t,\alpha) \equiv \int_t^{\infty} z^{\alpha-1} e^{-z} dz, \quad t > 0, \; \alpha \in \Re. \tag{4.4}$$

Note that $\Gamma(t,\alpha) < \infty$. Next, the cumulant generating function, denoted as $\Psi_x(u)$, is defined as

$$\Psi_x(u) \equiv \log_e\{M_x(u)\} = um - ux + \sigma\mu_x + \log_e\{\Gamma(\sigma\mu_x, 1 + u\sigma)\}. \tag{4.5}$$

We use (4.5) in section 6 to derive an expression for the complete expectation of life. Let $\delta \geq 0$ be the force of interest and let $\overline{A}_x$ denote the net single premium for a continuous whole-life insurance. If we let $u = -\delta$ in (4.3), then we get

$$\overline{A}_x = \exp\{\sigma\mu_x + \delta(x - m)\} \times \Gamma(\sigma\mu_x, 1 - \delta\sigma). \tag{4.6}$$

Consider the continuous life annuity, expressed as a Stieltjes integral,

$$\overline{a}_x \equiv \int_0^{\infty}\int_0^{t} e^{-\delta z} dz\, d({}_tq_x) = \int_0^{\infty} e^{-\delta t}\, {}_tp_x\, dt. \tag{4.7}$$

Consulting Appendix C, we find that the last identity for $\overline{a}_x$ is true regardless of any continuity assumptions on ${}_tp_x$. Calculating further, we find that

$$\overline{a}_x = \sigma \times \exp\{\sigma\mu_x + \delta(x - m)\} \times \Gamma(\sigma\mu_x, -\delta\sigma). \tag{4.8}$$

Once again, we get an expression that is based on the left-truncated gamma function. Next, consider the net level premium $\overline{P}_x \equiv \overline{A}_x / \overline{a}_x$. Using (4.8) and (4.6) we find that

$$\overline{P}_x = \frac{\Gamma(\sigma\mu_x, 1 - \delta\sigma)}{\sigma \times \Gamma(\sigma\mu_x, -\delta\sigma)} \; . \tag{4.9}$$

Also, consider the net level premium reserve ${}_t\overline{V}(\overline{A}_x) = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}$. Using (4.8) we find that

$${}_t\overline{V}(\overline{A}_x) = 1 - \exp\{\sigma(\mu_{x+t} - \mu_x) + \delta t\} \frac{\Gamma(\sigma\mu_{x+t}, -\delta\sigma)}{\Gamma(\sigma\mu_x, -\delta\sigma)} \; . \tag{4.10}$$

Let us prove that $\overline{A}_x \to 1$ as $x \to \infty$. This will imply that $\overline{P}_x \to 0$ and ${}_t\overline{V}(\overline{A}_x) \to 1$ as $x \to \infty$.

It is sufficient to prove that $\bar{a}_x \to 0$ as $x \to \infty$ because $\bar{A}_x = 1 - \delta \bar{a}_x$. Note that

$$0 \leq \bar{a}_x = \int_0^\infty e^{-\delta t} {}_tp_x \, dt \leq \int_0^\infty {}_tp_x \, dt,$$

and that ${}_tp_x = \exp\left\{ e^{(x-m)/\sigma}(1 - e^{t/\sigma})\right\} \to 0$ as $x \to \infty$ for all $t > 0$. Therefore, by the Monotone Convergence Theorem (Royden, 1968), $\int_0^\infty {}_tp_x \, dt \to 0$ as $x \to \infty$ and the result follows.

5. APPROXIMATING THE LEFT-TRUNCATED GAMMA FUNCTION

Actuaries are usually interested in calculating $\bar{A}_x$ or $\bar{a}_x$. The key to calculating these net single premiums is finding a good approximation to the left-truncated gamma function. This is easily done by using a composite numerical integration formula because we can write

$$\Gamma(t, \alpha) = \int_0^{\exp\{-t\}} |\log_e y|^{\alpha - 1} dy, \tag{5.1}$$

where $|\log_e y|^{\alpha - 1}$ is a continuous function that is analytic, bounded, and monotone on the interval $(0, \exp\{-t\})$. Using a composite Simpson's rule (Burden and Faires, 1993), we find that we can approximate $\Gamma(t, \alpha)$, $t > 0$, $\alpha < 1$, with the quadrature formula

$$\widehat{\Gamma}_N = \frac{e^{-t}}{6N}\left[t^{\alpha - 1} + 2\sum_{k=1}^{N-1} \left|\log_e\{k/N\} - t\right|^{\alpha - 1} + 4\sum_{k=1}^{N} \left|\log_e\{(k - .5)/N\} - t\right|^{\alpha - 1}\right], \tag{5.2}$$

where $N = 1, 2, \ldots$ It can be shown that $\widehat{\Gamma}_N \to \Gamma$ as $N \to \infty$. If $\alpha > 1$ then we can apply the recursive formula

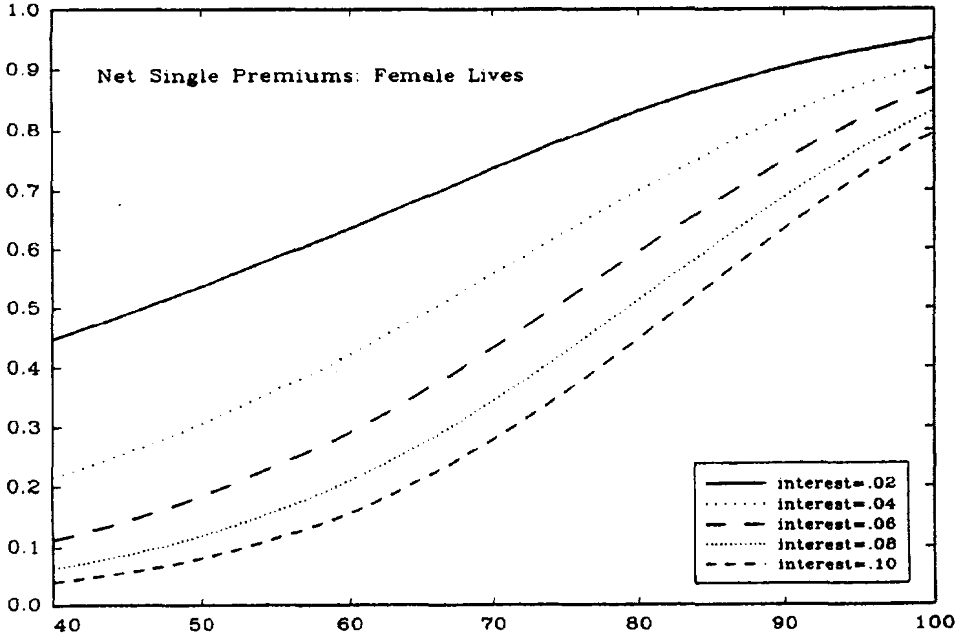
$$\Gamma(t, \alpha) = t^{\alpha - 1} e^{-t} + (\alpha - 1)\Gamma(t, \alpha - 1). \tag{5.3}$$

until the parameter α is less than 1 and then use (5.2). Using $\widehat{\Gamma}_N$ with $N = 500$ in conjunction with (4.6), we plotted $\bar{A}_x$ for $x = 40, \ldots, 100$ assuming that $m = 87.281$ and $\sigma = 10.478$. These parameter values are the estimates based on the female data from the Basic Tables. Figure 3 shows the plot of $\bar{A}_x$, assuming a force of interest of $\delta = .02$, .04, .06, .08 and .10.

FIGURE 3

The Net Single Premium for a Continuous Whole-Life Insurance under Gompertz's Law

This is a plot of $\bar{A}_x$ versus the age x under various assumptions for δ.



6. THE COMPLETE EXPECTATION OF LIFE

Let us derive an expression for the complete expectation of life $\overset{\circ}{e}_x \equiv E\{T(x)\} = \int_0^\infty {}_tp_x\,dt$. Appendix C shows that this identity is true, regardless of the continuity of ${}_tp_x$. Using the cumulant generating function shown in (4.5), we find that

$$\overset{\circ}{e}_x = \frac{\partial}{\partial u}\Psi_x(u)\,|_{u=0} = m - x + \sigma \times \frac{\frac{\partial}{\partial \alpha}\Gamma(\sigma\mu_x, \alpha)\,|_{\alpha=1}}{\Gamma(\sigma\mu_x, 1)}$$

$$= m - x + \sigma\, e^{\sigma\mu_x} \int_{\sigma\mu_x}^{\infty} \log_e(z)\, e^{-z}\, dz. \tag{6.1}$$

As a special case, let us investigate $\overset{\circ}{e}_0$. Typical values for m and σ are 87.3 and 10.5, respectively. This means that $\sigma\mu_0 = e^{-m/\sigma} \approx 0$ and $e^{\sigma\mu_0} \approx 1$ because σ is usually small relative to m. A bit of analysis shows that $1 > \exp\{e^{-m/\sigma}\} > 1 + \exp\left\{\frac{\sigma}{m} - \frac{m}{\sigma}\right\}$ when $m > 0$ because $e^{-m/\sigma} < m/\sigma$. This means that $\exp\{e^{-m/\sigma}\} = 1 + O(e^{-m/\sigma})$. The function $e^{-m/\sigma}$ converges to 0 very quickly because $\sigma^{-p} e^{-m/\sigma} \to 0$ as $\sigma \to 0$ for all $p > 0$; in other words $e^{-m/\sigma} = O(\sigma^p)\ \forall p > 0$. Consulting the appendix, we find that

$$\int_{\sigma\mu_0}^{\infty} \log_e(z)\, e^{-z}\, dz \approx \int_0^{\infty} \log_e(z)\, e^{-z}\, dz = -\gamma$$

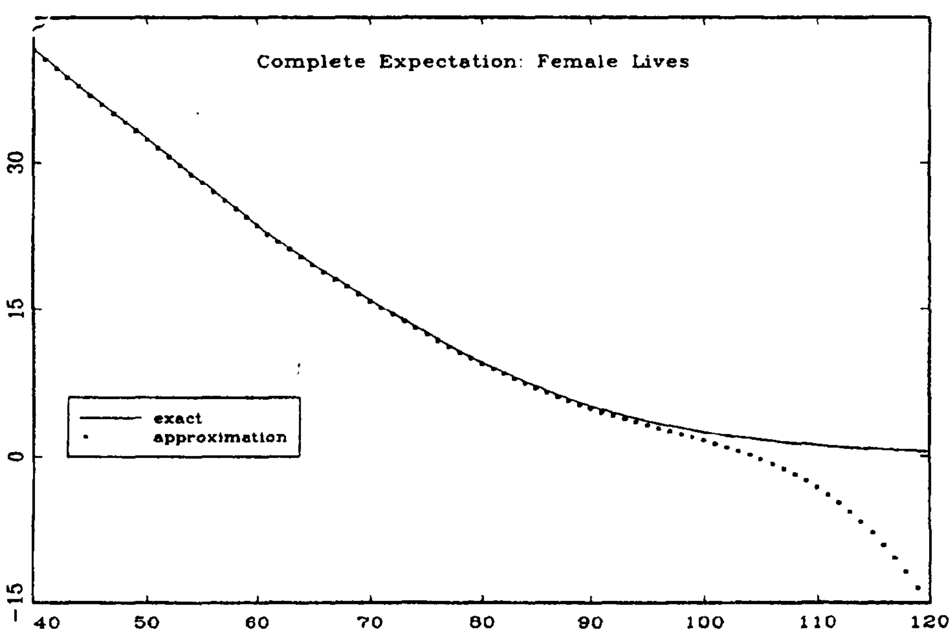
where $\gamma = .577\ldots$ is Euler's constant. Therefore, the mean of the Gompertz survival function given in (2.2) is

$$\mathring{e}_0 \approx m - \sigma\gamma. \tag{6.2}$$

FIGURE 4

A Comparison of the Actual Complete Expectation to its Approximation

This is a plot of $\mathring{e}_x$ and $\hat{e}_x$ versus the age x, assuming a Gompertz law.



Let us give another expression for (6.1). Using integration by parts, we find that

$$\mathring{e}_x = \sigma\, e^{\sigma \mu_x} \Gamma(\sigma \mu_x, 0). \tag{6.3}$$

Again, we have an expression based on the left-truncated gamma function that can easily be approximated to any level of accuracy.

Let us compare (6.3) with an approximation formula developed by Pollard (1980) that is based on the Euler-Maclaurin expansion found in Kellison (1976). Let $U_x \equiv \exp\{-\sigma \mu_x\}$, then Pollard recommends the following approximation:

$$\hat{e}_x = \frac{\sigma}{12 U_x}\{U_x \log_e(U_x) - 12 \log_e(1 - U_x) - 7 U_x\}. \tag{6.4}$$

Figure 4 is a plot of $\mathring{e}_x$ and $\hat{e}_x$, assuming that $m = 87.281$ and $\sigma = 10.478$. These parameter values are the estimates based on the female data from the Basic Tables. Consulting Figure 4, we find that if $x \leq m$ then (6.4) provides a good approximation of (6.3) but if $x > m$ then Pollard's approximation can be very poor.

7. VARIANCE, SKEWNESS AND KURTOSIS

Consider the random variables $T(0)$ and $\sigma Z + m \sim G(\cdot)$. The distribution of $T(0)$ is simply the conditional distribution of $\sigma Z + m$ given that $Z > -m/\sigma$. We know that $Pr(Z > -m/\sigma) \approx 1$ and so the distribution of $T(0)$ is approximately equal to that of $\sigma Z + m$. Therefore $E\{T(0) - E[T(0)]\}^k \approx \sigma^k E\{Z - E[Z]\}^k$. Using the central moments of Z, as shown in Appendix A, we find that

$$Variance\{T(0)\} \approx \frac{\sigma^2 \pi^2}{6}, \tag{7.1}$$

$$Skewness\{T(0)\} \approx -1.1395\ldots, \tag{7.2}$$

$$Kurtosis\{T(0)\} \approx 2.4\,. \tag{7.3}$$

Finally, let us derive an expression for the variance $Var\{T(x)\}$. No simple expression exists for the skewness and kurtosis of $T(x)$. Using the cumulant generating function shown in (4.5), we find that

$$Var\{T(x)\} = \sigma^2 \times \left\{ e^{\sigma\mu_x} \int_{\sigma\mu_x}^{\infty} [\log_e(z)]^2 \, e^{-z} dz - \left[e^{\sigma\mu_x} \int_{\sigma\mu_x}^{\infty} \log_e(z) \, e^{-z} dz \right]^2 \right\}. \qquad (7.4)$$

8. THE GOMPERTZ LAW AS A FRACTIONAL AGE ASSUMPTION

Usually, actuaries will use the probabilities q_x from a valuation table for pricing insurances and annuities. In this case, x is an integer and so if the actuary requires the mortality rate ${}_tq_x$ or the force of mortality μ_{x+t} at the fractional period $t \in [0,1)$ then an assumption like *uniform distribution of deaths* (UDD) is used for interpolation. We show that using a Gompertz assumption at the fractional ages is much more accurate than the UDD assumption, especially at the adult ages. Assume that q_x is known, then a Gompertz assumption at the fractional ages would yield the relations

$${}_tq_x = 1 - \left(1 - q_x\right)^{\frac{1 - \exp\{t/\sigma\}}{1 - \exp\{1/\sigma\}}}, \qquad (8.1)$$

$${}_{1-t}q_{x+t} = 1 - \left(1 - q_x\right)^{\frac{\exp\{t/\sigma\} - \exp\{1/\sigma\}}{1 - \exp\{1/\sigma\}}}, \qquad (8.2)$$

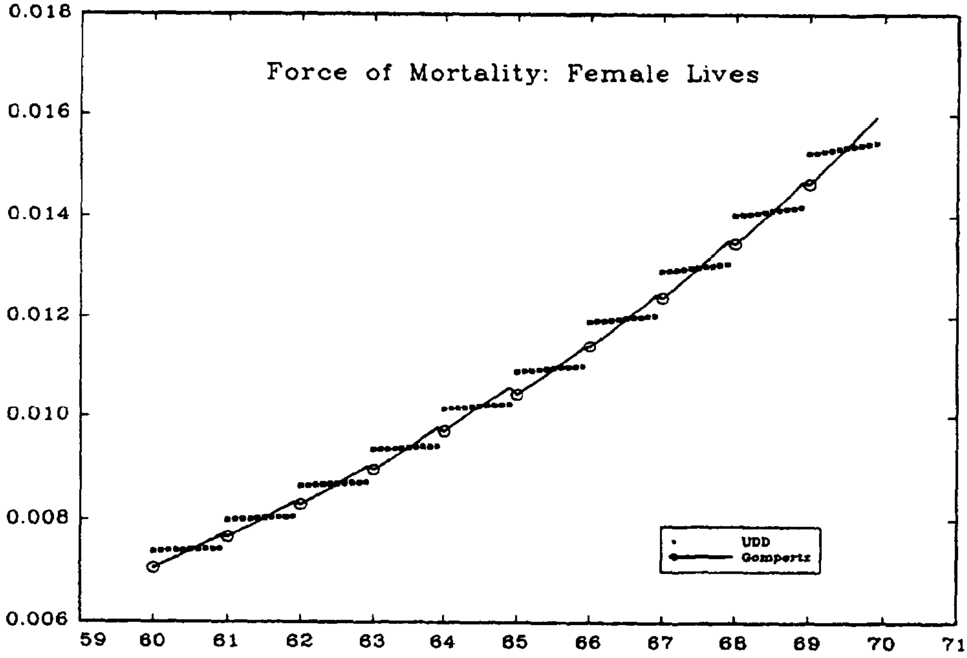
$$\mu_{x+t} = \frac{e^{t/\sigma} \log_e(1 - q_x)}{\sigma\left(1 - e^{1/\sigma}\right)}. \qquad (8.3)$$

In comparison, the UDD assumption yields ${}_tq_x = t \times q_x$, the Balducci assumption yields ${}_{1-t}q_{x+t} = (1-t) \times q_x$, and the constant force assumption yields $\mu_{x+t} = -\log_e(1 - q_x)$. These classical assumptions yield very simple formulas but they do not model reality very well. Let us compare the force of mortality calculated with (8.3) to that calculated with $\mu_{x+t} = \frac{q_x}{1 - t \times q_x}$ (UDD) where q_x is a graduated rate from the female Basic Tables. Figure 5 is a plot of μ_{x+t}, $0 \leq t < 1$, $x = 60, \ldots, 69$ using UDD and the Gompertz assumption at the fractional ages. Using UDD yields a step function which does not interpolate well. The same type of pattern would be exhibited by a Balducci or constant force assumption. On the other hand, the Gompertz assumption gives a good approximation at the fractional ages.

FIGURE 5

A Comparison of the Gompertz and UDD Assumptions for Fractional Ages

This is a plot of μ_{x+t}, $0 \leq t < 1$, $x = 60, \ldots, 69$.



APPENDIX A

PROPERTIES OF THE GUMBEL

In this appendix, we give some well-known properties of the standard Gumbel distribution for minima, whose random variable is denoted as Z. Specifically, we give expressions for the moment and cumulant generating functions along with the moments and cumulants of Z. These facts may be found in Kotz and Johnson (1983).

i) The moment generating function:

$M_Z(t) \equiv E\{e^{tZ}\} = \int_{-\infty}^{\infty} e^{tz}\exp\{z - e^z\}\,dz = \int_0^{\infty} y^t e^{-y}\,dy \equiv \Gamma(1+t)$, $t > -1$. Calculating the moments we find that $E(Z^k) = \frac{d^k}{dt^k}\Gamma(1+t)\big|_{t=0} = \int_0^{\infty} \{\log_e(y)\}^k e^{-y}\,dy$, $k = 0, 1, 2, \ldots$ As a special case we find that $E(Z) = -\gamma$ where $\gamma \equiv .5772157\ldots = \int_0^{\infty} \log_e(y^{-1})\,e^{-y}\,dy = -\int_0^1 \log_e|\log_e(y)|\,dy$ is Euler's constant.

ii) The cumulant generating function:

$\Psi_Z(t) \equiv \log_e\{M_Z(t)\} = \log_e\{\Gamma(1+t)\} = \sum_{i=1}^{\infty} \frac{\kappa_i t^i}{i!}$ where $\kappa_1 = E(Z) = -\gamma$, $\kappa_2 = Var(Z) = \pi^2/6$, $\kappa_3 = E\{Z - E(Z)\}^3 = -2.40411\ldots$, $\kappa_4 = E\{Z - E(Z)\}^4 - 3\{Var(Z)\}^2 = \pi^4/15$. This means that the coefficient of skewness is $\kappa_3/(\kappa_2)^{3/2} = -1.1395\ldots$ and the coefficient of kurtosis is $\kappa_4/(\kappa_2)^2 = 12/5$.

APPENDIX B

THE INVERSE-GOMPERTZ DISTRIBUTION

Pollard and Valkovics (1993) also present the Gumbel distribution for maxima, $1 - G(-x)$, which is a mirror-image of the distribution in (2.5). The Gompertz distribution is a truncated Gumbel distribution for minima. Similarly, we can define an Inverse-Gompertz distribution as a truncated Gumbel distribution for maxima. This definition yields the survival function

$$s_I(x) \equiv \frac{1 - \exp\left\{-e^{-(x - m_I)/\sigma}\right\}}{1 - \exp\left\{-e^{m_I/\sigma}\right\}},\ x \geq 0.$$

Note that m_I is the mode of the Inverse-Gompertz density when $m_I > 0$ and the limit in (2.3) holds for the density of $s_I(x)$. The Inverse-Gompertz survival function was first defined by Carriere (1992).

APPENDIX C

A MATHEMATICAL EXPECTATION RESULT

This appendix presents a generalization of Theorem 3.1 in Bowers, *et al* (1986). Consider the distribution function, ${}_tq_x$, $t \geq 0$, that corresponds to the random variable $T(x)$. In this discussion, we make no specific assumptions about ${}_tq_x$ and so it can be discontinuous. Consider the expectation $E\{g(T(x))\} = \int_0^\infty g(t)\, d({}_tq_x) < \infty$, expressed as a Stieltjes integral. Now, suppose that $g(\cdot)$ admits the representation

$$g(t) = g(0) + \int_0^t g'(s)\, ds,$$

so that $g(\cdot)$ is absolutely continuous. Using Fubini's Theorem (Royden, 1968), we find that we can write

$$E\{g(T(x))\} = g(0) + \int_0^\infty g'(t)\, {}_tp_x\, dt.$$

Unlike the result in *Actuarial Mathematics* (1986), this expression for the mathematical expectation does not require $g(\cdot)$ to be a non-negative, monotonic and differentiable function. In the case of a continuous annuity, we find that $g(t) = (1 - e^{-\delta t})/\delta$. Therefore the net single premium for this annuity is $\bar{a}_x \equiv \delta^{-1} E\left\{1 - e^{-\delta T(x)}\right\} = \int_0^\infty e^{-\delta t}\, {}_tp_x\, dt$. Letting $\delta = 0$, we get $\mathring{e}_x = \int_0^\infty {}_tp_x\, dt$, regardless of any continuity assumptions about ${}_tp_x$.

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Jacques F. Carriere
Warren Centre for Actuarial Studies & Research,
Faculty of Management, The University of Manitoba,
Drake Centre, 181 Freedman Crescent,
Winnipeg, Manitoba, Canada R3T 5V4