The Economic Capital and Risk Adjustment Performance for VA with Guarantees with an Example of GMAB

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Abstract

Economic capital (EC) and risk-adjusted performance measurement (RAPM) are becoming increasingly important criteria in product development and performance evaluation for insurance companies. The EC framework can assist in strategic decision making and increase capital allocation efficiency. However, there are very few precise and detailed definitions for EC and RAPM in life insurance case studies. This paper proposes a general definition for EC that can be applied to various types of insurance products in a consistent manner. Also, we will illustrate how to calculate RAPM by using a fair value framework.

We apply the EC framework using a specific example: a simple variable annuity (VA) with guaranteed minimum account benefit (GMAB). The paper also illustrates how to adjust EC for a longer time horizon and how hedging affects the EC and overall performance. We conclude by showing how the EC-based RAPM framework for the GMAB can be extended to evaluate various types of VA guarantees, such as a guaranteed minimum income benefit (GMIB). We also show how to increase the efficiency of capital usage.

1. Introduction

Economic capital (EC) and risk-adjusted performance measurement (RAPM) are becoming increasingly important criteria in product development and performance evaluation for insurance companies. Even though the regulatory capital requirement provides sufficient protection to policyholders, it fails to capture a company's "true" risk, which results in inefficient allocation of capital. Compared with regulatory capital requirements, the EC can improve the efficiency of capital usage since:

- EC captures the diversification benefits among business.

- EC allows the company to use firm-specific data and model to evaluate its own risk.

- EC can reflect all of the available financial resources that may not be captured under regulatory capital requirements, such as hedging instrument and the mark-to-market fair value of liabilities.

- EC's risk horizon corresponds to the company's budgeting process (which is usually done annually).

- EC provides a consistent framework under which the performance of different lines of business can be evaluated.

In spite of the promising benefits of EC, there are very few detailed case studies of its use in life insurance. The main purpose of this paper is to provide a concrete methodology for EC calculation. We use an example of a VA with a GMAB to illustrate the methodology. We also define a RAPM in this context. Finally, we discuss the critical issues to be resolved in order to apply the EC framework in a practical setting.

2. Economic Capital for the Variable Annuity with Guarantees

We use two random variables to calculate EC for the VA with guarantees: 1) fair value; and 2) loss function. Under the economic framework, both assets and liabilities should be valued under the market-consistent pricing measure. Determining the market value of most assets is straightforward, but there is no general methodology for calculating the fair value of a liability. We propose a methodology to calculate the fair value for EC modeling. And, then, we will show how to derive the loss distribution to calculate EC.

2.1 Valuation Formula for the VA with GMAB

The fair value is equivalent to the mark-to-market value at a certain time t. To provide the exact formula for it, we will use the following assumptions for the GMAB.

Underlying Stock

Total Return: $dS_t/S_t = \mu \cdot dt + \sigma \cdot dW_t$
where μ: the expected total return
σ: the volatility term for the *P-measure*
W_t: the standard Brownian motion process.

Fund Process

$F_t = F_0/S_0 \cdot S_t \cdot e^{-q \cdot t}$,
where fees are deducted proportionally from the fund at the continuous rate of
$q = m + \varepsilon + \delta$
and m: the management and expense fee
ε: the economic cost for the GMAB
δ: the spread to the GMAB charge.
The values of m, ε and δ are constants.

Actuarial Assumption

G: the guaranteed value
T: the maturity for the GMAB
ω^l: the continuous force of lapse
ω^d: the continuous force of mortality

Note that there is a spread (δ) to the GMAB charge. If the GMAB is priced within a no-arbitrage framework, δ should be zero; however, in practice, GMAB writers usually commit a spread over the market price of the GMAB as a cushion for market uncertainty. Under the assumption described above, the fair value (*FV*) for the VA with guarantees at time t is calculated as

$$FV(t) = E_Q[PV[Fee(t)]] - E_Q[[G - F_T]^+ \cdot e^{-r_f(T-t)}]$$
where $E_Q[\cdot]$: Expected value under the market consistent pricing measure or
Q-measure
r_f : Risk –free rate

T: GMAB maturity

$PV[Fee(t)]$: Present value of the fee income evaluated at time t.

For any type of guarantee, the analytical formula for the expected fees is:

$$E_Q[PV[Fee]] = E_Q[\int_t^T (\varepsilon + \delta) \cdot F_s \cdot e^{-r_t(s-t)} \cdot {}_{s-t}p_{x+t}^\tau ds] \cdot {}_t p_x^\tau$$

$$= E_Q[\int_t^T (\varepsilon + \delta) \cdot F_s \cdot e^{-r_t(s-t)} \cdot e^{-(\omega^l + \omega^d)\cdot(s-t)} ds] \cdot e^{-(\omega^l + \omega^d)t}$$

$$(\text{let } u = s - t)$$

$$= E_Q[\int_0^{T-t} (\varepsilon + \delta) \cdot F_{t+u} \cdot e^{-r_t \cdot u} \cdot e^{-(\omega^l + \omega^d)\cdot u} du] \cdot e^{-(\omega^l + \omega^d)t}$$

$$= [\int_0^{T-t} (\varepsilon + \delta) \cdot E_Q[F_{t+u}] \cdot e^{-r_t \cdot u} \cdot e^{-(\omega^l + \omega^d)\cdot u} du] \cdot e^{-(\omega^l + \omega^d)t}$$

$$= [\int_0^{T-t} (\varepsilon + \delta) \cdot F_t \cdot e^{(r_t - q)\cdot u} \cdot e^{-r_t \cdot u} \cdot e^{-(\omega^l + \omega^d)\cdot u} du] \cdot e^{-(\omega^l + \omega^d)t}$$

$$= [\int_0^{T-t} (\varepsilon + \delta) \cdot F_t \cdot e^{(r_t - q)\cdot u} \cdot e^{-r_t \cdot u} \cdot e^{-(\omega^l + \omega^d)\cdot u} du] \cdot e^{-(\omega^l + \omega^d)t}$$

$$= [\int_0^{T-t} (\varepsilon + \delta) \cdot F_t \cdot e^{-(q + \omega^l + \omega^d)\cdot u} du] \cdot e^{-(\omega^l + \omega^d)t}$$

$$= \frac{\varepsilon + \delta}{q + \omega^l + \omega^d}[1 - e^{-(q + \omega^l + \omega^d)(T-t)}] \cdot F_t \cdot e^{-(\omega^l + \omega^d)t}$$

where ${}_t P_x^\tau$: The probability that a policyholder aged x will stay in-force at time $x+t$.

For the benefit part, a VA with GMAB can be priced with the Black-Scholes-Merton option pricing formula because its payoff—excluding the decrements—is equivalent to a put option. With the decrement assumption, the GMAB can be priced with the following formula:

$$E_Q[[G - F_T]^+ \cdot e^{-r_f(T-t)}] = Put[F_t, G, T-t, q] \cdot e^{-(\omega^l + \omega^d)\cdot(T-t)} \cdot e^{-(\omega^l + \omega^d)\cdot t}$$

where $Put[S, X, T, d]$ is the Black-Scholes-Merton price formula given a stock price S, the exercise price X, the time to maturity T and dividend yield d.

It is important to stress that the FV of a GMAB typically decreases with time. As time-to-maturity decreases, cash inflows (GMAB fees) and outflows both decrease, but

the cash outflows decrease at a slower rate. Since the GMAB has no interim cash outflows before maturity, the expected cash outflow from the GMAB may exceed the expected cash inflows. As an extreme example, the FV immediately before maturity should be less than or equal to zero because there are no longer fees or charges but the company may need to pay the GMAB benefits. The following table shows this natural decrease in the FV of a VA with GMAB. Each value is the expected value at each time based on 5,000 simulations.

TABLE 1
Expected Fair Value at Various Time Horizons ($, Thousands)

Year	FV	PV[GMAB]	PV[GMAB Fee]
Year 0	72.51	95.28	167.80
Year 1	70.18	89.78	159.96
Year 2	66.95	83.73	150.68
Year 3	62.66	77.14	139.80
Year 4	56.92	70.16	127.07
Year 5	49.56	62.81	112.37

2.2 Loss Distribution and Economic Capital

The general definition of EC is a capital amount to cover future uncertainty over a risk horizon with a certain confidence level. To calculate EC, we have to define a random variable that can capture the future's uncertainty or generate the distribution of risks. Given the distribution for the random variable, we can determine EC based on a relevant risk measure such as value-at-risk (VaR) or conditional tail expectation (CTE). For example, the VaR can be defined as:

$$prob\big(Loss(t) \geq T\widetilde{R}]\big) < 1 - \alpha \quad (1)$$

where *Loss(t)*: a loss random variable at time *t evaluated under a realistic measure(P-measure)*

$T\widetilde{R}$: a target risk tolerance or value-at-risk
α : a confidence level

Loss distribution can be derived from the "return distribution" since the loss is equivalent to the negative return. The return, by definition, is the sum of the income gain and the capital gain. For the GMAB, income gain is defined as:

Income Gain = Income – Expense
Income = GMAB Charge + Interim Cash
Expense = GMAB + Commission & Expenses
Return = Income Gain + ΔFV
ΔFV = Change in Fair Value

Hence, loss can be calculated as

Loss(t)   = - Income Gain – ΔFV
= - Income Gain – [FV(t) – FV(0)]
= FV(0) – [Income Gain(t) + FV(t)]

Also, in this equation, embedded value (EV) at time t can be defined as

EV(t) = Income Gain(t) + FV(t)

With the definition of EV, the equation (1) becomes

$$prob\left(Loss(t) \geq T\widetilde{R}]\right) < 1 - \alpha$$
$$= prob\left(FV(0) - [Income\ Gain(t) + FV(t)] \geq T\widetilde{R}]\right) < 1 - \alpha$$
$$= prob\left(FV(0) - EV(t) \geq T\widetilde{R}]\right) < 1 - \alpha$$

This formula is exactly same as that for VaR calculation. Hence, EC can be calculated based on VaR as:

$$prob\left(EV(0) - EV(t) \geq EC(t,\alpha)]]\right) < 1 - \alpha$$
where *EV(0) = FV(0)*

EC based on the CTE can be calculated as

$$EC(t,\alpha) = \frac{\int_{-\infty}^{VaR} l \cdot f(l)dl}{1-\alpha}$$
where VaR: Value-at-Risk at a confidence level α
l: the loss random variable at time t
$f(l)$: the distribution function for the loss random variable l

Note that since income gain from the GMAB is path-dependent variable, there is no analytical formula to calculate EC for any type of guarantee, including GMAB. For this reason, a realistic simulation (*P-measure*) is required to calculate EC.

Given the economic value distribution, we can estimate the VaR for the VA with GMAB at each risk horizon. The risk horizon will be determined depending on a company's decision-making process. Instead of discussing which risk horizon is relevant, we introduce an issue related to risk horizons. Risk, by nature, does increase as the risk horizon gets longer. For example, the uncertainty over 10 years is clearly greater than that over one day or one year. For this reason, the confidence level for each different risk horizon should be adjusted according to target ratings and historical default probabilities. The following table shows the historical default probabilities by each rating group.

TABLE 2
Average Default Rates 1981 to 2004(%)[2]

Risk Horizon	1-Year	3-Year	5-Year
AAA	0.00%	0.04%	0.12%
AA	0.01%	0.08%	0.26%

The company may or may not directly use historical default probabilities as the confidence interval. However, it is clear that confidence interval should be adjusted for each risk horizon; otherwise risk may be either overestimated or underestimated compared to the "true" risk.

Another issue on the confidence interval is how we can estimate an extreme left tail value. As seen in the table, the historical confidence interval for AA group over one-year risk horizon is 99.99 percent. If we want to use this confidence interval and estimate the left-tail value based on the simulations, at least 100,000 simulations are required to estimate the left-tail value, requiring a huge amount of computing resources and time. Because of the resources required, we need to develop an alternative methodology to reduce the number of simulations. Extreme value theory or variance reduction techniques can be used for this purpose.

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[2] Source: Standard & Poor's Global Fixed Income Research.

2.3 Economic Capital with Hedging Portfolio

A hedging portfolio does not reduce required regulatory capital; however a hedging portfolio can substantially reduce VaR, so it can also reduce EC requirements. With a hedging portfolio, the loss is defined as:

$$Loss(t) = - \text{Income Gain} - \Delta FV - \Delta MV \text{ of Hedging}$$
$$= FV(0) - [\text{Income Gain}(t) + FV(t)] - \Delta MV \text{ of Hedging}$$
$$= FV(0) - EV(t) - \Delta MV \text{ of Hedging}$$
$$\text{where } \Delta MV \text{ of Hedging} = \text{Change in Market Value of Hedging Portfolio}$$

With the loss distribution, EC with a hedging portfolio can be estimated based on a relevant measure as explained in the previous section.

2.4 Risk-Adjusted Performance Measurement

As explained in Section 2.2, EV is the value created for the business at a certain time. Hence, values created from the product can be defined as

$$\text{Income Gain}(t) = \text{Income} - \text{Expense}$$
$$EV(t) = \text{Income Gain}(t) + FV(t)$$
$$\text{Economic Value Added}(EVA)(t) = EV(t) - \text{Cost of Capital}(t)$$
$$\text{where Cost of Capital}(t) = EC(t) \cdot [e^{c \cdot t} - 1]$$
$$c: \text{cost of capital}(\%)$$

The embedded value is sometimes called available capital or the amount of available financial resources. It is mainly composed of two parts: 1) realized income gain over the past; and 2) unrealized pricing margin which will be realized over the future. In the economic viewpoint, the two values are additive and a gain to shareholders. It is important to understand how EV affects performance measurements. We will demonstrate this with a simple example.

TABLE 3
Business Unit 1, Which Has Initial Pricing Margin

Year	0	1	2	3	4	5
FV	10	9	8	7	6	5
ΔFV	-	-1	-1	-1	-1	-1
Income Gain	-	1	1	1	1	1
Sum(Income Gain)	-	1	2	3	4	5
Return(=IG + ΔFV)	-	0	0	0	0	0
EV(=FV + Sum(IG))	10	10	10	10	10	10

TABLE 4
Business Unit 2, Which Has No Initial Pricing Margin

Year	0	1	2	3	4	5
FV	0	-1	-2	-3	-4	-5
ΔFV	-	-1	-1	-1	-1	-1
Income Gain	-	1	1	1	1	1
Sum(Income Gain)	-	1	2	3	4	5
Return(=IG + ΔFV)	-	0	0	0	0	0
EV(=FV + Sum(IG))	0	0	0	0	0	0

Tables 3 and 4 show that the fair value decreases with the passage of time, causing capital loss between each time. Due to the capital loss, total return for each time period becomes zero; for example, total return between time 0 and time 1 is composed of income gain(1) + capital loss(-1). If we evaluate the performance just based on income gain, both businesses create exactly the same value; however, in the economic sense, the two business units have different EV at each time (10 vs. 0), even though they have the same return and income gain. Actually, the first business has the positive financial resources at each time. This is because Business Unit 1 sold the same product at the higher price than Business Unit 2 did. The EV concept gives appropriate credit to Business Unit 1 for its superior performance. Hence, if we evaluate the performance based on the EV concept, we can get consistent performance measurement since we can evaluate how much value is added by a particular product regardless of time horizons. The concept of EV can be shown as:

FV   **Income Gain**

Accumulated Income Gain   *t*   **Discounted Future Value**
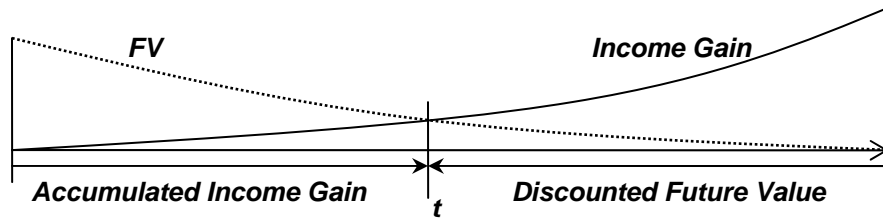
EV = Income Gain + FV

Accordingly, the concept of fair value is crucial to calculating the risk-adjusted return. FV exists due to the excess of fees over expected benefits and is gradually released as income over the remaining life of the product. An issue of performance measurement is how to assess FV. Under the fair value framework, this results in an initial unrealized capital gain and produces high volatility in the RAPM. To resolve this issue, we need to devise a smoothing methodology to reflect the economic sense of return. Recall that the EV is composed of: 1) realized income gain; and 2) unrealized pricing margin. It is clear that realized income gain should be added to the return from the business; the unrealized pricing margin will be realized over the remaining lifetime. Therefore, we can apply the amortization concept to reflect the remaining pricing margin as the return for each time horizon by:

FV on Risk-Adjusted Capital (FVORAC)
$$= [[1 + FV(t)/EC]^{1/\tau} - 1]$$
Where τ: Time-to-Maturity ($= T - t$)

This is an annual rate of return we can expect for the remaining life of time of the product. With the concept of FVORAC, various RAPMs are defined as:

Return on Risk-Adjusted Capital(RORAC)
  = Income Gain(t) / EC(t,α)
Adjusted Return on Risk-Adjusted Capital(Adj. RORAC)
  = RORAC + FVORAC
Risk-Adjusted Return on Risk-Adjusted Capital(RARORAC)
  = Adj. RORAC – Cost of Capital (%)

3. Case Study: VA with GMAB

In this section, we will provide the illustrative example for the GMAB. To provide a concrete example about EC and RAPM, we compare two businesses: 1) one with initial spread (1 percent) to the GMAB fee; and 2) the other with no spread. For simulation, we assume that there is no gain or losses from management and expense fees, so that we are evaluating risks only from the GMAB business. In addition, income gain for each period is accumulated at the risk-free rate.

TABLE 5
Valuation Assumptions for Each Business Unit

Underlying Asset	μ: 10%, σ: 15%, dividend: 1.5%, risk free rate: 5%, Implied Vol.: 20%	
Pricing Assumption	Premium: $1,000,000 GMAB Guarantee: 100%, Maturity: 10-year, M&E Fee: 1.5%	
	Business 1	Business 2
	GMAB Fee: 1.31%, Spread to GMAB Charge: 1% → GMAB Fee + Spread= 2.31% Total Fee: 3.81%	GMAB Fee: 0.98%, Spread to GMAB Charge: 0% → GMAB Fee + Spread= 0.98% Total Fee: 2.48%
Decrements	Mortality: 1%, Lapse: 2%	
Expense	Initial Commission: 5%, M&E Fee: 1.5%	
Number of Simulations	5,000 realistic scenarios for the underlying stock index	

Initial Pricing Margin

Based on the assumption above, each business has an initial pricing margin as follows:
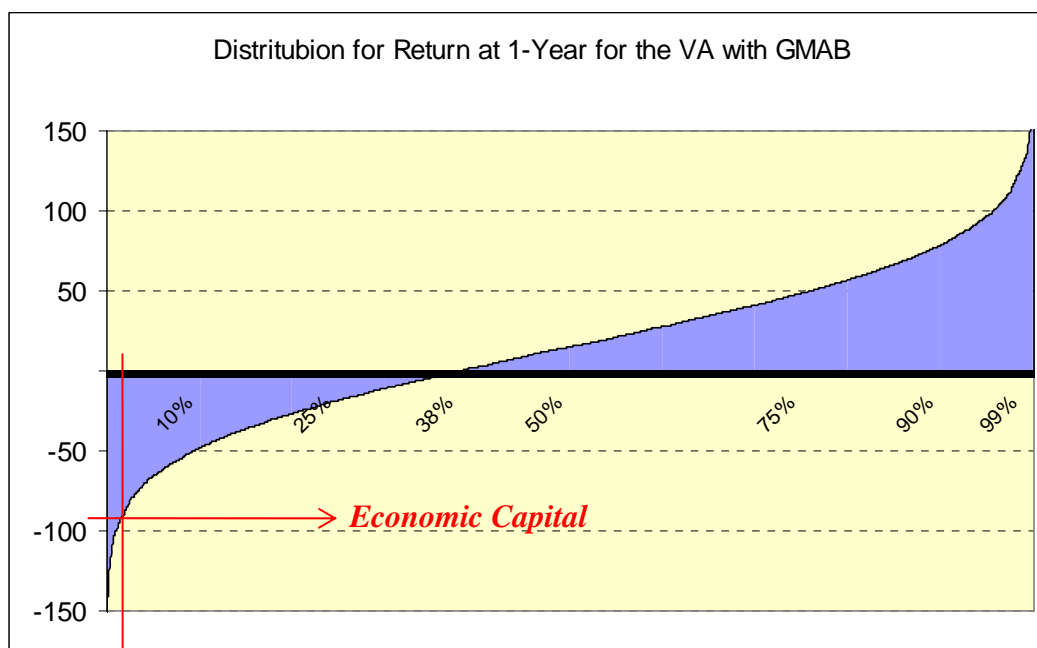
TABLE 6
Initial Pricing Margin for Each Business ($, Thousands)

	PV[Fee Income]	PV[GMAB]	Pricing Margin or FV(0)
Business 1	167.80	95.28	72.51
Business 2	75.23	75.23	0.00

Economic Capital and RAPM

As explained above, EC can be set based on the loss (return) distribution. The following chart shows an example for the BU1 over a one-year risk horizon.

Chart 1
Return Distribution at 1-year Horizon from the BU1

Distritubion for Return at 1-Year for the VA with GMAB

The following table summarizes EC and RAPM for each business unit. For simplicity, we use confidence intervals as follows instead of using an extremely high percentile value. Also, the result is based on 5,000 simulations and EC is determined by VaR measure. Income gain and fair value are the expected values across all of the 5,000 simulations.

TABLE 7
EC and RAPM for Each Business at Several Risk Horizons ($, Thousands)

Risk Horizon	1-Year		3-Year		5-Year	
C.L. (%)	99.00%		98.00%		97.00%	
Business Unit	BU1	BU2	BU1	BU2	BU1	BU2
EC	99.91	73.86	140.10	111.69	150.67	125.00
Income Gain	17.59	3.78	57.49	13.21	104.36	25.50
RORAC	17.60%	5.11%	12.14%	3.80%	11.10%	3.78%
FV(t)	70.18	2.64	62.66	6.71	49.56	7.86

Risk Horizon	1-Year		3-Year		5-Year	
C.L. (%)	99.00%		98.00%		97.00%	
Business Unit	BU1	BU2	BU1	BU2	BU1	BU2
FVORAC	6.09%	0.39%	5.42%	0.84%	5.85%	1.23%
EV	87.77	6.41	120.14	19.92	153.91	33.36
Adj. RORAC	23.69%	5.50%	17.57%	4.63%	16.95%	5.01%
Cost at 10%	10.51	7.77	49.01	39.08	97.74	81.09
EVA	77.26	-1.36	71.13	-19.16	56.17	-47.73
RARORAC	13.69%	-4.50%	7.57%	-5.37%	6.95%	-4.99%

Economic Capital with Hedging

The following table summarizes hedging analysis for the Business Unit 1 at the one-year risk horizon. The hedging amount is defined as the percentage of the initial GMAB price. In this example, the initial market value of the GMAB is $95.28 thousands. Also, for the hedging instruments, we use the at-the-money option with a constant implied volatility of 20 percent. The capital gain from the hedging instruments is added to the income gain.

TABLE 8
Hedging Analysis for BU1 with Different Hedging Budgets ($, Thousands)

Hedging (% of GMAB Fee)	0%	10%	20%	30%	40%
EC	99.91	93.33	86.75	80.17	73.36
Income Gain + Cap Gain from Hedge	17.59	16.28	14.98	13.67	12.37
RORAC	17.60%	17.45%	17.27%	17.06%	16.86%
FV(t)	70.18	70.18	70.18	70.18	70.18
FVORAC	6.09%	6.43%	6.81%	7.24%	7.74%
EV	87.77	86.47	85.16	83.86	82.55
Adj. RORAC	23.69%	23.88%	24.08%	24.29%	24.60%
Cost at 10%	10.51	9.82	9.12	8.43	7.72
EVA	77.26	76.65	76.04	75.42	74.84
RARORAC	13.69%	13.88%	14.08%	14.29%	14.60%

The hedging analysis shows:

- Hedging is effective under the EC framework since hedging portfolio increases capital efficiency; for example, over the one-year risk horizon, the RARORAC without hedging is 13.69 percent, while that with 40bps hedging is 14.60 percent.

- Since the hedging strategy requires a smaller amount of EC, the company can utilize this excess capital to generate more gains.

- If we use RORAC as the performance measurement, then the no-hedging strategy dominates other strategies; for example, RORAC with no-hedging is 17.60 percent, while that with 40-hedging is only 16.86 percent. This example demonstrates the importance to add FVORAC to the performance measurement. Adjusted RORAC or RARORAC can reflects the company's true economic value created from the business.

4. Economic Capital for other Types of Guarantees

Thus far, we have evaluated EC and have presented RAPMs for the VA with GMAB. The EC model for the GMAB can be expended to more complicated types of guarantees, such as guaranteed minimum withdrawal benefits (GMWBs) or guaranteed minimum income benefits (GMIBs). Recall that EC for the VA with guarantees is defined as
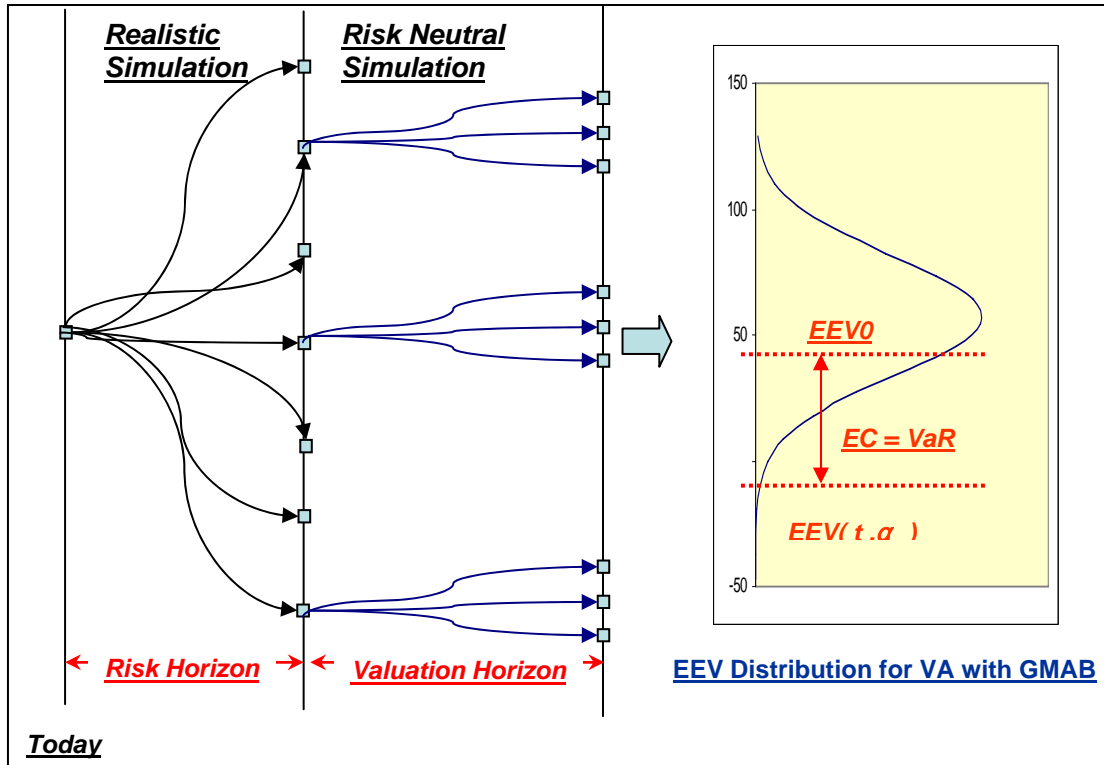
$$prob\big(FV(0) - [Income\ Gain(t) + FV(t)] \geq EC(t,\alpha]]\big) < 1 - \alpha$$

where *Income Gain(t)* is evaluated under a realistic measure(*P-measure*) and *FV(t)* is the fair value at time t evaluated under a risk-neutral measure (Q-measure).

For the GMAB, we did a simulation using the *P-measure* and calculated *FV(t)* based on an analytical formula. The basic idea for other types of guarantees is essentially same; however, the challenging issue for other types of guarantees is that there is no analytical solution for the fair value calculation because of the path-dependent property of benefit outflows (GMDB or GMWB) and the uncertainty from the future interest rates curve (GMIB). The fair value can only be calculated with a risk-neutral simulation (*Q-measure*). As a result, a two-tier simulation is required. The first tier uses realistic scenarios over the risk horizon, and the second tier uses risk-neutral scenarios from the risk horizon up to the product maturity or the valuation horizon. The chart below shows

this concept graphically. If we want to do risk analysis, such as a profit test based on the EC, we need one more outer realistic scenario for simulations.

Chart 2
Two-Tier Simulation Framework

Realistic Simulation   *Risk Neutral Simulation*

← Risk Horizon →   **← Valuation Horizon →**   **EEV Distribution for VA with GMAB**

Today

EEV0

EC = VaR

EEV(t .α)

One difficulty with the two-tier simulation framework is that pricing at a certain risk horizon requires a set of market-consistent risk-neutral scenarios at the risk horizon. The problem is that we may not have market instruments at the risk horizon, so it's difficult to calibrate and generate risk-neutral scenarios; hence, we have to devise a method to calibrate the risk-neutral scenario at each risk horizon.

5. Conclusion

This paper introduces a methodology to evaluate EC for VA with guarantees. Loss and fair value random variables are crucial to the calculation of EC. Using these two key random variables, we introduce how to evaluate EC for VA with GMAB, and we define various RAPMs. Also, we discuss some critical issues that must be resolved in order to apply the EC framework to real business. The model introduced in this paper can be extended to other life insurance products, such as VAs with other types of guarantees.