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The implementation of Bayesian Credibility models by a sampling-resampling technique.

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Abstract

The implementation of credibility theories is often hampered by the computational difficulties of evaluating integrals such as in Bayes' theorem. This paper presents a simple but powerful sampling-resampling technique which allows the assessment of the posterior distribution for arbitrary likelihood function and prior distribution.

Introduction

The Bayesian approach to credibility, which was introduced to actuaries in the pioneering work of Bailey (1945,1950) and later by Mayerson (1964), is now widely accepted. This is partly because Bayesian estimation offered, under particular assumptions, a sound justification to the famous credibility formula for the adjusted estimate of claims, i.e.,

$z.x + (1-z).m$, where z is the credibility factor, x the risk of an individual-a member of a risk collective- θ is the risk parameter and $m(\theta)$ is the fair premium of the individual. (see Bühlmann (1970) for further details). Given $f_X(x|\theta)$, the distribution of the risk, $g_\Theta(\theta)$, the prior distribution of θ (the so called structure distribution) and n years individual experience $\mathbf{x} = (x_1,...,x_n)$, the estimation of the fair premium of an individual risk is given by $E(m(\theta)|\mathbf{x})$. The calculation of the posterior mean requires the integration of the product $f_X(\mathbf{x}|\theta).g_\Theta(\theta)$ which is often difficult to obtain.

Fortunately,Bayesian analysis is both tractable and produces the desirable linear estimator for a wide range of models. This typically requires $f_X(x|\theta)$ to be a member of the exponential family and $g_\Theta(\theta)$ a conjugate prior to sampling from $f_X(x|\theta)$.(See Jewell (1974) Ericson (1969) and Diaconis & Ylvisaker (1979) for more details) The adoption of assumptions on the underlying distributions with a view to obtaining linear Bayes estimators led to criticism of the Bayesian approach. The difficulties in calculating the Bayesian estimator for arbitrary likelihood and prior distribution has directed research to alternative methods of calculations based on numerical and analytic approximations. Recent advances in this area [Naylor and Smith (1988),Smith et al (1985,1987),Tierney and Kadane(1986), shaw (1988) and Geweke(1988)] produced solutions to hitherto intractable problems. However, these methods require sophisticated expertise and therefore could not be used by most practitioners. In a recent paper Smith & Gelfand (1990) suggested a sampling-resampling technique for evaluating the posterior distribution which can be easily adopted by actuaries. The method is introduced in the next section.

The sampling-resampling approach.

The technique proposed here is aimed at generating a sample from the posterior distribution which can be used, if required,for estimating this distribution and evaluating the moments of the underying variables. The technique is based on the 'rejection method' (von Neumann (1951)) for generating random variates.The method which we now briefly outline is discussed in Devroy (1986),Ripley (1987) and Dagpunar (1988).

Suppose it is desirable to generate random variates X having a p.d.f $q_X(x)$ and c.d.f $Q_X(x)$. We need an arbitrary 'dominating' distribution with p.d.f $r_Y(y)$ and c.d.f $R_Y(y)$ and a known constant M such that $M = \max_X\{q_X(x)/r_Y(x)\}$.

Step 1-*Generation*-Generate Y from $r_Y(.)$ and U ,a random variable uniformly distributed on [0,1].

Step 2-*Acceptance*-If $U \le q_X(y)/[M.r_Y(y)]$ then X = Y. Go to step 1.

Step 3-*Rejection*-If $U > q_X(y)/[M.r_Y(y)]$ then go to step 1.

Smith and Gelfand (1991) suggested using this method for the generation of a posterior sample from a prior sample. From Bayes formula we have $g_\Theta(\theta|\mathbf{x}) \propto f_X(\mathbf{x}|\theta).g_\Theta(\theta) \propto l(\theta;\mathbf{x}).g_\Theta(\theta)$, where $l(\theta;\mathbf{x})$ is the likelihood of θ in the presence of the data $\mathbf{x}$. Let θ^* be the maximum likelihood estimate of θ and $M = l(\theta^*;\mathbf{x})$. Clearly,if we let $q'_\Theta(\theta) = l(\theta;\mathbf{x}).g(\theta)$ and $r_\Theta(\theta) = g_\Theta(\theta)$ and adopt the rejection method with $g_\Theta(\theta)$ as the dominating distribution and $M = \max_\theta\{q'_\Theta(\theta)/g_\Theta(\theta)\} = l(\theta^*;\mathbf{x})$,the generated sample is distributed $g_\Theta(\theta|\mathbf{x})$. Smith and Gelfand demonastrated that Bayes theorem can be regarded as a 'mechanism for generating a posteior sample from a prior sample' whereby each θ in the prior sample is accepted into the posterior sample with probability $q'_{\theta}(\theta)/[M.g(\theta)] = l(\theta;\mathbf{x})/l(\theta^*;\mathbf{x})$ which increases with the likelihood of θ.

In Makov (1991) the method is demonstrated in a rather complicated set-up where claim size is exponentially distributed and the prior distribution of the reciprocal of the mean claim size is a generalized inverse Gaussian. A comparison of the resampling technique with numerical integartion favours the former approach in terms of ease of implementation.

References.

Ahrens, J.H. and Dieter, U. (1974). "Computer methods for sampling from gamma,beta,Poisson and binomial distribution". Computing **12**, 223-46.

Bailey, A. L. (1945). "A generalized theory of credibility". Proc. Casual Actuar. Soc., **32**, 13-20.

Bailey, A.L. (1950) "Credibility Procedures, LaPlaces's generalization of Bayes' rule,and the combination of collateral knowledge with observed data" Proc. Casual. Actuar, Soc., **37**, 7-23.

Bühlmann, H. (1970). *Mathematical methods in risk theory.* Springer-Verlag. NY.

Dagpunar, J. (1988). *Principles of random variate generation.* clarendon Press. Oxford.

Devroy, L. (1986). *Non-uniform random variate generation.* Springer Verlag. NY.

Diaconis, P. and Ylvisaker, D. (1979). "Conjugate priors for exponential families". Ann. Statist.,7, 269-281.

Ericson, W.A. (1969). "A note on the posterior mean of a population mean". J. Roy. Statist. Soc.,B, **37**, 402-405.

Geweke, J. (1988). "antithetic acceleration of Monte Carlo integation in Bayesian Statistics". J. Econometrics, **38**, 73-90.

Jewell, W.S. (1974). "Credible means are exact Bayesian for exponential families#". ASTIN Bull.,**8**, 77-90.

Jorgensen. B. (1982). *Statistical properties of the generalized inverse Gaussian distribution*. Springer. NY.

Makov, U.E. (1991). "The implementation of Bayesian models by sampling-resampling technique", Submitted for publication.

Mayerson, A.L. (1964). "A Bayesian view of credibility",Proc. Casual. Actuar. Soc., **51**.

Naylor, J.C. and Smith, A.F.M. (1988). "Applications of a method for the efficient computation of posterior distributions". Appl. Statist., 31, 214-225.

Ripley, B.D. (1987) *Stochastic simulation*. Wiely, NY.

Shaw, J.E.H. (1988). "A quasirandom approach to integration in bayesian statistics". Ann. Statist., **16**. 895-914.

Smith, A.F.M.,Skene, A.M., Shaw, J.E.H. and Naylor, J.C. (1987). "Progress with numerical and graphical methods for Bayesian statistics". The Statistician, **36**, 75-82.

Smith, A.F.M. and Gelfand, A.E. (1991). "Bayesian statistics without tears: a sampling-resampling perspective". To appear in the American Statistician.

Tierney, L. and Kadane, J. (1986). "Accurate approximations of posterior moments and marginal densities". J. Amer. Statist. Assoc.,**81**, 82-86.