ACTUARIAL RESEARCH CLEARING HOUSE
1993 VOL. 3

A CHANCE CONSTRAINED PROGRAMMING APPROACH TO PENSION PLAN MANAGEMENT

by

Patrick L. Brockett

Abraham Charnes

Li Sun

A CHANCE CONSTRAINED PROGRAMMING APPROACH TO PENSION PLAN MANAGEMENT

1. Introduction

The objective of providing retirement income is becoming increasingly important for both workers and employers. One reason for this is the fact that most people can anticipate living long enough to enjoy their retirement life. Indeed, life tables for the United States show that the life expectancy at all ages exceeds the typical retirement age, and often the probability of survival to age 65 considerably exceeds the probability of death. The development of a good retirement plan is also important to employers because a good pension plan can be useful in attracting and keeping excellent employees.

Recently, providing retirement income for employees through the employee benefit mechanism has become a very common practice with more and more companies providing a pension plan. There are, of course, several employee benefit plans available to choose among in order to provide retirement income. These include pension plans, profit-sharing plans, thrift or saving plans, employee stock ownership plans, plans for the self-employed, group individual retirement account plans, simplified employee pension plans, tax-deferred annuities and others. For the purpose of this paper, and in order to show how one can define a model to trade off the various investment goals, tax considerations and contribution factors involved, we will restrict ourselves to the analysis of defined benefit plans.

Every employee wants more retirement income, and most employers would like to use as little money as possible in order to provide the best possible pension plan. The employer's goal involves many uncertain factors, such as the life expectancy of employees for ages after retirement; the pension funds; returns on investments which will accrue to money markets, stocks or bonds and the rates of the changes in cost of living, salaries and inflation. How to optimally decide on contribution levels necessary in order to satisfy the employer's conflicting goals is a complicated problem and is the subject of this article.

The existence of uncertainty (randomness) causes further complications in the analysis. The usual method for handling uncertainty in actuarial analysis is to take expected values and proceed as if the process were deterministic. If the probability distribution under investigation is symmetric, there is a 50% chance of falling below or above this expected value, and for pension plans this can necessitate remedial funding actions at the end of the year in order to preserve qualification. In this article, we introduce a new policy-making model for pension plan funding which openly represents the uncertainty. The word "policy" is used here to indicate a strategy which is intended to be followed most (but not necessarily all) of the time. In unusual circumstances, the policy will have to be modified at the end of the year to retain qualification; however, the frequency with which this remedial action is needed is controllable by the actuary (unlike the method of using expected values). The simple substitution of expected values of random quantities produces an unknown frequency of last minute remedial actions at possibly disastrous investment costs to the pension plan. However, with our techniques, once the actuary acknowledges the uncertainty associated with the various aspects of the pension plan, he/she can explicitly trade off the costs associated with remedial actions, tax penalties, etc. with the investment returns in order to meet the pension plan goals.

The method that this paper introduces is a new Chance-Constrained Programming approach to selecting annual contribution levels and asset allocation (holding other factors fixed) that we believe will better fit the employer's goals while recognizing the many risks and constraints associated with defined benefit plans. The chance-constrained programming approach can require less funding from the employer while giving more benefits to employees, can provide a better guarantee that most employees will have the expected retirement income at their normal retirement dates, and can offer greater reduced benefits for those electing early retirement. While the model presented in the paper does not incorporate all the intricacies of pension funding, like any good model, it suggests actions or policies and simultaneously points the way towards more innovative and new approaches to previously investigated problems. Further "bells and whistles" extensions can be provided in a sequel paper. The purpose of this paper is to concisely present this entirely new approach and examine its ramifications.

2. Some Concepts for the Defined Benefit Plan

A defined benefit plan is one that provides participants with a specified defined benefit at retirement [7]. The essence of this type of plan is that the retirement benefit is a stated or fixed factor, while the contributions to the plan necessary to produce this defined benefit are the variable factors. With such a plan, employers can utilize pension plan benefit formulas to achieve specific retirement income benefit targets in a relatively accurate fashion. Such plans also may be easier for employers to administer because contributions and investment income do not have to be allocated to individual employee participants.

As mentioned above, the retirement benefit is the fixed factor, and benefit formulas are used to calculate it. For the defined benefit pension plan, the basic formulas usually correspond to some combination of benefits of a flat amount, a flat percentage of final average earnings, a flat amount per year of service or a flat percentage of earnings per year of service.

This paper will consider retirement benefits that correspond to "the fixed percentage of earnings per year of service" integrated with Social Security rules and participation. The generalization to other, more complicated plans can be addressed by techniques similar to those discussed herein.

Under "the fixed percentage of earnings per year of service," the monthly retirement benefit equals a fixed percentage times the number of years of credited service. For example, if a plan provides for a 1½% benefit for an employee who has 25 years of credited service and average earnings of $2,000 per month, the monthly retirement benefit would be $750 (1.5% × $2,000 × 25).

Combining Social Security with the benefit formula could help redress the current biasing of Social Security benefits in favor of the lower paid workers. Through integration with social security benefits, a more reasonable, relatively larger benefit can be paid to certain higher paid employees.

There are three basic methods used to integrate pension plans with Social Security: (1) the offset method, (2) the excess method and (3) the step-rate excess method. Most private plans use the offset method together with the fixed percentage of earnings per year of service. For instance, a private plan might provide a monthly benefit (formula) of: (1) 1½% of the Average Monthly Compensation, multiplied by the number of years of vested service, minus (2) 50% of the Primary Social Security benefit.

For tax purposes, employer contributions to a pension fund are viewed as a necessary and reasonable business expense; therefore, the employer contributions are tax-deductible, provided that the contributions meet certain IRS guidelines. In such situations, the plan is called a qualified pension plan. To be a qualified defined benefit plan, the plan must provide enough money for the pension fund to be able to operate; i.e., it must provide a minimum fund. In order to discourage the employer from using the pension fund as a tax shelter for company profits, the IRS also has a regulation which sets an upper limit on contributions. If an employer contributes more to a defined benefit plan and exceeds the limits specified by the IRS, then this excess contribution must be carried over and deducted in later years. As a practical consequence of this rule, employers usually do not want to contribute more than the tax deductible amount in a given year. In addition, a qualified pension plan cannot discriminate in favor of officers, stockholders or highly compensated employees in terms of contributions or benefits. Such a qualified plan should meet either of two requirements--the "70-percent rule" or the"discretionary rule." All of these requirements are represented in our model's constraints.

For the numerical illustrations in this paper, we shall use the 1983 Group Annuity Mortality Table with separate rates for males and females in order to model the probability structure of mortality. Clearly, other lifetables could be used in conjunction with our technique if deemed more appropriate for a particular employee benefit application.

A final concept we shall consider in our analysis is the method of investment used for the contributions to the plan. For simplicity, we shall consider only three investment vehicles: a stock or equity fund, a diversified bond fund and a money market fund for short-term investments. To simplify the model, we shall assume that we do not transfer money between funds after the initial allocation of the employer's contributions each year.

3. Structure of Chance-Constrained Pension Funding Model

In this section, we shall show how to use a multivariate objective function with chance-constraints to produce a model for determining funding and investment strategy for an n-year defined benefit pension plan.

The Objective Function

For defined benefit plans, the employer desires to use as little money as possible while providing a qualified pension plan to the employees. Clearly, minimizing the cost of the plan is one of the components in the employer's objective function. These employer contributions are also subject to several internally and externally imposed constraints including those resulting from IRS regulations (such as those of the Employee Retirement Income Security Act (ERISA) of 1974). Actually, because the corporate decision to commit funds to the defined benefit plan is an investment decision, the money commitment must be made wisely and in consistency with other internal corporate objectives. It is not always true that the smaller the contribution the better.

In a wise design of pension plan contributions, the employer would want to take advantage of tax deductibility and perhaps also accumulate assets in the fund for future costs. Moreover, because the assets of a qualified plan can grow tax-free, some motivation exists for the employer to overfund in order to shelter current income. Overfunding, however, can lead to undesirable social consequences such as plan terminations with the employer reclaiming the excess funds in a later, perhaps lower tax year period. These large build-ups of overfunded plans can also lead to termination by a hostile corporate raider who can then use the excess pension funds to capitalize the take-over. It has been estimated that there are over $600 billion in overfunded pension funds in the U.S. [2][1].

For the preceeding reasons, there is a limit to the tax-deductibility of contributions. No deduction for tax purposes is permitted for contributions that exceed certain specified limits. Thus, a first part of the employer's objective might be to maximize the expected plan assets (ending balance), subject to constraints on the tax-deductible pension contributions. On the other hand, as mentioned previously, the employer would want to minimize the expected cost. Considering the limits on the tax-deductible pension contributions, the expected cost would be the total contributions in n years plus the product of the tax rate with the contributions that exceed the tax-deduction limit. Thus, an employer's second objective may be to minimize the sum of the contributions plus the expected tax paid on the amount exceeding the maximum funding requirement.

Because the development of an appropriate objective for the pension plan is intrinsically

---

[1] It was pointed out to one of the authors (Patrick L. Brockett) in a personal communication by Dr. James Hickman [Obberwolfach Germany Risk Theory Seminar, (1990)] that nominal overfunding in pension plans occurs when there are higher real interest rates and relatively lower salary rate changes (such as occurred in the 1980s). This is because of the formulas relating salary to promised future benefits. Similarly, when interest rates are low and salary rate changes appear relatively higher, pension fund liabilities will appear larger than the accumulated value of current assets plus premiums; i.e., pension funds will appear to be "underfunded" as in the late 1970s. In any case (whether real or perceived) the under- or over-funding of pension plans can create adverse incentives and should be controlled.

multidimensional, the model we use involves a vector or multi-objective optimization. The ordering relation on the plane R^2 corresponding to the vector extremization process is that of Pareto optimality; i.e., no improvement is possible in one dimension without causing a sacrifice in another dimension.

Put mathematically, the multi-objective may be written as:

Determine the value of contributions each year, $W = (W_1, \ldots, W_n)$, so as to achieve the vector maximum (V-Max) of:

$$(1) \qquad V\text{-}MAX\ (f_1(W),\ -f_2(W)).$$

Here the first component in the maximization, $f_1(W)$, corresponds to the ending balance in the pension fund:

$$(2) \qquad f_1(W) = E(Y_n),$$

where E denotes the mathematical expectation operator and Y_n denotes the ending balance of the pension plan in the nth year; i.e., the nth year's plan assets. This ending balance depends, of course, on the amounts of contributions, the experienced investment returns and the normal cost of the plan.

In order to delineate this relationship further, let Y_0 be the initial funding (a known constant), and let Y_i denote the ith year's ending balance. These balances may be further broken down into the amount of funds invested in each of the three investment vehicles described earlier. Thus, we set:

(3) $$Y_0 = \sum_{j=1}^{3} Y_{0j} ,$$

where the subscript j indicates the jth investment (j = 1, 2, 3). Thus:

(4) $$Y_i = \sum_{j=1}^{3} Y_{ij} , \qquad i = 1,2,\ldots,n.$$

Further, the financial dynamics between year i and i-1 give the balance sheet identity:

(5) $$Y_{ij} = (Y_{i-1\,j} + W_{ij})\times(1 + r_{ij}) - 1/3\times(\xi_i + m_i + c_i),$$

where r_{i1} = the rate of return on stocks at the ith year,

r_{i2} = the rate of return on bonds at the ith year,

r_{i3} = the rate of return on money market at the ith year,

ξ_i = the actual cost of the plan at the ith year,

m_i = the minimum amortization charges of the ith year,

c_i = the annual expense assumption of the ith year, such as trustee fees, recordkeeping fees and legal fees.

W_{ij} is the amount of the employer's contribution which is invested in the jth type investment fund in the ith year, so that:

$$W_i = \sum_{j=1}^{3} W_{ij}, \quad i = 1,2,\ldots,n$$

and r_{i1}, r_{i2}, r_{i3} and ξ_i are random variables. The expected value of ξ_i is the normal cost of the plan (cf., [5] pg. 266), which represents the allocation of costs to year i if all the actuarial assumptions hold exactly. The variables c_i and m_i are constant (i=1, 2, ..., n), and the final term (involving the factor of 1/3 for each fund) is the cost of paying benefits and expenses, which we assume to be paid equally from each of the three funds.

The value of the first component of the objective function may be increased by increasing the contribution levels ($W_1, W_2, \ldots, W_n$); however, this action increases costs and risks potential non-deductibility of the contributions. This leads to the trade-off encountered in the second component of the objective function.

The second component of the objective function (1) provides for minimizing plan costs and is expressed via the cost function:

$$(6) \qquad f_2(W) = \sum_{i=1}^{n} W_i + \textit{Expected cost of exceeding tax deduction limit.}$$

The second part of (6) can be expressed further as:

$$(7) \qquad \sum_{i=1}^{n} t_i \max \{W_i - E(\xi_i) - c_i - M_i + E(B_i),\ 0\},$$

where t_i is the tax rate during the ith year, c_i is the expense assumption for year i, B_i is the balance in the funding standard account for the ith year (to be discussed subsequently), $E(\xi_i)$ is the normal cost for current service and M_i is the maximum allowable amortization charge for year i. Clearly, the recursion relations:

$$B_1 = B_0,\ B_0 \textit{ is a known constant,}$$

$$B_i = B_{i-1} - \xi_{i-1} - m_{i-1} - c_{i-1} + \sum_{j=1}^{3} W_{i-1,j}\,(1 + r_{i-1,j}),$$

hold where m_i is the minimum amortization charge for year i. Equation (7) can be interpreted to mean that when contributions are less than the maximum funding requirement, there is no tax paid on the contribution. Tax need only be paid on the amount that exceeds the maximum funding requirement, and this maximum funding standard depends upon the contribution in relation to the normal cost, the maximum allowable contribution and the funding standard account.

For the purpose of using mathematical programming to obtain optimal funding amounts, it is useful to rewrite the expected plan cost equation for year i:

$$\max \{W_i - E(\xi_i) - c_i - M_i + E(B_i),\ 0\}$$

as

$$= \frac{1}{2}\{|W_i - E(\xi_i) - c_i - M_i + E(B_i)| + W_i - E(\xi_i) - c_i - M_i + E(B_i)\}.$$

Accordingly, the second objective can be rewritten as:

$$\sum_{i=1}^{n} W_i + \frac{1}{2}\sum_{i=1}^{n} t_i\{|W_i - E(\xi_i) - c_i - M_i + E(B_i)| + W_i - E(\xi_i) - c_i - M_i + E(B_i)\}.$$

In our chance-constrained programming formulation, the stochastic decision rules provide for the determination of the W_{ij}. These rules need to be "informationally feasible" in the sense that they are functions of the previous observations and the decision rules used in the previous periods. In a sense of Bayesian updating of the basic equation if another possibility which is "informationally feasible." Our chance-constrained method does not require the assessment of subjective "prior" distributions, and more importantly, is directed towards optimal behavior in most (but not all) situations. The Bayesian methods use means of posterior distributions in an "average" technique which might require last minute remedial funding action at the end of the year in order to preserve qualification of the plan, and this can occur an undetermined proportion of the time in the Bayesian setting. The "learning" aspects of Bayesian updating can be incorporated into our model by using higher order decision rules and conditional probability distributions. Note that once the W_{ij}s are determined, the employer contribution level $W_i = \Sigma W_{ij}$ and the asset allocation levels ΣW_{ij} are determined. The second component of the objective function is a convex function of the W_{ij}, so that minimizing this objective results in a convex programming problem.

Further Constraints

a. Minimum Funding Requirement

One of the significant regulatory requirements imposed by ERISA is the minimum funding requirement. Under this requirement, the minimum annual amount that the employer may contribute to a qualified plan equals the sum of the annual normal cost for current service, the annual amortization payment of initial unfunded past service liability (cost relating to participants' service before establishment of the plan) amortized over 30 years, the annual amortization payment of increases or decreases in past service liability due to plan amendments amortized over 30 years, the annual amortization of gain and/or loss experience amortized over 15 years, the annual amortization of gains and/or losses resulting from changes in actuarial assumptions amortized over 30 years, the annual amortization of any waived contributions amortized over 15 year, and the annual amortization of certain other amounts. In this model, we shall consolidate all of these amortization costs together into a single item we shall call the amortization charge. Although the actual implementation of this procedure could easily treat these components individually, we treat them collectively here for simplicity of presentation.

If all variables involved in the analysis were deterministic, the minimum funding requirement would be:

$W_i \geq \xi_i + m_i + c_i - B_i$. However, because ξ_i is a random variable, instead of specifying that W_i must exceed the actual cost and minimum amortization charges under all conceivable situations in our formulation, we require instead that $W_i \geq \xi_i + m_i + c_i - B_i$ at least some prespecified proportion of the time (say 95%). Thus, in place of a deterministic minimum funding constraint, we should have a chance constraint of the form:

(8) $P\{W_i \geq \xi_i + m_i + c_i - B_i\} \geq \alpha_i, \qquad i = 1, 2, \ldots, n,$

where α_i is some preassigned probability, $0 \leq \alpha_i \leq 1$. The parameter α is essentially an index of reliability of the funding policy determined using our method. This is the notion of chance-constrained programming invented by Charnes and Cooper (cf., [3]). This notion has also been used by McCabe and Witt [4]. When the precise unveiling of the random variables occurs, it is possible that the constraint (8) will not be satisfied (in fact, this is expected to occur α% of the time). In these situations, remedial action outside the model is necessitated. This same situation occurs in traditional actuarial techniques; however, the important difference here is that the funding and investment policy is constructed such that this does not occur too often. In fact, if one required equation (8) to hold all the time (with probability 1) and if ξ_i was an unbounded random variable, then the funding level W_i would also need to be unbounded. In essence, the tail of ξ_i, no matter how unlikely, would be the determining factor for W_i. By introducing the probability constraint instead of a deterministic constraint, we keep "the tail from wagging the dog."

b. Limits on Employer Tax Deductions

The IRS imposes limits on the amount of tax-deductible annual employer contributions to defined benefit plans. The tax deduction is limited to the normal cost for the year plus the maximum amortization charges. The employer will pay tax on contributed amounts which exceed this maximum funding requirement. The second term in the vector optimization formulation of the chance-constrained programming model for pension funding (1) incorporates this tax-induced charge.

Summarizing, the mathematical model with funding level decision variables (W_{ij}) is to determine the vector maximum of the function:

$$V\text{-}MAX\ (E(Y_n),\ -(\sum_{i=1}^{n} W_i + \frac{1}{2}\sum_{i=1}^{n} t_i\{|W_i - E(\xi_i) - c_i - M_i + E(B_i)| + W_i$$

$$- E(\xi_i) - c_i - M_i + E(B_i)\})),$$

subject to the delineated constraints:

$$P\{W_i \geq \xi_i + m_i + c_i - B_i\} \geq \alpha_i,$$

(9)

$$W_i = \sum_{j=1}^{3} W_{ij},$$

$$Y_0 = \sum_{j=1}^{3} Y_{0j},$$

$$Y_i = \sum_{j=1}^{3} Y_{ij},$$

$$Y_{ij} = (Y_{i-1,j} + W_{ij})(1 + r_{ij}) - 1/3(\xi_i + m_i + c_i),$$

$$B_i = B_{i-1} - \xi_{i-1} - m_{i-1} - c_{i-1} + \sum_{j=1}^{3} W_{i-1j}\,(1 + r_{i-1j}),$$

$$W_{ij} \geq 0,\ Y_{0i} \geq 0,\ 0 \leq \alpha_i \leq 1.$$

Within this formulation, the variables r_{ij} and ξ_i are allowed to be random (i=1, 2, ..., n and j=1, 2, 3. This specifies the chance-constrained bi-objective n-stage investment model we shall use for pension funding and investment vehicle levels. We shall explicitly investigate the case of zero order stochastic decision rules for W_{ij}, whose selection does not depend on unknown variables; however, generalization to linear decision rules is also possible. We shall discuss the mathematical treatment in the next section.

Mathematical Treatment

In this section, we transform the previously described stochastic model into an equivalent deterministic form which lends itself to simple numerical solutions. We use the Charnes-Cooper test [3] for vector extremality to convert the bi-objective problem to a set of equivalent single dimensional objective problems.

The chance-constraints in model (9) may be converted into deterministic equivalents. This can be done regardless of the precise probability distributions involved, as long as the distribution functions are known. In this paper, for illustrative purposes, we assume that ξ_i and r_{ij} are mutually independent random variables with known normal distributions, and we take the W_{ij} as zero order stochastic decision rules. To see how this conversion to deterministic equivalent constraints is accomplished, we first convert the constraints (8) to their deterministic equivalents. The first component of the objective function $E(Y_n)$ can be expressed in a similar fashion.

By the equation:

$$B_1 = B_0,$$

we can reduce this system to:

$$B_i = B_{i-1} - \xi_{i-1} - m_{i-1} - c_{i-1} + \sum_{j=1}^{3} W_{i-1,j}\,(1 + r_{i-1,j}),$$

$$B_1 = B_0,$$

$$B_2 = B_0 - \xi_1 - m_1 - c_1 + \sum_{j=1}^{3} W_{1j}\,(1 + r_{1j}),$$

$$B_3 = B_0 - \xi_1 - m_1 - c_1 + \sum_{j=1}^{3} W_{1j}\,(1 + r_{1j}) - \xi_2 - m_2 - c_2 + \sum_{j=1}^{3} W_{2j}\,(1 + r_{2j})$$

$$= B_0 - \sum_{k=1}^{2} \xi_k - \sum_{k=1}^{2} m_k - \sum_{k=1}^{2} c_k + \sum_{k=1}^{2}\sum_{j=1}^{3} W_{kj}\,(1 + r_{kj}),$$

$$B_i = B_0 - \sum_{k=1}^{i-1} \xi_k - \sum_{k=1}^{i-1} m_k - \sum_{k=1}^{i-1} c_k + \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}\,(1 + r_{kj})$$

$$= B_0 - \sum_{k=1}^{i-1} \xi_k - \sum_{k=1}^{i-1} m_k - \sum_{k=1}^{i-1} c_k + \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj} + \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}\, r_{kj}.$$

Let us define:

$$\chi(i) = \sum_{k=1}^{i} \xi_k - \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}\, r_{kj},$$

$$m(i) = \sum_{k=1}^{i} m_k,$$

$$c(i) = \sum_{k=1}^{i} c_k.$$

The constraints become:

$$W(i) = \sum_{k=1}^{i}\sum_{j=1}^{3} W_{kj} = \sum_{k=1}^{i} W_k,$$

$$P\{W_i \geq \xi_i + m_i + c_i - B_0 + m(i-1) + c(i-1) + \chi(i) - \xi_i - W(i-1)\} \geq \alpha_i,$$

or:

$$P\ \{\chi(i) \leq W_i + W(i-1) - m(i) - c(i) + B_0\} \geq \alpha_i,$$

or:

(10) $$P\ \{\chi(i) \leq W(i) - m(i) - c(i) + B_0\} \geq \alpha_i.$$

Since ξ_i, $W_{ij}r_{ij}$ are mutually independent random variables having,

respectively, the normal distributions $N(\mu_{\xi_i}, \sigma_{\xi_i}^2)$, $N(W_{ij}\mu_{r_{ij}}, W_{ij}^2\sigma_{r_{ij}}^2)$, *the*

random variable $\chi(i) = \sum_{k=1}^{i} \xi_k - \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}\, r_{kj}$ *is normally distributed with*

mean $\sum_{k=1}^{i} \mu_{\xi_k} - \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}\, \mu_{r_{kj}}$ *and variance* $\sum_{k=1}^{i} \sigma_{\xi_k}^2 + \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}^2\sigma_{r_{kj}}^2$, *so*

that $\chi(i)$ *is* $N(\sum_{k=1}^{i} \mu_{\xi_k} - \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}\, \mu_{r_{kj}}, \sum_{k=1}^{i} \sigma_{\xi_k}^2 + \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}^2\sigma_{r_{kj}}^2)$.

After standard normalization, (10) may be rewritten as:

$$P\{Z \le \frac{W(i) - m(i) - c(i) + B_0 - E(\chi_i)}{\sigma_{x(i)}}\} \ge \alpha_i,$$

where $Z = \dfrac{\chi(i) - E(\chi(i))}{\sigma_{x(i)}}$ *is a standard normal random variable; i.e.,*

$$\phi\{\frac{W(i) - m(i) - c(i) + B_0 - E(\chi_i)}{\sigma_{x(i)}}\} \ge \alpha_i,$$

where ϕ denotes the N(0,1), the standard Normal N(0,1) distribution function,

$$or \quad \frac{W(i) - m(i) - c(i) + B_0 - E(\chi_i)}{\sigma_{x(i)}} \ge \phi^{-1}(\alpha_i)$$

Therefore, the chance-constraint (10) is equivalent to the deterministic constraint:

$$(11) \qquad W(i) - m(i) - c(i) + B_0 - E(\chi_i) \ge \sigma_{x(i)}\phi^{-1}(\alpha_i).$$

(Note that this process would have worked for any two parameter location-scale family of distributions, not just the normal distribution. More general distributions are also possible as well.)

Although (11) could be used directly, we have found that the computation involved in using standard nonlinear optimization computer codes can be further simplified by introducing the so called "spacer" variable v_i to separate inequality (11) into two inequalities:

$$W(i) - m(i) - c(i) + B_0 - E(\chi_i) \ge v_i \ge \sigma_{x(i)}\phi^{-1}(\alpha_i).$$

The v_i variable separates the double inequality into a linear inequality plus a non-linear inequality; i.e.,

$$W(i) - m(i) - c(i) + B_0 - E(\chi_i) \geq v_i$$

$$\sigma_{x(i)}\phi^{-1}(\alpha_i) \leq v_i.$$

If α_i is sufficiently large so that $\phi^{-1}(\alpha_i) > 0$, then vi > 0, and, since $v_i \geq 0$, the constraint $\phi^{-1}(\alpha_i)\sigma_{X(i)} \leq v_i$ can be rewritten as:

$$\sigma_{x(i)}^2[\phi^{-1}(\alpha_i)]^2 - v_i^2 \leq 0.$$

This quadratic constraint in the W_{ij} with $v_i \geq 0$ defines a convex set in these variables. Also, the function $E(Y_n)$ is linear in the W_{ij}, thereby achieving a bi-objective convex programming problem.

Using the Charnes-Cooper test [3] and the Ben-Israel, Ben-Tal and Charnes theory [1], the vector maximization of the bi-objective function $(f_1(W), -f_2(W))$ is equivalent to the univariate extremization $(\lambda_1 f_1(W) - \lambda_2 f_2(W))$, where $\lambda_i \geq 0$, $\lambda_1 + \lambda_2 = 1$ in the sense that each of the extremum (under Pareto optimality of the bi-objective function) corresponds to the univariate extremum for some choice of λ. Depending upon the relative priority given to the two objectives in the vector objective function, different sets of λ_i may reveal different vector extremal decision rules. From these, we can choose the decision rule which best fits the employer's goals to contribute to the pension fund in a manner which maximizes the security of achieving the highest possible asset value and which minimizes the costs associated with the plan.

Using our previously developed formulas for f_1 and f_2 in the univariate equivalent maximization problem results in the objective:

$$\lambda_1 E(Y_n) - \lambda_2 (\sum_{i=1}^{n} W_i + \frac{1}{2}\sum_{i=1}^{n} t_i \{ | W_i - E(\xi_i) - c_i - M_i + E(B_i) | + W_i - E(\xi_i)$$

$$- c_i - M_i + E(B_i)\}),$$

which is a concave function of W_{ij}. Maximizing this concave function subject to the convex constraints gives us a convex programming problem.

This problem may be further simplified by replacement of B_i by:

$$W_i - E(\xi_i) - c_i - M_i + E(B_i)$$

$$= W_i + W(i-1) - M(i) - c(i) - B_0 - E(\chi(i)) = W(i) - M(i) - c(i) - B_0 - E(\chi(i)),$$

$$where\ M(i) = \sum_{k=1}^{i} M_k.$$

Let us now find an appropriate reformulation for Y_n. From:

$$Y_i = \sum_{j=1}^{3} Y_{ij} \quad and$$

$$Y_{ij} = (Y_{i-1j} + W_{ij})(1 + r_{ij}) - 1/3(\xi_i + m_i + c_i),$$

we have:

$$Y_{1j} = (Y_{0j} + W_{1j})(1 + r_{1j}) - 1/3(\xi_1 + m_1 + c_1)$$

$$= Y_{0j} \times (1 + r_{1j}) + W_{1j} \times (1 + r_{1j}) - 1/3 \times (\xi_1 + m_1 + c_1),$$

$$Y_{2j} = [Y_{0j}(1+r_{1j}) + W_{1j}(1+r_{1j}) - 1/3(\xi_1 + m_1 + c_1)](1+r_{2j}) + W_{2j}(1+r_{2j}) - 1/3(\xi_2 + m_2 + c_2)$$

$$= Y_{0j}\ (1 + r_{1j})(1 + r_{2j}) + W_{1j}(1 + r_{1j})(1 + r_{2j}) + W_{2j}\ (1 + r_{2j})$$

$$- 1/3\,(\xi_1 - m_1 - c_1)(1 + r_{2j}) - 1/3\,(\xi_2 + m_2 + c_2),$$

$$Y_{ij} = Y_{0j}\prod_{k=1}^{i}(1 + r_{kj}) + \sum_{k=1}^{i} W_{kj}\prod_{h=k}^{i}(1 + r_{hj}) - 1/3\sum_{k=1}^{i}\xi_k\prod_{h=k+1}^{i}(1 + r_{hj})$$

$$- 1/3\sum_{h=k+1}^{i} m_k\prod_{h=k+1}^{i}(1 + r_{hj}) - 1/3\sum_{k=1}^{i} c_k\prod_{h=k+1}^{i}(1 + r_{hj}).$$

Let us define:

$$\prod_{h=k}^{i}(1 + r_{hj}) = R_k(i,j), \quad \text{if } k > i, \quad R_k(i,j) = 1,$$

$$\sum_{k=1}^{i} W_{kj}R_k(i,j) = RW(i,j),$$

$$\sum_{k=1}^{i} \xi_k R_{k+1}(i,j) = R\xi\,(i,j),$$

$$\sum_{k=1}^{i} m_k R_{k+1}(i,j) = Rm(i,j),$$

$$\sum_{k=1}^{i} c_k R_{k+1}(i,j) = Rc(i,j).$$

Therefore,

$$Y_n = \sum_{j=1}^{3} Y_{nj},$$

and hence,

$$E(Y_n) = \sum_{j=1}^{3} [Y_{0j}E(R_1(n,j)) + E(RW(n,j)) - 1/3E(R\xi(n,j))$$

$$- 1/3E(Rm(n,j)) - 1/3E(Rc(n,j)],$$

Thus, the deterministic equivalent of the model (9) with zero order decision rules is:

$$MAX(\lambda_1\{\sum_{j=1}^{3}[Y_{0j}E(R_1(n,j)) + E(RW(n,j)) - 1/3E(R\xi(n,j)) - 1/3E(Rm(n,j))$$

$$- 1/3E(Rc(n,j))]\} - \lambda_2\{\sum_{i=1}^{n} W_i + \frac{1}{2}\sum_{i=1}^{n} \iota_i\{|W(i) - E(\chi(i)) - c(i) - M(i) + B_0| +$$

$$+ W(i) - E(\chi(i)) - c(i) - M(i) + B_0\}\}),$$

subject to:

$$-W(i) + m(i) + c(i) - B_0 + E(\chi_i) + v_i \leq 0,$$

$$\sigma^2_{\chi(i)}[\phi^{-1}(\alpha_i)]^2 - v_i^2 \leq 0$$

where:

$$W_i = \sum_{j=1}^{3} W_{ij},$$

$W_{ij} \geq 0$, $v_i \geq 0$.

$$(12) \quad Y_0 = \sum_{j=1}^{3} Y_{0j},$$

$$E(\chi(i)) = \sum_{k=1}^{i} E(\xi_k) - \sum_{k=1}^{i-1}\sum_{j=1}^{3} W_{kj}\, E(r_{kj}),$$

$$m(i) = \sum_{k=1}^{i} m_k,$$

$$c(i) = \sum_{k=1}^{i} c_k,$$

$$M(i) = \sum_{k=1}^{i} M_k,$$

$$W(i) = \sum_{k=1}^{i}\sum_{j=1}^{3} W_{kj} = \sum_{k=1}^{i} W_k,$$

$$E(R_k(i,j)) = E(\prod_{h=k}^{i} (1 + r_{hj})),$$

$$E(RW(i,j)) = \sum_{k=1}^{i} W_{kj} E(R_k(i,j)),$$

$$E(R\xi(i,j)) = \sum_{k=1}^{i} E(\xi_k) E(R_{k+1}(i,j))$$

$$E(Rm(i,j)) = \sum_{k=1}^{i} m_k E(R_{k+1}(i,j))$$

$$E(Rc(i,j)) = \sum_{h=1}^{i} c_k E(R_{k+1}(i,j)).$$

Where Y_{0j}, B_0 are given non-negative constants, and ξ_i, r_{ij} are mutually independent random variables with known distributions, $0 \le \alpha_i \le 1$, j = 1, 2, 3 and i = 1, 2, ..., n.

Theoretically, we can use the following Kuhn-Tucker conditions [8] to find global optimal solutions. The Kuhn Tucker Theorem is as follows: consider the primal non-linear programming problem:

$$(P) \quad \begin{array}{c} Max\ f(x), \\ g_I(x) \leq 0, \\ g_{II}(x) \leq 0, \\ x \geq 0. \end{array}$$

Then x^* is a global optimal solution of (P) if here exist $\lambda^* \geq 0$, μ^* such that (x^*, λ^*, μ^*) satisfy the conditions:

$$\begin{array}{c} \nabla f(x) - \lambda^T \nabla g_I(x) - \mu^T \nabla g_{II}(x) \leq 0, \\ [\nabla f(x) - \lambda^T \nabla g_I(x) - \mu^T \nabla g_{II}(x) \leq 0] x = 0, \\ \lambda^T \nabla g_I(x) = 0, \end{array}$$

where $\nabla f(x)$ is the gradient of f, f(x) is a concave function, $g_I(x)$ is a vector convex function for which some x^0 satisfies $g_I(x^0) < 0$,and $g_{II}(x)$ is a vector linear function.

Because the objective function of our model is a concave function, and the constraints are convex (or linear) functions of W_{ij}, the Kuhn-Tucker Theorem implies that when these Kuhn-Tucker conditions are met, the local optimal found is indeed a globally optimal solution of the mathematical programming problem. It should also be noted that higher order decision rules and/or conditional probability functionals (instead of the unconditional probabilities used here) can be used to incorporate learning and past experience into the chance-constrained programming models of this paper.

4. Illustrative Example and Discussion

The previous section developed a bi-objective expectational model of n-stage investment under chance-constraints for the purpose of making decisions on how much money to contribute into the pension fund and how to allocate these contributions for investment purposes. In this section, starting with the deterministic equivalent form of the mathematical programming formulation as developed in the last section, we solve this problem explicitly for a prototypical situation.

Consider the following scenario: ABC Oil & Gas Company has a defined benefit plan and would like to make investment decisions for the next two years, with respect to how much money they need to contribute to the fund each year and how to invest it. The company provides the following information concerning the plan:

(i) Initial funding Y_0 (i.e., last year's ending balance of the plan)

$$Y_{01} = \$406,854,\ Y_{02} = \$1,232,374,\ Y_{03} = \$1,150,110;$$

(ii) Initial balance in the funding standard account: $B_0 = \$27,434$;

(iii) Plan expenses: $c_1 = c_2 = \$22,500$;

(iv) Tax rate: $t_1 = 0.28$, $t_2 = 0.33$;

(v) Maximum amortization: $M_1 = \$157,308$, $M_2 = \$159,405$;

(vi) Minimum amortization: $m_1 = \$89,546$, $m_2 = \$92,645$;

(vii) Actuarial assumptions (see Appendix A);

(viii) The number of employees and their age, compensation (see Appendix B & C);

(ix) Benefit formula: monthly benefit = (35% × the Average Monthly Compensation (up to 5 years) - 70% × Primary Social Security benefit) × number of years of Vested Service ÷ number of years from the Date of Hire to the Normal Retirement Date.

From (vii) - (ix), we can calculate the expected annual normal cost $E\xi_i$ from the following formula (cf., [9]):

$$(13) \qquad E(\xi_i) = \sum_{j=1}^{m} (B_x \ {}_{65-x}p_x^{(T)} \ v^{65-x} \ \ddot{a}_{65})_{ij},$$

where B_x = benefit accrued up to age x, calculated by using the benefit formula given in (ix),

${}_{65-x}p_x^{(T)}$ = probability of surviving in service from age x to age 65 (This depends upon the mortality rate, the turnover rate, the disability rate and the retirement rate. See Appendix A),

v^{65-x} = interest discount from x to age 65 (from Appendix A, we assume v = 7.5% in this problem),

$\ddot{a}_{65}$ = life annuity valued at age 65,

i = the ith year of the plan,

m = the number of employees participating in the plan,

j = the jth employee in the plan.

By using the information in Appendices A, B and C, the benefit formula in (ix), the formula in (12) and the actuarial code LYNCHVAL (produced by Lynchval Systems Inc.), we obtain $E(\xi_1)$ = \$66,720, $E(\xi_2)$ = \$184,084.

For convenience, we list all random variables with their distribution functions and parameters in Table I. Also, for convenience, these distributions are assumed normal in this illustrative example. However, other distributions (even the empirical distributions) could be used. Likewise, the independence assumption can be relaxed using results from joint chance-constrained programming.

Table I

Random Variable	Distribution Function	Parameters
ξ_1 Actual Annual cost first year	$N(\mu_1,\sigma_1)$	μ_1=\$166,720, σ_1=\$1,000
ξ_2 Actual Annual cost second year	$N(\mu_2,\sigma_2)$	μ_2=\$184,084, σ_2=\$1,000
r_{11} Return rate on Stocks in period 1	$N(\mu_3,\sigma_3)$	$\mu_3 = 0.20$, $\sigma_3 = 0.20$
r_{21} Return rate on Stocks in period 2	$N(\mu_4,\sigma_4)$	$\mu_4 = 0.15$, $\sigma_4 = 0.20$
r_{12} Return rate on Bonds in period 1	$N(\mu_5,\sigma_5)$	$\mu_5 = 0.15$, $\sigma_5 = 0.10$
r_{22} Return rate on Bonds in period 2	$N(\mu_6,\sigma_6)$	$\mu_6 = 0.12$, $\sigma_6 = 0.10$
r_{13} Return rate on Money Market in period 1	$N(\mu_7,\sigma_7)$	$\mu_7 = 0.085$, $\sigma_7 = 0.005$
r_{23} Return rate on Money Market in period 2	$N(\mu_8,\sigma_8)$	$\mu_8 = 0.09$, $\sigma_8 = 0.005$

Substituting all these numbers in the formulas for M_i, m_i, Y_{0j}, B_0, $E(\xi_i)$, $E(r_{ij})$, and t_i of model (13), and letting $\alpha_1 = \alpha_2 = 0.95$, we have the following model:

$$\text{Max } \{ \lambda_1 [1.356W_{11} + 1.288W_{12} + 1.18265W_{13} + 1.13W_{21} + 1.12W_{22} + 1.09W_{23} + 2{,}889{,}580] - \lambda_2[0.14|W_{11} + W_{12} + W_{13} - 319{,}094| + 0.165|1.2W_{11} + 1.15W_{12} + 1.085W_{13} + W_{21} + W_{22} + W_{23} - 685{,}083| + 1.338W_{11} + 1.32975W_{12} + 1.319025W_{13} + 1.165W_{21} + 1.165W_{22} + 1.165W_{23} - 157{,}711]\},$$

(14) s.t.

$$-W_{11} - W_{12} - W_{13} + v_1 \leq -251{,}332,$$

$$-1.2W_{11} - 1.15W_{12} - 1.085W_{13} - W_{21} - W_{22} - W_{23} + v_2 \leq -550{,}561,$$

$$-v_1^2 \leq -2{,}705{,}531,$$

$$-0.04W_{11}^2 - 0.001W_{12}^2 - 0.000025W_{13}^2 - 0.36961v_2^2 \leq -2{,}000{,}000,$$

$$W_{ij} \geq 0,$$

$$v_i \geq 0, \quad i = 1, 2, \quad j = 1, 2, 3.$$

By varying the parameters λ_1 and $\lambda_2 = (1 - \lambda_1)$, the bi-objective function of maximizing benefits while minimizing costs can be reduced to the univariate extremization involving an unknown trade-off λ_1 between costs and benefits. For various choices of λ_1 and λ_2, we obtain the results presented in Table II:

Table II

$\alpha_1 = \alpha_2 = 0.95$

Variable	$\lambda_1 = 0$ $\lambda_2 = 1$	$\lambda_1 = 0.2$ $\lambda_2 = 0.8$	$\lambda_1 = 0.3$ $\lambda_2 = 0.7$	$\lambda_1 = 0.4$ $\lambda_2 = 0.6$	$\lambda_1 = 0.5$ $\lambda_2 = 0.5$
W_{11}	$17,485	$138,305	$143,554	$150,453	$157,823
W_{12}	$121,000	$105,166	$103,979	$102,533	$142,683
W_{13}	$114,490	$76,433	$72,371	$66,928	$131,204
W_{21}	$0	$61,643	$62,551	$63,862	$64,314
W_{22}	$0	$60,801	$61,059	$61,444	$63,577
W_{23}	$176,620	$58,276	$56,584	$54,191	$61,366
v_1	$1,644	$37,389	$37,389	$37,389	$42,827
v_2	$0	$0	$0	$0	$0
W_1	$252,975	$319,904	$319,904	$319,904	$431,710
W_2	$176,620	$180,720	$180,194	$179,497	$189,257

Objective \$-186,857 \$300,255 \$701,199 \$1,102,228 \$2,945,900

With $\alpha_1 = \alpha_2 = 0.95$ fixed, varying the parameters λ_1 and λ_2 means we vary the priorities of the first objective (of maximizing return) versus the second objective (of minimizing costs). From Table II we see that if we are only concerned with minimizing cost ($\lambda_1 = 0$), we should invest more money in the stock market and not pay more than the minimum funding requirement. Moreover, since cost is the determinant attribute and these costs increase with ove-r or under-funding in the second year, all contributions go into the safer investment (Money Markets). The effect on return of this conservative strategy is not put into the objective function when $\lambda_1 = 0$. When λ_1 increases so that return becomes more important, we can see from the row of W_1, that we would increase the employer's contributions. When $\lambda_2 = 0.8$, 0.7 or 0.6, we would pay only the IRS tax limit (319,904); however, when $\lambda_2 = 0.5$, we would pay an amount which exceeds the IRS tax limit in order to increase the funding balance and hence, increase return and security.

It is natural, when you are concerned most about accumulating your assets, that you make contributions to the fund in excess of the IRS limits. On the other hand, if you are more concerned about reducing the cost, you might pay only the lower limit permissible for tax-deductibility (or even less). In the real business world, the following are some factors that may affect an employer's decision:

1) Cash Flow Considerations. Employers may want to maintain reasonable stability in the cash flow required to fund their pension plans. This factor depends, in part, on the financial strength and stability of the employer.
2) Nature of the Industry and Competitive Conditions. An employer's funding policy

will naturally be influenced by the competitive conditions under which he/she must operate. For example, if profit margins in a particular industry are quite narrow, it may be important for an employer to keep pension costs as low as possible. The reverse may be true in more affluent industries when tax and/or deferral considerations might prevail.

3) The Employer's Internal Rate of Return. If the after-tax rate of return the employer can earn on funds within his/her own business is relatively high compared to the tax-free return from qualified pension fund investments, the employer may be inclined toward funding his/her pension obligations around the minimum permitted level.

4) Matching Contribution to Accrued Benefits. Employers may establish certain goals for funding plans, such as an providing funding of an amount at least equal to the total present value of accrued benefits, or of accrued vested benefits.

Since our model solves for a bi-objective strategy which can be altered by adjusting the λ values to emphasize one versus the other objective, it provides flexibility to better meet various employers' desires.

Additionally, the analysis can be sharpened by adding a constraint on investment in stock. From the results presented in Table II, W_{i1} is higher than the return rate in bonds and the money market. Usually the trustees who manage a pension fund do not invest a high percentage of assets of the fund in stocks because of the high risk of stock (and some pension funds prohibit extensive investment in certain stocks). In order to incorporate such desires or contractual obligations, the actuary need only add a constraint which limits the percentage of contributions that can be invested in stocks, to say 40%; i.e., add constraints $W_{i1} \leq 0.4$

W_i. The results of the computations exemplified in the previous illustration in light of this new constraint are presented below. They are compared with the optimal allocation which results in the absence of this constraint.

Table III

$\lambda = 0.4,\ \alpha_1 = \alpha_2 = 0.95$

Variable	Stock Unconstrained	Stock Constrained
W_{11}	\$150,453	\$127,961
W_{12}	\$102,523	\$191,942
W_{13}	\$66,928	\$0
W_{21}	\$63,862	\$60,509
W_{22}	\$61,444	\$115,763
W_{23}	\$54,191	\$0
v_1	\$37,389	\$1,644
v_2	\$0	\$0
W_1	\$319,904	\$319,633
W_2	\$179,497	\$176,272
Objective	\$1,102,228	\$1,005,604

Table III shows the effects of constraining the percentage of contributions that can be invested in stocks. W_{11} drops from \$150,452 to \$127,961, and the optimal objective function value drops from \$1,102,228 to \$1,005,604. In this situation, investors of pension assets need to invest capital in bonds instead of in the money market in order to make a higher return. If regulators put a constraint on the percentage that can be invested in "junk bonds" (as there is some talk of doing), then this can easily be handled within this framework by designating a fourth investment vehicle type and including a constraint on the amount which can be invested therein.

5. Conclusion

The chance-constrained model (9) appears to give realistic and sensible results as well as offer flexibility for optimally determining funding levels in a defined benefit plan within the constraints imposed by regulators, the IRS and corporate objective goals. The chance-constrained methods also allow the plan administrator to meet the basic requirements of design of the defined benefit plan. For a large class of additional special cases, we could add new constraints to meet the additional special needs; i.e., $g_1(x) \geq 0$ while having $g_1(x)$ a vector convex function. Our procedure of finding a global optimal solution to a non-linear programming problem by employing the Kuhn-Tucker optimality conditions to a convex programming problem is also valid in the larger context of the employee benefit plans.

6. References

1. Ben-Isreal, A., Ben-Tal, A. and Charnes, A., "Necessary and Sufficient Conditions for a Pareto Optimum in Convex Programming," Econometrica, Vol. 45, No.4, (May 1977), 811-820.
2. Business Week, (November 6, 1989), 154
3. Charnes, A. and Cooper, W. W. Management Models & Industrial Application of Linear

Programming, Vol. 1, John Wiley & Son, Inc., New York, (1960).

4. McCabe, G. M. and Witt, R. C., "Insurance Pricing and Regulation Under Uncertainty: A Chance Constrained Approach," The Journal of Risk and Insurance, Vol. 47, No. 4, (1980), 607-635.

5. McGill, D.M. and Grubbs, Jr., D.S., Fundamental of Private Pensions, Sixth Edition, Pension Research Council, (1989).

6. Rao, C. R., "A Note on a Generalized Inverse of a Matrix with Applications to Problems in Mathematical Statistics," Journal of the Royal Statistical Society, 24, Series B, (1962), 152-158.

7. Rosenbloom, J.S. and Hallman, G.V., Employee Benefit Planning, Prentice-Hall, Englewood Cliffs, (1986).

8. Saaty, Thomas L. and Bram, Joseph, Nonlinear Mathematics,????

9. Winklevoss, H. E., Pension Mathematics: with Numerical Illustrations, Richard D. Irwin, Inc. (1977).

Appendix A

STATEMENT OF ACTUARIAL METHODS AND ASSUMPTIONS

Valuation Method

Entry Age Normal Method -- In general, this method assumes that level annual funding payments (as a percentage of compensation) are made to the Plan for each member during his entire period of employment. These assumed payments are called "Normal Costs."

Asset Valuation Method

The actuarial value of Plan assets is set equal to the fair market value of Plan assets determined as of the valuation date.

Actuarial Assumptions

Rate of Return: 7½% per annum, compounded annually
Salary Scale: 5½% per annum, compounded annually

Assumed Retirement Age: 65.

Mortality Rates for Active, Disabled and Retired Members: Rates are those of the 1983 Group Annuity Mortality Table. There are separate tables for males and females. Sample mortality rates are:

	Mortality Rate	
Age	Male	Female
25	0.05%	0.03%
30	0.06	0.03
35	0.09	0.05
40	0.12	0.07
45	0.22	0.10
50	0.39	0.16
55	0.61	0.25
60	0.92	0.42

Rates of Incidence of Disability: A scale of disability rates consistent with insured long-term disability rates developed by the Society of Actuaries from 1972 to 1976 experience. Sample rates are:

Disability Rates

Age	Male	Female
25	0.083%	.106%
30	0.083	.106
35	0.083	.106
40	0.176	.303
45	0.341	.407
50	0.638	.597
55	1.119	.921
60	1.541	1.093

Turnover Rates: A scale of termination rates consistent with the following sample rates:

Age	Male and Female
25	9.75%
30	9.40
35	8.84
40	7.95
45	6.71
50	4.87
55	2.59
60	1.70

Expenses: $22,500 per year

Social Security Benefits: The law as amended through January 1, 1989 with assumed:

a. Cost-of-living increases:	3% per annum, compounded annually
b. Wage base and national average earnings increases:	5% per annum, compounded annually
c. Salary increases prior to the valuation date:	5.5% per annum, compounded annually

Maximum Retirement Benefit:	$98,064 annually, payable as a straight life annuity at the participant's Social Security retirement age (per Section 415 of the Internal Revenue Code)
Marital Status:	a. Percentage Married: 100%
	b. Age Difference: Males are assumed to be three years older than their spouses.
Eligible Earnings:	1988 W-2 earnings were adjusted by the salary scale, as applicable, in determining all costs and actuarial liabilities under the Plan for the Plan Year beginning January 1, 1989. However, actual salary histories were used to determine a participant's accrued benefits as of January 1, 1989.

APPENDIX B

AGE, SERVICE AND COMPENSATION DISTRIBUTION
AS OF JANUARY 1, 1990

ANNUAL EARNINGS BY AGE GROUPS

	MALE			FEMALE			ALL		
AGE GROUP	NUMBER OF PEOPLE	TOTAL ANNUAL EARNINGS	AVERAGE ANNUAL EARNINGS	NUMBER OF PEOPLE	TOTAL ANNUAL EARNINGS	AVERAGE ANNUAL EARNINGS	NUMBER OF PEOPLE	TOTAL ANNUAL EARNINGS	AVERAGE ANNUAL EARNINGS
0-19	0	$ 0	$ 0	0	$ 0	$ 0	0	$ 0	$ 0
20-24	15	203,817	13,588	12	169,579	14,132	27	373,396	13,829
25-29	36	712,758	19,799	23	367,107	15,961	59	1,079,865	18,303
30-34	46	1,177,820	25,605	21	445,188	21,199	67	1,623,008	24,224
35-39	49	1,500,534	30,623	9	188,313	20,924	58	1,688,846	29,118
40-44	35	1,502,156	42,919	19	409,048	21,529	54	1,911,203	35,393
45-49	25	898,159	35,926	6	109,470	18,245	31	1,007,629	32,504
50-54	22	635,340	28,879	7	133,011	19,002	29	768,351	26,495
55-59	20	1,015,912	50,796	6	122,265	20,378	26	1,138,177	43,776
60-64	6	242,233	40,372	6	111,284	18,547	12	353,517	29,460
65-69	3	253,583	84,528	1	21,501	21,501	4	275,084	68,771
70-74	0	0	0	0	0	0	0	0	0
75-79	0	0	0	0	0	0	0	0	0
80-84	0	0	0	0	0	0	0	0	0
85 +	0	0	0	0	0	0	0	0	0
TOTAL	257	$ 8,142,312	$ 31,682	110	$2,076,765	$ 18,880	367	$10,219,076	$ 27,845

Appendix C

RECONCILIATION OF PLAN PARTICIPANTS AS OF JANUARY 1, 1990

	Active	Terminated Vested	Retired or Beneficiaries
Number of Plan Participants on April 1, 1988	390	23	21
Data Adjustments	0	-1	0
New Participants	2	0	0
Participants Terminated			
* Vested	-9	9	0
* Non-Vested	-74	0	0
Participants Retired	-12	-1	13
Participants Who Received Lump Sum Distributions from the Plan	0	-6	0
Number of Plan Participants on January 1, 1989	367	24	34