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Mixed Lognormal Distributions

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Abstract

Consider an insurance policy where the claim amount has a mixed lognormal distribution. In this paper, we assume that the individual distribution of claim amount, given the value of the parameter for the median of that distribution, follows a lognormal distribution. To model the heterogeneity in the population, we assume that the median is disbursed according to normal, Laplace, uniform, power function and gamma distributions. *The exact ultimate forms for the probability density function of the* portfolio claim amount are given for the former three distributions. Whereas, a difference equation representation is given for the latter two distributions.

Keywords : difference equations.

1. Introduction

Suppose an insurance claim amount with random variable X follows a lognormal distribution. Denote its probability density function by $f(x|\mu)$ where

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}\sigma x}\exp\left[-\frac{1}{2}\left(\frac{\log_e(x)-\mu}{\sigma}\right)^2\right], \quad \forall x \in \Re^+, \mu \in \Re. \sigma \in \Re^+. \tag{1.1}$$

As a convention, we write $X|\mu \sim LN(\mu, \sigma)$. Note that μ and σ are the parameters for the lognormal distribution but not the mean and standard deviation for it. We assume that the heterogeneity in the insurance portfolio is due to variability in the parameter μ.

Therefore, we assume that μ is random and that it follows a known continuous distribution. Throughout the discussion we assume that σ is known. The probability density function (pdf) of μ will be denoted as $g(\mu)$. The distribution function corresponding to $g(\mu)$ is often called a *mixing* or *risk* distribution. The unconditional pdf of the portfolio claim amount is denoted by $h(x)$ and it is equal to

$$h(x) = \int_{\Re} f(x|\mu)\ g(\mu)\ d\mu\ , \qquad \forall x \in \Re^+ . \tag{1.2}$$

The purpose of this paper is to find $h(x)$ based on several assumed distributions for the risk distribution. The mean, mode and median of a lognormal distribution are as follows:

$$\begin{aligned} \text{mean} &= e^{\mu + \frac{1}{2}\sigma^2}, \\ \text{mode} &= e^{\mu - \sigma^2} \text{ and,} \\ \text{median} &= e^{\mu}. \end{aligned} \tag{1.3}$$

2. Recursive results

In this section, we derive some ordinary difference equations that will be useful later. Consider the following integral for non-negative integers of n. Let

$$I_n = \int_0^{\theta} \mu^n \exp\left\{-\tfrac{1}{2}\left(\tfrac{\mu - a}{b}\right)^2\right\} d\mu\ , \qquad \forall \theta \in \Re^+,\ b \in \Re^+,\ a \in \Re. \tag{2.1}$$

Using the substitution $y = \frac{\mu - a}{b}$, we get

$$I_n = \int_{-a/b}^{(\theta - a)/b} (a + b\,y)^n\ b\ e^{-\frac{y^2}{2}}\ dy. \tag{2.2}$$

When $n = 0$,

$$I_0 = \int_{-a/b}^{(\theta - a)/b} b\ e^{-\frac{y^2}{2}}\ dy = \sqrt{2\pi}\ b\left[\Phi\left(\tfrac{\theta - a}{b}\right) - \Phi\left(-\tfrac{a}{b}\right)\right] \tag{2.3}$$

where $\Phi(z)$ is the cumulative distribution function for a standard normal random variable. When $n = 1$,

$$I_1 = \int_{-a/b}^{(\theta - a)/b} (a + b\,y)\ b\ e^{-\frac{y^2}{2}}\ dy$$

$$= \sqrt{2\pi}\; a\; b \left[\Phi\left(\frac{\theta - a}{b}\right) - \Phi\left(-\frac{a}{b}\right) \right] + b^2 \left\{ \exp\left[-\frac{1}{2}\left(\frac{a}{b}\right)^2 \right] - \exp\left[-\frac{1}{2}\left(\frac{\theta - a}{b}\right)^2 \right] \right\}. \tag{2.4}$$

When $n \geq 2$, we get the recursive equation

$$I_n = a\, I_{n-1} + (n-1)\, b^2\, I_{n-2} - b^2\, \theta^{n-1} \exp\left[-\frac{1}{2}\left(\frac{\theta - a}{b}\right)^2 \right]. \tag{2.5}$$

As $\theta \to \infty$, we find that

$$I_0 = \sqrt{2\pi}\; b \left[1 - \Phi\left(-\frac{a}{b}\right) \right],$$

$$I_1 = \sqrt{2\pi}\; a\; b \left[1 - \Phi\left(-\frac{a}{b}\right) \right] + b^2 \exp\left[-\frac{1}{2}\left(\frac{a}{b}\right)^2 \right],$$

$$I_n = a\, I_{n-1} + (n-1)\, b^2\, I_{n-2}, \quad \forall n \geq 2. \tag{2.6}$$

3. Mixing distribution of μ : μ is real

In this section, we consider the case where μ has support on all the real numbers. Two mixing distributions will be employed, the normal distribution and the Laplace distribution. A comprehensive list of statistical distributions can be found in Patel, Kapadia and Owen (1976).

Normal distribution

A random variable μ is said to follow a normal distribution with mean a and variance b^2 if its pdf has the form

$$g(\mu) = \frac{1}{\sqrt{2\pi} b} \exp\left[-\frac{1}{2}\left(\frac{\mu - a}{b}\right)^2 \right], \quad \forall \mu \in \Re,\ a \in \Re,\ b \in \Re^+. \tag{3.1}$$

Note that if $\mu \sim N(\mathrm{a}, b^2)$, then $e^\mu \sim LN(\mathrm{a}, b)$. Recall from (2.1) that e^μ is the median of the mixed lognormal distribution. The joint pdf of X and μ is obtained by taking the product of the pdf's given by (1.1) and (3.1). Integrating with respect to x yields

$$h(x) = \frac{1}{x\sqrt{2\pi\,(b^2 + \sigma^2)}} \exp\left[-\frac{1}{2}\left(\frac{\log_e(x) - a}{\sqrt{b^2 + \sigma^2}} \right)^2 \right], \quad \forall x > 0. \tag{3.2}$$

In other words, the unconditional distribution of X follows a lognormal distribution with parameters a and $\sqrt{b^2 + \sigma^2}$.

Laplace distribution

If μ has a Laplace distribution then its probability density function is:

$$g(\mu) = \frac{1}{2\beta}\exp\left\{-\frac{|\mu-\alpha|}{\beta}\right\}, \qquad \forall\mu\in\Re,\ \alpha\in\Re \text{ and } \beta\in\Re^{+}. \tag{3.3}$$

To find the corresponding unconditional probability density function of X, we have to split the integral from $-\infty$ to $+\infty$ into two integrals. One of the integrals is evaluated from $-\infty$ to α while the other integral is evaluated from α to $+\infty$. The unconditional pdf of X is

$$h(x) = \frac{\exp\left(\frac{\sigma^2}{2\beta^2}\right)}{2\beta}\times$$

$$\left\{e^{-\frac{\alpha-1}{\beta}}\,\Phi\left(\frac{\alpha-\log_e(x)-\sigma^2/\beta}{\sigma}\right) + e^{\frac{\alpha-1}{\beta}}\left[1-\Phi\left(\frac{\alpha-\log_e(x)+\sigma^2/\beta}{\sigma}\right)\right]\right\}. \tag{3.4}$$

4. Gamma mixing distribution

In this section, the gamma distribution will be used as the mixing distribution. The pdf of the gamma is

$$g(\mu) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}e^{-\beta\mu}\mu^{\alpha-1}, \qquad \forall\mu\in\Re^{+}\cup\{0\} \text{ and } \alpha,\beta\in\Re^{+}. \tag{4.1}$$

Therefore, we obtain an expression for $h(x)$

$$h(x) = \frac{\exp\left(-\frac{A_2}{2\sigma^2}\right)}{\sqrt{2\pi}\sigma x}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\left\{\int_0^{\infty}\mu^{\alpha-1}\exp\left[-\frac{1}{2}\left(\frac{\mu-A_1}{\sigma}\right)^2\right]d\mu\right\}, \qquad \forall x\in\Re^{+}, \tag{4.2}$$

where

$A_1 = \log_e(x) - \beta\sigma^2$ and

$A_2 = 2\beta\sigma^2\log_e(x) - \beta^2\sigma^4$.

We only consider the case where α belongs to the set of natural numbers. Utilizing the results derived earlier in section 2, the unconditional probability density function of X can be found easily for $\alpha = 1$ and 2. In this case, we have $\theta = \infty$, $n = \alpha + 1$, $a = A_1$ and $b = \sigma$. When $\alpha = 1$,

$$h(x) = \frac{\beta\exp\left(\frac{\beta^2\sigma^2}{2}\right)}{x^{\beta+1}}\left[1-\Phi\left(\frac{\beta\sigma^2-\log_e(x)}{\sigma}\right)\right]. \tag{4.3}$$

When $\alpha = 2$,

$$h(x) = \frac{\beta^2 \exp\left(\frac{\beta^2\sigma^2}{2}\right)}{x^{\beta+1}} \times$$

$$\left\{(\log_e(x) - \beta\sigma^2)\left[1 - \Phi\left(\frac{\beta\sigma^2 - \log_e(x)}{\sigma}\right)\right] + \sqrt{2\pi}\sigma \exp\left[-\frac{1}{2}\left(\frac{\log_e(x) - \beta\sigma^2}{\sigma}\right)^2\right]\right\}. \tag{4.4}$$

When $\alpha \geq 3$, we have to evaluate the integral recursively. For a fixed value of α, the integral inside the curly brackets in (4.2) can be found recursively using the expression in (2.6). Denote the left hand side of (4.2) by h_α, then

$$h_\alpha = \frac{\beta A_1}{\alpha - 1} h_{\alpha-1} + \frac{\beta^2\sigma^2}{\alpha - 1} h_{\alpha-2}, \quad \forall \alpha \geq 3, \tag{4.5}$$

with the initial boundary values of h_1 and h_2 given by (4.3) and (4.4) respectively.

5. Mixing distribution of μ : μ is bounded

Finally, we consider the case when μ is a bounded random variable. Both the uniform and power function distributions are chosen in turn as the mixing distribution of μ.

Uniform distribution

The uniform distribution is the simplest distribution among all the known statistical distributions. Its pdf is given by

$$g(\mu) = \frac{1}{b-a}, \quad \forall \mu \in (a,b) \text{ and } a,\ b \in \Re. \tag{5.1}$$

The unconditional pdf of X is

$$h(x) = \frac{1}{(b-a)x}\left[\Phi\left(\frac{b - \log_e(x)}{\sigma}\right) - \Phi\left(\frac{a - \log_e(x)}{\sigma}\right)\right], \quad \forall x \in \Re^+. \tag{5.2}$$

Power function distribution

The power function distribution has a pdf given by

$$g(\mu) = \frac{c}{\theta^c} \mu^{c-1}, \quad \forall \mu \in (0,\theta) \text{ and } \theta,\ c \in \Re^+. \tag{5.3}$$

The unconditional pdf of X is

$$h(x) = \int_0^{\theta} \frac{c}{\sqrt{2\pi}\sigma x \theta^c} \mu^{c-1} \exp\left[-\frac{1}{2}\left(\frac{\mu - \log_e(x)}{\sigma}\right)^2\right] d\mu, \qquad \forall x \in \Re^+. \tag{5.4}$$

We consider only those positive integral values of c. When $c = 1$, it can easily be identified that (5.3) is the pdf for a uniformly distributed random variable over the interval $(0,\theta)$. Thus, by putting $b = \theta$ and $a = 0$ in (5.2), we get

$$h(x) = \frac{1}{\theta x}\left[\Phi\left(\frac{\theta - \log_e(x)}{\sigma}\right) - \Phi\left(\frac{-\log_e(x)}{\sigma}\right)\right], \qquad \forall x \in \Re^+. \tag{5.5}$$

When $c = 2$, making use of (2.4), we obtain

$$h(x) = \frac{2}{\sqrt{2\pi}\, x\theta^2}\left\{\sqrt{2\pi}\log_e(x)\left[\Phi\left(\frac{\theta - \log_e(x)}{\sigma}\right) - \Phi\left(-\frac{\log_e(x)}{\sigma}\right)\right]\right.$$
$$\left. + \sigma\left[\exp\left\{-\frac{1}{2}\left(\frac{\log_e(x)}{\sigma}\right)^2\right\} - \exp\left\{-\frac{1}{2}\left(\frac{\theta - \log_e(x)}{\sigma}\right)^2\right\}\right]\right\}, \qquad \forall x \in \Re^+. \tag{5.6}$$

When $c \geq 3$, we can use the recursive formula in (2.5) and get

$$h(x;\, c) = \frac{\log_e(x)}{\theta}\frac{c}{c-1}\, h(x;\, c-1) + \frac{c\sigma^2}{\theta^2}\, h(x;\, c-2) + r, \tag{5.7}$$

where $r = -\frac{c\sigma}{\sqrt{2\pi} x\theta^2}\exp\left[-\frac{1}{2}\left(\frac{\theta - \log_e(x)}{\sigma}\right)^2\right]$.

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