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Approximations of Ruin Probability by Di-atomic or Di-exponential Claims

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ABSTRACT

Given the three moments of the claim amount, we find the diatomic and the diexponential distributions that fits the given three moments. We use the well known ruin probability formulas when the claim amount is discete or is a combination of exponentials to compute ruin probabilities based on the claim amount distribution being the diatomic and the diexponential which match the first three moments of the original claim amount distribution. We then compare the approximations with the exact values of $\psi(u)$ for three examples drawn from fire (large spread), individual life (medium spread) and group life (small spread) insurance data.

KEYWORDS

Ruin probability; diatomic distribution; diexponential distribution.

1. INTRODUCTION

Parametric representation of claim data and exact calculation of ruin probabilities has a long history. In the classical work of Cramér (1955, p.43), we find that the following claim amount distribution was used to represent data from Swedish non-industry fire insurance covering the years 1948-1951:

$$p(x) \;=\; 4.897954\, e^{-5.514588\,x} \;+\; 4.503\,(x+6)^{-2.75}\,, \qquad 0 < x < 500. \tag{1}$$

and that exact ruin probabilities were computed by numerically solving

$$\psi(u) \;=\; \frac{\lambda}{c}\int_0^u [1-P(y)]\,\psi(u-y)\,dy \;+\; \frac{\lambda}{c}\int_u^\infty [1-P(y)]\,dy\,, \tag{2}$$

which was a nontrivial numerical task at that time (Cramér 1955, p.45). A modern reference for the above integral equation is Exercise 12.11 in Bowers *et alii* (1986).

A much easier numerical task even now is to approximate (1) by a distribution for which there is a readily executable formula for its ruin probability. In this paper, we choose our approximants form special types of claim amount distributions on which there is recent renewed interest: For combination of exponential distributions, there is the Tãcklind (1942) type formulas. See Shiu (1984) Dufresne and Gerber (1988) (1989) (1991) and Chan (1990). For discrete distributions (mixture of atomic distirbutions), there is the Takács (1967) type formulas. See Beekman(1968), Shiu (1989) and Kass (1991). In particular, we consider the special cases of mixture of two atoms (diatomic) and of combination of two exponentials (diexponential).

2. THREE MOMENT FIT OF DIATOMIC AND DIEXPONENTIAL DISTRIBUTIONS

A moment's reflection would tell us that for a given set of mean, variance, and third central moment, written as μ, σ^2, and κ^3, there fits a unique diatomic distribution. A little algebra leads to the following proposition for which the straightforward verification would not be reproduced here.

PROPOSITION 1:
Given mean, variance, and third central moment written as μ, σ^2, and κ^3, there is a unique diatomic distribution fitting these moments. the locations of the two atoms are:

$$\mu - \frac{\sqrt{\kappa^6 + 4\sigma^6} - \kappa^3}{2\sigma^2}, \qquad \mu + \frac{\sqrt{\kappa^6 + 4\sigma^6} + \kappa^3}{2\sigma^2}; \tag{3}$$

and the probabilities are:

$$\frac{\sigma^2}{\left(\frac{\sqrt{\kappa^6 + 4\sigma^6} - \kappa^3}{2\sigma^2}\right)^2 + \sigma^2}, \qquad \frac{\left(\frac{\sqrt{\kappa^6 + 4\sigma^6} - \kappa^3}{2\sigma^2}\right)^2}{\left(\frac{\sqrt{\kappa^6 + 4\sigma^6} - \kappa^3}{2\sigma^2}\right)^2 + \sigma^2}. \tag{4}$$

In addition, if $\mu > 0$ and non-negative atoms are desired, then one must have

$$\kappa^3 \geq \frac{\sigma^2(\sigma^2 - \mu^2)}{\mu}.$$

For mixture (with positive coefficients A and $1-A$) or combination (the coefficients allowed to be one positive and one negative or both positive) of two exponentials, the resulting distribution is a three parameter family:

$$p(x) = A\,\beta\,e^{-\beta x} + (1-A)\,\gamma\,e^{-\gamma x} \quad \text{for } x \geq 0 \text{ where } 0 < \beta \leq \gamma .$$

That is, we label the coefficient going with the smaller exponential parameter A. Although one of A or $1-A$ can be negative, A must be positive otherwise $p(x)$ would be negative for large x. When $A > 1$ and $1-A$ is negative, we must have

$$\gamma \leq \frac{A}{A\text{-}1}\beta$$

since $\frac{A}{A\text{-}1}\beta < \gamma$ would lead to negative $p(x)$ for small x. We thus describe this three parameter family as:

$$p(x) = A\,\beta\,e^{-\beta x} + (1-A)\,\gamma\,e^{-\gamma x} \text{ for } x \geq 0 \quad \begin{array}{l}\text{where } 0 < \beta \leq \gamma,\ 0 \leq A \leq 1; \\ \text{or } 0 < \beta \leq \gamma \leq \frac{A}{A-1}\beta, 1 < A.\end{array} \tag{5}$$

We would call a distribution described by (5) a ***diexponential distribution***. Note that when $\beta{=}\gamma$ it degenerates to a single exponential. When $\gamma{=}\frac{A}{A\text{-}1}\beta$ we have $p(0) = 0$ and it becomes a two parameter family:

$$p(x) = A\,\beta\,e^{-\beta x} - A\,\beta\,e^{-\frac{A\beta}{A-1}x} \text{ for } x \geq 0, \text{ where } 0 < \beta,\ 1 < A. \tag{6}$$

This distribution is usually known as the Erlang distribution, which is the independent sum of two exponentials of parameters β and γ and usually parametrized as:

$$p(x) = \frac{\beta\,\gamma}{\gamma-\beta}\left(e^{-\beta x} - e^{-\gamma x}\right) \text{ for } x \geq 0$$

$$= \frac{\gamma}{\gamma-\beta}\,\beta\,e^{-\beta x} - \frac{\beta}{\gamma-\beta}\,\gamma\,e^{-\gamma x} \quad \text{for } x \geq 0 \tag{7}$$

where $0 < \beta < \gamma$.

How big is the family of diexponential distributions? Can a diexponential be found to fit up to the third moment? This translates to the solution of the following system of equations

$$\begin{aligned} \frac{A}{\beta} + \frac{1-A}{\gamma} &= E(X) = \mu \\ \frac{A}{\beta^2} + \frac{1-A}{\gamma^2} &= \frac{E(X^2)}{2} = \frac{\sigma^2-\mu^2}{2} \\ \frac{A}{\beta^3} + \frac{1-A}{\gamma^3} &= \frac{E(X^3)}{6} = \frac{\kappa^3+3\sigma^2\mu+\mu^3}{6} \end{aligned} \tag{8}$$

The answer is different from the case of diatomic distrtibution where any μ,σ^2, and κ^3 would find a fit. Again, we omit the computational but straightforward proof of:

PROPOSITION 2:

A. Necessary and sufficient conditions for finding a fit: For a fixed $\mu > 0$,

(i.a) if $\sigma^2 > \mu^2$ and $\kappa^3 > \frac{\mu^4 + 3\sigma^4}{2\mu}$, then a mixture of exponentials with $A < 1$ fits.

(i.b) if $\sigma^2 > \mu^2$ and $\kappa^3 \leq \frac{\mu^4 + 3\sigma^4}{2\mu}$, then it is not a diexponential.

(ii) if $\sigma^2 = \mu^2$ then it is one single exponential and $\kappa^3 = \frac{\mu^4 + 3\sigma^4}{2\mu} = 2\mu^3$

(iii.a) if $\frac{\mu^2}{2} < \sigma^2 < \mu^2$ and $6\mu\sigma^2 - 4\mu^3 + \sqrt{18\,(\mu^2 - \sigma^2)^3} < \kappa^3 < \frac{\mu^4 + 3\sigma^4}{2\mu}$, then a combination of exponentials with $A > 1$ fits.

(iii.b) if $\frac{\mu^2}{2} < \sigma^2 < \mu^2$ and $\kappa^3 \leq 6\mu\sigma^2 - 4\mu^3 + \sqrt{18\,(\mu^2 - \sigma^2)^3}$ or $\frac{\mu^4 + 3\sigma^4}{2\mu} \leq \kappa^3$, then it is not a diexponential.

(iv) if $\sigma^2 = \frac{\mu^2}{2}$ then it is one single gamma $(2, 1/\mu)$ and we must have

$$\kappa^3 = 6\mu\sigma^2 - 4\mu^3 + \sqrt{18\,(\mu^2 - \sigma^2)^3} = 2\mu^3$$

(v) if $\sigma^2 < \frac{\mu^2}{2}$ then it is not a diexponential.

B. The fit: When the three moments μ, σ^2, κ^3 satisfies the above conditions to give a fitting diexponential, the appropriate parameters are:

$$\frac{1}{\beta} = \frac{\kappa^3 - 2\mu^3 + \sqrt{\kappa^6 - 4\kappa^3\mu(3\sigma^2 - 2\mu^2) + 18\sigma^6 - 18\sigma^4\mu^2 + 6\sigma^2\mu^4 - 2\mu^6}}{6(\sigma^2 - \mu^2)}$$

$$\frac{1}{\gamma} = \frac{\kappa^3 - 2\mu^3 - \sqrt{\kappa^6 - 4\kappa^3\mu(3\sigma^2 - 2\mu^2) + 18\sigma^6 - 18\sigma^4\mu^2 + 6\sigma^2\mu^4 - 2\mu^6}}{6(\sigma^2 - \mu^2)}$$

$$A = \frac{\beta(\gamma\mu - 1)}{\gamma - \beta}.$$

3. RUIN PROBABILITIES FOR DIATOMIC AND DIEXPONENTIAL DISTRIBUTIONS

The ruin probabities for a discrete claim amount distribution has been given by Schmitter(1990). See Kass (1991, p.136). For similar formulas see Shiu (1989). For the diatomic case,

$$\psi(u) = 1 - \frac{\theta}{1+\theta} \sum_{k_1, k_2} (-z)^{k_1 + k_2} \quad e^z \quad \frac{p_1^{k_1} p_2^{k_2}}{k_1! k_2!}, \tag{9}$$

$$\text{where } z = \frac{(u - k_1 x_1 - k_2 x_2)+}{(1+\theta)\mu}.$$

Proof for the atomic case and a reference to Feller (1971) is found in Shiu (1987).

The theory of ruin probability for mixture and combination of exponentials is well known. See Shiu(1984), Dufresne and Gerber (1988) (1989) (1991), and Chan (1990). In the case when there are only two exponentials, the adjustment coefficient equation

$$(1+\theta)\ \mu = \frac{M_X(r) - 1}{r}$$

is quadratic and has solutions:

$$R, r_2 = \frac{1}{2}\left(\beta + \gamma - \frac{1}{\mu(1+\theta)} \mp \sqrt{\left(\frac{1}{\mu(1+\theta)} - (\beta+\gamma)\right)^2 - \frac{4\beta\gamma\theta}{1+\theta}}\right), \tag{10}$$

and

$$\psi(u) = C_1 e^{-Ru} + C_2 e^{-r_2 u} \tag{11}$$

where C_1,C_2 are found by the Täcklind formula ($r_1 = R$):

$$C_k = \prod_{\substack{i \neq k \\ i=1}}^{2} \frac{r_i}{r_i - r_k} \prod_{i=1}^{2} \frac{\beta_i - r_k}{\beta_i} \quad k = 1,2. \tag{12}$$

4. DIATOMIC AND DIEXPONENTIAL AS APPROXIMANTS

In this section, we study three claim amount distirbutions and compute ruin probabilities of approximating diatomic and diexponential with matching first three moments and compare the approximations with the exact values of $\psi(u)$. In the first example (Cramér's fire) the claim amount distribution has a large spread, none of the approximations is very close to the exact value, and there we point out the run-off error problem encountered in the Takács type formulas. In the second example (Reckin, Schwark, and Snyder's individual life) the claim amount distribution has a medium spread, both of the diatomic and diaexponential give good approximations. In the third example (Mereu's group life) the claim amount distribution has a small spread, the diatomic gives an excellent approximation, and the spread is so small that there is no diexponential fit.

Example 1: We consider Cramér's fire insurance data, the one mentioned in the introduction. In the following table, the exact values of $\psi(u)$ for $\theta = 0.3$, and the values for the Cramér-Lundberg approximation is from Cramér (1955, p.45). The values for the Beekman-Bowers approximation is from Beekman (1969, p.279). The ruin probability for diatomic claims, (9), encounters convergence problems when u is large; it is indicated in the table below by **. Our experience echoes with that reported in Seah (1990, §4). For values of u close to and above 30 times μ, run-off error takes over and we obtain probabilities less than zero or greater than one.

TABLE 1 Cramér's Fire Insurance

$\mu = 1$, $\sigma^2/\mu^2 = 42.20323069$, $\kappa^3/\sigma^3 = 27.69286626$

u	$\psi(u)$	CL	BB	diatom	diexp	CL/$\psi(u)$	BB/$\psi(u)$	dia/$\psi(u)$	die/$\psi(u)$
20	.5039	.4524	.5140	.4133	.4666	0.898	1.020	0.820	0.926
40	.3985	.3904	.4079	**	.4010	0.980	1.028	**	1.006
60	.3280	.3370	.3369	**	.3447	1.027	1.027	**	1.051
80	.2757	.2909	.2812	**	.2962	1.055	1.020	**	1.074
100	.2346	.2511	.2369	**	.2546	1.070	1.010	**	1.085

Example 2: In this example, we consider the individual life insurance date from Reckin, Schwark, and Snyder (1984). This is also the claim distribution in Example 3 of Seah (1990). The claim amount X is discrete with support {1,2,3,4,5,7,8,10,12,13,15,16} and probabilities (in order) {.5141, .3099, .0639, .0220, .0194, .0096, .0276, .0036, .0041, .0019, .0013, .0226}. Since the claim amount distribution is more spread out, (i.a) of Proposition 2 is satisfied and we have a diexponential fit.

TABLE 2.1 $\psi(u)$ by Seah for RSS's Individual Life Insurance Data
$\mu = 2.2896$, $\sigma^2/\mu^2 = 1.43257300$, $\kappa^3/\sigma^3 = 3.60560786$

	$\theta = .1$	$\theta = .2$	$\theta = .3$	$\theta = .4$	$\theta = .5$
$u = 0$	.909091	.833333	.769231	.714286	.666667
$u = 10$	.644361	.450722	.334890	.260412	.209732
$u = 20$	.469129	.254324	.152965	.099371	.068466
$u = 30$	.341528	.143813	.070341	.038430	.022840
$u = 40$	.248408	.081101	.032173	.014735	.007526
$u = 50$	.180700	.045752	.014725	.005654	.002482

TABLE 2.2 diatomic approximant/$\psi(u)$ for RSS's Data

	$\theta = .1$	$\theta = .2$	$\theta = .3$	$\theta = .4$	$\theta = .5$
$u = 0$	1	1	1	1	1
$u = 10$	1.013	1.029	1.045	1.060	1.073
$u = 20$	1.003	1.007	1.012	1.015	1.018
$u = 30$	1.001	1.000	0.996	0.990	0.981
$u = 40$	1.001	0.999	0.992	0.982	0.968
$u = 50$	1.001	0.997	0.988	0.974	0.957

TABLE 2.3 diexponential approximant/$\psi(u)$ for RSS's Data

	$\theta = .1$	$\theta = .2$	$\theta = .3$	$\theta = .4$	$\theta = .5$
$u = 0$	1	1	1	1	1
$u = 10$	0.997	0.984	0.966	0.947	0.928
$u = 20$	0.994	0.985	0.979	0.978	0.984
$u = 30$	0.995	0.991	0.997	1.016	1.047
$u = 40$	0.996	1.000	1.022	1.066	1.132
$u = 50$	0.998	1.009	1.048	1.119	1.224

Example 3: In this example, we consider the group insurance data from Mereu (1972). This is also the claim distribution in Example 2 of Seah (1990). The claim amount X is discrete with support {4,6,8,10,12,14,16,20,25} and probabilities (in order) {.15304533960, .07882237436, .11199119040, .10432698260, .09432769021, .10925807990, .09727308107, .18073466720, .07022059474}.

TABLE 3.1 $\psi(u)$ by Seah for Mereu's Group Life Insurance Data
$\mu = 12.61243786$, $\sigma^2/\mu^2 = 0.25079144$, $\kappa^3/\sigma^3 = 0.30556145$

	$\theta = .25$	$\theta = .5$	$\theta = .75$	$\theta = 1$
$u = 0$	.8	.666667	.571429	.5
$u = 25$	.433995	.232316	.141606	.094198
$u = 50$	.222739	.072766	.030113	.014607
$u = 75$	.114114	.022685	.006349	.002236
$u = 100$	.058463	.007072	.001339	.000342

TABLE 3.2 diatomic approximant/$\psi(u)$ for Mereu's Group Life Insurance Data

	$\theta = .25$	$\theta = .5$	$\theta = .75$	$\theta = 1$
$u = 0$	1	1	1	1
$u = 25$	0.9995	0.9992	0.9986	0.9977
$u = 50$	1.0003	1.0004	0.9988	0.9962
$u = 75$	1.0000	0.9978	0.9929	0.9857
$u = 100$	0.9997	0.9962	0.9888	0.9795

The diatomic approximant is producing excellent values! Since the variance is quite small, there is no diexponential fit as indicated by our Proposition 2, (v).

5. THE SCHMITTER PROBLEM

The Schmitter problem asks: Given θ, u, μ, σ^2, and the range $[0,b]$, is there a distribution with support on $[0,b]$ which would maximize the ruin probability $\psi(u)$? See Brockett, Goovaerts, and Taylor (1991) and Kass (1991). Schmitter's conjecture of diatomic being the ones giving the extremal ruin probability inspires us to use diatomic as approximants. The conjecture, however, has been disproved by Kass (1991).

The general question is the stability of $\psi(u)$ when $p(x)$ is under pertubation. Schmitter specialized to the question of extreme value of $\psi(u)$ for fixed θ, u, μ, σ^2, and range $[0,b]$. We would ask another specialized question: Find the extreme value of $\psi(u)$ for fixed θ, u, μ, σ^2, and κ^3. Like the Schmitter problem, our question may not have a complete solution. Our question is related to the practical problem: When the true claim amount distribution is estimated by the sample, a discrete distribution, or a parametrized estimation based on the sample, for example, a mixture of exponentials, how robust is the ruin probability? In this paper we have found computational tools to address the stability of $\psi(u)$ while $p(x)$ is diatomic or diexponential and with fixed given first three moments.

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