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Nonparametric Tests for Heterogeneity of Risk

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ABSTRACT

Consider a portfolio of insurance policies where the mean frequency of claims for each policy may vary. This heterogeneity of risk in the portfolio may be modeled as a distribution function $F(\lambda)$ on the mean frequency λ. Let N be the frequency of claims for a policy. We know that if the conditional distribution of N given λ is Poisson and $F(\lambda)$ is gamma then N has a negative binomial distribution. Let N_i for $i=1, \ldots, T$ be the observed frequencies from a portfolio with T policies. Based on this data we may want to test the hypothesis that $F(\lambda)$ is gamma. We present a test for this and other hypotheses that is based on *nonparametric estimates of certain statistical parameters of* $F(\lambda)$.

Keywords: Risk heterogeneity, Nonparametric estimation, Tests of hypothesis.

1. Introduction

Consider a portfolio of insurance policies where the mean frquency of claims for each policy may vary. This heterogeneity of risk may be modeled as a cumulative distribution function (cdf) $F(\lambda)$ on the mean frequency λ. A reasonable and usual assumption is that the number of claims N for a policy is a Poisson random variable with a mean λ. We will show how to test this hypothesis when we observe the frequencies N_i for $i=1,\ldots,T$ from T policies. When the conditional distribution of N given λ is Poisson and $F(\lambda)$ is gamma then the unconditional distribution of N is a negative binomial. We will present some nonparametric tests for this and other hypothesis about the risk distribution function.

The discussion starts by showing that knowing the probability density function (pdf) of N is equivalent to knowing the risk distribution function $F(\lambda)$. Next we show how to estimate statistical parameters of $F(\lambda)$ and we give some asymptotic properties of these estimates. Using this theory we show how to test the Poisson hypothesis and we apply the test to some motor vehicle data given in Johnson and Hey (1971). Next we give a chi-squared test for the hypothesis that $F(\lambda)$ is in some parametric class of distributions. Using the Johnson and Hey data we find that we can reject the hypothesis that the risk distribution is gamma.

2. An Equivalence Result

In this section we will derive some relationships between the risk distribution and the claim frequency distribution. Suppose that the number of claims N for a policy can be modeled with the Poisson pdf

$$p(n|\lambda) = \frac{e^{-\lambda}\,\lambda^n}{n!} \tag{1}$$

where $\lambda>0$ is the mean frequency and $n=0,1,2,\ldots$ In a heterogeneous collection of policies the frequency λ may have a cdf $F(\lambda)$ for the whole population of risks. This means that the unconditional pdf of N is equal to the Lebesque integral

$$p(n)=\int_{(0,\infty)} p(n|\lambda)\ dF(\lambda). \tag{2}$$

We will demonstrate that knowing $p(n)$ is equivalent to knowing $F(\lambda)$ by showing that the moment generating function (mgf) of λ can be derived from the mgf of N and vice versa. See Bhat (1981) for a proof that knowing a cdf is equivalent to knowing an mgf whenever it exists. Let $t\varepsilon\Re$ and let $M_N(t)$ be the mgf of N and let $M_\lambda(t)$ be the mgf of λ, then

$$\mathrm{M}_N(t)=M_\lambda(e^t-1) \tag{3}$$

because $E(e^{tN})=E\Big(E(e^{tN}|\lambda)\Big)=E\Big(exp(\lambda(e^t-1))\Big)$. The expression in (3) gives M_N in terms of M_λ. To get M_λ in terms of M_N we let $t=ln(1+s)$ and find that

$$M_\lambda(s)=M_N(ln(1+s)). \tag{4}$$

We have just proved that knowing $p(n)$ is equivalent to knowing $F(\lambda)$, at least in theory. Although actually calculating the risk cdf from $p(n)$ is not a trivial matter when $F(\lambda)$ is not absolutely continuous. Using (4) we can immediately deduce that if N is a negative binomial random varaible then λ must have a gamma cdf. Moreover, if N is Poisson then the measure associated with the cdf of λ must put all of its mass at a single point. If M_λ exists then the moment

$$\mu_k = E(\lambda^k) \tag{5}$$

exists for $k=1,2,\ldots$ Define

$$N_{(k)} = N(N-1)\cdots(N-k+1) \tag{6}$$

for $k=1,2,\ldots$ Then

$$E(N_{(k)}) = E\Big(E(N_{(k)}|\lambda)\Big) = E(\lambda^k).$$

This relation will prove useful later. Let $g(\lambda)=\lambda^k$ and $N_{(k)}=h(N)$, then we can write

$$E\Big(h(N)\Big) = E\Big(g(\lambda)\Big). \tag{7}$$

We will prove that the identity in (7) holds whenever $g(\lambda)$ is integrable and

$$g^{(k)}(\lambda) = \frac{d^k}{d\lambda^k}\, g(\lambda) \tag{8}$$

exists for $k=0,1,2,\ldots$ and $\forall\, \lambda \,\varepsilon\, \Re$. Moreover we will show that

$$h(N) = \sum_{k=0}^{N} \binom{N}{k} g^{(k)}(0). \tag{9}$$

To prove this result we will use Fubini's theorem as given in Royden (1968). The expectation $E(h(N))$ can be written as

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} g^{(k)}(0)\; p(n). \tag{10}$$

By interchanging the summations we get

$$\sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \sum_{n=k}^{\infty} \frac{n!\; p(n)}{(n-k)!}. \tag{11}$$

Using equation (2) and interchanging the integral with the sums we get

$$\int_{(0,\infty)} \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} \sum_{n=k}^{\infty} \frac{n!\, p(n|\lambda)}{(n-k)!} \quad dF(\lambda). \tag{12}$$

Using the definition in (1) we find that

$$\lambda^k = \sum_{n=k}^{\infty} \frac{n!\, p(n|\lambda)}{(n-k)!}. \tag{13}$$

Noting that

$$g(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} g^{(k)}(0) \tag{14}$$

we find that

$$E\Big(h(N)\Big) = \int_{(0,\infty)} g(\lambda) \quad dF(\lambda) \tag{15}$$

and so the identity in (7) is true whenever $g(\lambda)$ is integrable and $h(N)$ is defined as (9) and the condition in (8) holds. From this result we find that if $g(\lambda)=p(n|\lambda)$ then $h(N)=I(N=n)$. Also if $g(\lambda)=e^{t\lambda}$ then $h(N)=(1+t)^N$ and if $g(\lambda)=\lambda^k$ then $h(N)=N_{(k)}$.

3. Estimation of Statistical Parameters

In this section we will present a nonparametric estimate of a p-dimensional vector of statistical parameters and we will describe the asymptotic behavior of this estimate. These results will also be useful later for constructing tests of hypothesis. Suppose we want to estimate $\underset{\sim}{\theta}=(\theta_1,\ldots,\theta_p)'$ where $\underset{\sim}{\theta}\,\varepsilon\,\Re^p$ and

$$\theta_k = \int_{(0,\infty)} g_k(\lambda) \quad dF(\lambda) \tag{16}$$

for $k=1,\ldots,p$. We will call θ_k a statistical parameter because it will be estimable when we observe the frequencies N_i for $i=1,\ldots,T$ from T policies in some insurance portfolio. Assume N_1, N_2, ... are independent and identically distributed random variables with a common pdf $p(n)$. An unbiased and strongly consistent estimate of $p(n)$ is

$$\hat{p}(n) = \frac{1}{T}\sum_{i=1}^{T} I(N_i=n). \tag{17}$$

Replacing $p(n)$ with $\hat{p}(n)$ in (10) yields an empirical estimate of θ_k equal to

$$\hat{\theta}_k = \sum_{n=0}^{\infty} h_k(n) \quad \hat{p}(n) \tag{18}$$

where

$$h_k(n) = \sum_{l=0}^{n} \binom{n}{l} g_k^{(l)}(0). \tag{19}$$

Using the definition in (17) we can also write (18) as

$$\hat{\theta}_k = \frac{1}{T} \sum_{i=1}^{T} h_k(N_i) \tag{20}$$

Note that $\hat{\theta}_k$ is an unbiased and strongly consistent estimate of θ_k. Let $\hat{\underset{\sim}{\theta}}=(\hat{\theta}_1,\ldots,\hat{\theta}_p)'$. Using a multivariate central limit theorem as given in Lehmann (1983) we find that if $T\rightarrow\infty$ then

$$\sqrt{T}(\ \hat{\underset{\sim}{\theta}}-\underset{\sim}{\theta}\ ) \xrightarrow{d} N_p(\ \underset{\sim}{0}, \Sigma\ ) \tag{21}$$

where $\xrightarrow{d}$ means convergence in distribution and $N_p(\ \underset{\sim}{0},\ \Sigma\ )$ is a p-dimensional normal random variable with a mean of $\underset{\sim}{0}$ and a variance-covariance matrix $\Sigma=\{\sigma_{kl}\}$ whose coordinates are equal to

$$\sigma_{kl} = E\Big(h_k(N_1)\ h_l(N_1)\Big) - \theta_k\ \theta_l \tag{22}$$

whenever all expectations exist. Consider a transformation

$$\underset{\sim}{f}(\underset{\sim}{\theta}) = \Big(f_1(\underset{\sim}{\theta}),\ \ldots\ , f_q(\underset{\sim}{\theta})\Big)' \tag{23}$$

where $\underset{\sim}{f}$ is continuously differentiable in an open ball centered at $\underset{\sim}{\theta}$. Let

$$D(\underset{\sim}{\theta}) = \left\{ \frac{\partial}{\partial\theta_k} f_j(\underset{\sim}{\theta}) \right\} \tag{24}$$

be a $p\times q$ matrix of the partial derivatives of $\underset{\sim}{f}$. Suppose D is nonsingular, then by a theorem given in Lehmann (1983) we find that if $T\rightarrow\infty$ then

$$\sqrt{T}\Big(\underset{\sim}{f}(\hat{\underset{\sim}{\theta}})-\underset{\sim}{f}(\underset{\sim}{\theta})\Big) \xrightarrow{d} N_q(\ \underset{\sim}{0},\ \Psi\ ) \tag{25}$$

where

$$\Psi = D'\Sigma D. \tag{26}$$

Now let $\hat{\Sigma} = \{\hat{\sigma}_{kl}\}$ with

$$\hat{\sigma}_{kl} = \frac{1}{T}\sum_{i=1}^{T} h_k(N_i)\, h_l(N_i) - \hat{\theta}_k\, \hat{\theta}_l \tag{27}$$

and let

$$\hat{\Psi} = \hat{D}'\hat{\Sigma}\hat{D} \tag{28}$$

where $\hat{D} = D(\hat{\underline{\theta}})$. If $T \to \infty$ then by a strong law of large numbers given in Bhat (1981) and a Mann-Wald theorem given in Serfling (1980) we must have $\|\hat{\Psi} - \Psi\| \xrightarrow{a.s.} 0$ where $\|\{a_{kl}\}\| = \underset{kl}{MAX}|a_{kl}|$ is a matrix norm. Moreover if $\hat{\Psi}$ and Ψ are positive definite and $T \to \infty$ then by using theorems in Billingsley (1968) we must have

$$\sqrt{T}\,\hat{\Psi}^{-1/2}\left(\underline{f}(\hat{\underline{\theta}}) - \underline{f}(\underline{\theta})\right) \xrightarrow{d} N_q(\underline{0}, I) \tag{29}$$

where I is a $q \times q$ identity matrix. Also if $T \to \infty$ then by using a Mann-Wald theorem given in Serfling (1980) we must have

$$T\left(\underline{f}(\hat{\underline{\theta}}) - \underline{f}(\underline{\theta})\right)'\hat{\Psi}^{-1}\left(\underline{f}(\hat{\underline{\theta}}) - \underline{f}(\underline{\theta})\right) \xrightarrow{d} \chi^2_{(q)} \tag{30}$$

where $\chi^2_{(q)}$ is a chi-squared random variable with q degrees of freedom. The approximations suggested by (30), (29), (25) and (21) will be useful for constructing tests of hypothesis.

4. A Test of the Poisson Assumption

In this section we will test the null hypothesis H_0 that the conditional distribution of N given λ is Poisson. That is we will test

$$H_0 : p(n|\lambda) = \frac{e^{-\lambda}\lambda^n}{n!}. \tag{31}$$

Let $\mu_k = E(N_{(k)})$. If $\mu_k < \mu_1^k$ for any $k=2,3,\ldots$ and $F(\lambda)$ is a cdf then (31) must be false otherwise $\mu_k \geq \mu_1^k$ $\forall k$. We can use this fact to construct a test of H_0 that has a level of significance of at most α. Let $g_k(\lambda) = \lambda^k$ and let $\theta_k = \mu_k$ in equation (16). Also let

$$h_k(n) = n_{(k)} = n(n-1)\cdots(n-k+1)$$

and let $\hat{\theta}_k = \hat{\mu}_k$ in (18). By (21) we know that if $T \to \infty$ then

$$\sqrt{T}\Big((\hat{\mu}_1,\hat{\mu}_k)' - (\mu_1,\mu_k)'\Big) \xrightarrow{d} N_2(\ (0,0)'\ ,\ \Sigma_k\ ) \tag{32}$$

where

$$\Sigma_k = \begin{bmatrix} E(N^2)-\mu_1^2 & E(NN_{(k)})-\mu_1\mu_k \\ E(NN_{(k)})-\mu_1\mu_k & E(N_{(k)}^2)-\mu_k^2 \end{bmatrix}. \tag{33}$$

Let $f(\mu_1,\mu_k) = \mu_1^k - \mu_k$. If $T \to \infty$ then by (25) we have

$$\sqrt{T}\Big((\hat{\mu}_1^k - \hat{\mu}_k) - (\mu_1^k - \mu_k)\Big) \xrightarrow{d} N_1(\ 0\ ,\ \Psi_k\ ) \tag{34}$$

where $\Psi_k = D_k' \Sigma_k D_k$ and $D_k = (k\mu_1^{k-1}, -1)'$. Let $\hat{\Psi}_k$ be an estimate of Ψ_k as suggested in (28). Then (29) says that if $T \to \infty$ then

$$\sqrt{T}\,\hat{\Psi}_k^{-1/2}\Big((\hat{\mu}_1^k - \hat{\mu}_k) - (\mu_1^k - \mu_k)\Big) \xrightarrow{d} N_1(\,0\,,1\,). \tag{35}$$

Let $z_{\alpha/p}$ be a value such that $\Phi(z_{\alpha/p}) = 1 - \alpha/p$ where

$$\Phi(z) = \int_{-\infty}^{z} \frac{e^{-t^2/2}}{\sqrt{2\pi}}\; dt. \tag{36}$$

Suppose we reject H_0 when

$$M_p = \underset{2 \le k \le p+1}{MAX} \;\; \frac{\sqrt{T}(\hat{\mu}_1^k - \hat{\mu}_k)}{\hat{\Psi}_k^{1/2}} \; > \; z_{\alpha/p}. \tag{37}$$

Then this is a test with a level of significance of at most α. This is true because asymptotically

$$Pr(M_p > z_{\alpha/p}) \le \alpha \tag{38}$$

when H_0 is true. The type of test given in (37) is sometimes called a Bonferroni multiple comparison test because a Bonferroni inequality is used to show that the level of the test is at most α. We will now give an explicit expression for (37) when $p=1$. First we find that the statistic $\hat{\mu}_k$ is equal to

$$\hat{\mu}_k = \sum_{n=0}^{\infty} n_{(k)}\; \hat{p}(n) \tag{39}$$

for $k=1,2,\ldots$ Using (33) we find that

$$\Sigma_2 = \begin{bmatrix} \mu_2+\mu_1-\mu_1^2 & \mu_3+2\mu_2-\mu_1\mu_2 \\ \mu_3+2\mu_2-\mu_1\mu_2 & \mu_4+4\mu_3+2\mu_2-\mu_2^2 \end{bmatrix} \tag{40}$$

and that

$$\Psi_2 = 4(1-\mu_1)(\mu_1^3-2\mu_2\mu_1+\mu_3)+\mu_4+2\mu_2-\mu_2^2. \tag{41}$$

Let $\hat{\Psi}_2$ be an estimate of Ψ_2 based on the estimators $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\mu}_3$, $\hat{\mu}_4$. Then we would reject H_0 if

$$M_1 \;=\; \frac{\sqrt{T}(\hat{\mu}_1^2-\hat{\mu}_2)}{\hat{\Psi}_2^{1/2}} \;>\; z_\alpha. \tag{42}$$

Let us try this test on some data given in Johnson and Hey (1971). With $T=421{,}240$ and

$$\begin{array}{ll} \hat{p}(0) = .879337 & \hat{p}(1) = .110495 \\ \hat{p}(2) = .009341 & \hat{p}(3) = .000753 \\ \hat{p}(4) = .000066 & \hat{p}(5) = .000007 \end{array} \tag{43}$$

we find that

$$\begin{array}{ll} \hat{\mu}_1 = .131735 & \hat{\mu}_2 = .024132 \\ \hat{\mu}_3 = .006522 & \hat{\mu}_4 = .002424 \\ \hat{\mu}_5 = .000840 & \hat{\mu}_6 = .000000. \end{array} \tag{44}$$

Therefore we can calculate $\hat{\Psi}_2^{1/2} = .2421$ and $M_1 = -18.17$. This means that we cannot reject H_0 at most levels of significance. Note that this test relies on the asymptotic properties of the test statistic. With $T = 421{,}240$ it is reasonable to assume that M_1 is appproximately a standard normal random variable.

5. Tests of Heterogeneity

In this section we will test hypothesis about the risk distribution function $F(\lambda)$. First, let us test the hypothesis

$$H_0 : F = F_0 \tag{45}$$

where F_0 is a completely specified cdf. Let $p(k) = \int p(k|\lambda)dF$ and let $p_0(k) = \int p(k|\lambda)dF_0$. If $p(k) \neq p_0(k)$ for any $k = 1,2,\ldots$ then (45) must be false. We can use this fact to construct a test of H_0 that has a level of significance of α. Let $g_k(\lambda) = e^{-\lambda}\lambda^k / k!$ and $\theta_k = p(k)$ in equation (16). Also let $h_k(n) = I(n = k)$ and $\hat{\theta}_k = \hat{p}(k)$ in (18). Let $\underset{\sim}{p} = (p(0), p(1), \ldots, p(r-1))'$ and $\hat{\underset{\sim}{p}} = (\hat{p}(0), \hat{p}(1), \ldots, \hat{p}(r-1))'$. Also let $\Sigma = \{\sigma_{kl}\}$ where $\sigma_{kl} = p(k-1)\Big(1 - p(k-1)\Big)$ when $k = l$ and $\sigma_{kl} = -p(k-1)p(l-1)$ when $k \neq l$ and $k, l = 1, \ldots, r$. Using (21) and a Mann-Wald theorem we find that if $T \to \infty$ then

$$T(\ \hat{\underset{\sim}{p}} - \underset{\sim}{p}\ )'\Sigma^{-1}(\ \hat{\underset{\sim}{p}} - \underset{\sim}{p}\ ) \xrightarrow{d} \chi^2_{(r)}. \tag{46}$$

Let $\underset{\sim}{p}_0$ and Σ_0 be equal to $\underset{\sim}{p}$ and Σ when H_0 is true. Let $\chi^2_{(\alpha,r)}$ be a value such that $G(\chi^2_{(\alpha,r)}) = 1 - \alpha$ where

$$G(x) = \int_0^x \frac{2^{-r/2}}{\Gamma(r/2)} y^{(r-2)/2} e^{-y/2} dy. \tag{47}$$

Suppose we reject H_0 when

$$T(\hat{\underline{p}} - \underline{p}_0)' \Sigma_0^{-1} (\hat{\underline{p}} - \underline{p}_0) > \chi^2_{(\alpha, r)}. \tag{48}$$

Then this is a test with a level of significance equal to approximately α. Note that another test statistic for (45) is one where $\hat{\underline{p}}$ and $\underline{p}_0$ in (48) are replaced with $\hat{\underline{\mu}}$ and $\underline{\mu}_0$ respectively. In this case the coordinate μ_{k0} of $\underline{\mu}_0$ is equal to $\int \lambda^k dF_0$ and the coordinate $\hat{\mu}_k$ of $\hat{\underline{\mu}}$ is equal to (39). Let us try the chi-squared test in (48) on the Johnson and Hey data presented in (43). Suppose F_0 is an exponential cdf with a mean equal to .15, then under this hypothesis we would find that

$$p_0(k) = .15^k / 1.15^{k+1}.$$

It seems reasonable to assume that the chi-squared approximation holds when $r=5$. In this case we find that the statistic in (48) is equal to 1295. Comparing this with $\chi^2_{(.005,5)} = 16.75$ we find that we would reject the null hypothesis at $\alpha = .005$.

Now suppose we wanted to test that $F(\lambda)$ is a gamma distribution or that it is a Pareto distribution. Let $\mathcal{F}$ be a parametric class of cdf's such as the gamma class. Our objective is to test the hypoyhesis

$$H_0 : F \epsilon \mathcal{F}. \tag{49}$$

The test that we present will reduce the degrees of freedom r, of a fully specified hypothesis, by the number of parameters that index members in the class $\mathcal{F}$. Suppose each element in $\mathcal{F}$ is equal to $F_{\underline{\eta}}$ for some $\underset{\sim}{\eta} \varepsilon \Re^n$. Then we can usually reparametrize the elements in $\mathfrak{F}$ with

$$\underset{\sim}{\theta} = (\mu_1, \ldots, \mu_n)' \tag{50}$$

where $\mu_k = E(\lambda^k)$. For example, if $\mathfrak{F}$ is a gamma class then $\underset{\sim}{\eta} = (\alpha, \beta)' \varepsilon (0, \infty)^2 \subset \Re^2$ and under the usual parametrization of the gamma class we find that

$$\alpha = \frac{\mu_1^2}{\mu_2 - \mu_1^2} \qquad \beta = \frac{\mu_1}{\mu_2 - \mu_1^2}. \tag{51}$$

So we can reparametrize the gamma class with $\underset{\sim}{\theta} = (\mu_1, \mu_2)'$. Under this moment parametrization we find that there exists a function m_k such that

$$\mu_k = m_k(\underset{\sim}{\theta}). \tag{52}$$

That is all moments are functions of the first n moments. If (52) is false for any $k = n+1, n+2, \ldots$ then (49) must be false. We can use this fact to construct a test of H_0 that has a level of significance of α. Let $g_k(\lambda) = \lambda^k$ and $\theta_k = \mu_k$ in equation (16). Also let $h_k(n) = n_{(k)}$ and $\hat{\theta}_k = \hat{\mu}_k$ in (18). Let $\underset{\sim}{\mu} = (\mu_1, \ldots, \mu_r)'$ and $\hat{\underset{\sim}{\mu}} = (\hat{\mu}_1, \ldots, \hat{\mu}_r)'$. Let $r > n$ and define $q = r - n$. Next consider the transformation

$$f_i(\underset{\sim}{\mu}) = \mu_{r+i} - m_{r+i}(\underset{\sim}{\theta}) \tag{53}$$

for $i = 1, \ldots, q$. Assume that $\underset{\sim}{f} = (f_1, \ldots, f_q)'$ is continuously differentiable in an open ball centered at $\underset{\sim}{\mu}$. For the gamma class this assumption holds because

$$\mu_k = \frac{\alpha}{\beta} \; \frac{\alpha+1}{\beta} \; \cdots \; \frac{\alpha+k-1}{\beta} \tag{54}$$

where α and β are given in (51). Using all the results in section 3 we can argue that rejecting H_0 when

$$T\underset{\sim}{f}(\hat{\underset{\sim}{\mu}})'\hat{\Psi}^{-1}\underset{\sim}{f}(\hat{\underset{\sim}{\mu}}) > \chi^2_{(\alpha,q)} \tag{55}$$

is a test with a level of significance of approximately α. Note that this test has $q=r-n$ degrees of freedom. So the degrees of freedom for a fully specified hypothesis $F=F_0$ was reduced by the number of estimated parameters. We will now show how to calculate the statistic in (55) when $q=1$ and $\mathfrak{F}$ is a gamma class. Using our definitions we find that

$$\underset{\sim}{f}(\hat{\underset{\sim}{\mu}}) = \hat{\mu}_3 - 2\hat{\mu}_2^2/\hat{\mu}_1 + \hat{\mu}_1\hat{\mu}_2 \tag{56}$$

and that $\hat{\Psi} = \hat{\mathrm{D}}'\hat{\Sigma}\hat{\mathrm{D}}$ where

$$\hat{D} = \begin{bmatrix} \hat{\mu}_2 + 2(\hat{\mu}_2/\hat{\mu}_1)^2 \\ \hat{\mu}_1 - 4\hat{\mu}_2/\hat{\mu}_1 \\ 1 \end{bmatrix} \tag{57}$$

and where $\hat{\Sigma} = \{\hat{\sigma}_{kl}\}$ with

$$\begin{aligned} \hat{\sigma}_{11} &= \hat{\mu}_2 + \hat{\mu}_1 - \hat{\mu}_1^2 & \hat{\sigma}_{12} &= \hat{\mu}_3 + 2\hat{\mu}_2 - \hat{\mu}_1\hat{\mu}_2 \\ \hat{\sigma}_{22} &= \hat{\mu}_4 + 4\hat{\mu}_3 + 2\hat{\mu}_2 - \hat{\mu}_2^2 & \hat{\sigma}_{13} &= \hat{\mu}_4 + 3\hat{\mu}_3 - \hat{\mu}_1\hat{\mu}_3 \\ \hat{\sigma}_{33} &= \hat{\mu}_6 + 9\hat{\mu}_5 + 18\hat{\mu}_4 + 6\hat{\mu}_3 - \hat{\mu}_3^2 & \hat{\sigma}_{23} &= \hat{\mu}_5 + 6\hat{\mu}_4 + 6\hat{\mu}_3 - \hat{\mu}_2\hat{\mu}_3. \end{aligned} \tag{58}$$

Let us try this chi-squared test on the Johnson and Hey data presented in (44). We will test that F is in a gamma class and use formulas (55) to (58). The statistic in (55) is equal to 6.01. Comparing this with $\chi^2_{(.025,1)}=5.024$ and with $\chi^2_{(.01,1)}=6.635$ we find that we would accept H_0 at $\alpha=.01$ but we would reject it at $\alpha=.025$. In other words the P-value is between .025 and .01.

6. Summary

This paper presents test statistics that are asymptotically normal or chi-squared. The construction of these tests relies on an equivalence relation between the observed claim frequency of an insurance portfolio and the risk distribution. We present a test for the assumption that the number of claims for each policy is Poisson. Using data from Johnson and Hey we find that we cannot reject this hypothesis. We also present a test for identifying risk distributions when they are completely specified. But more importantly we present a test for identifying that the risk distribution is in some parametric class like the gamma class. We also show that the degrees of freedom for this chi-squared test are reduced by the number of estimated parameters. Using this test on the Johnson and Hey data we find that we can reject the hypothesis that the risk distribution is gamma at a level of significance equal to 2.5%.

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