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Abstract

ANNUITIES FOR THE AGED

by

Cecil Nesbitt and Marjorie Rosenberg

This paper began with a study of the duality of individual risk theory for annual premium whole life insurance and for amortization of a unit loan by annual payments of $\frac{1}{\ddot{a}_x}$ while (x) survives. We refer to this annuity process as survivorship amortization. Actually, any annuity can be regarded as a set of periodic amortization payments, to be made while a given status exists, in order to discharge the obligation for an initial provision of capital. This idea could lead to a paper by itself but will be explored only briefly in the Introduction.

The possibility of converting property of an aged individual into a life income (from a survivorship annuity) appealed to the authors as being a matter of increasing public interest. That led us to consider individual risk theory for various forms of annuities for the aged.

Section 2 shows a tabular and graphical representation of probabilities of death in successive years for individuals initially aged 65,75,85 and 95. The probabilities are derived from the Blended 1983 α-D-Mortality Table.

Individual risk theory for whole life annuities issued at ages 65,75,85 and 95 is discussed mathematically, and illustrated, in Section 3. Some analysis is extended to annuities payable while a joint-and-last-survivor status exists. This is followed in Section 4 by individual risk theory for annuities modified by certain-periods.

Essentially, the paper is an individual-risk-theoretical examination of annuities for the aged. Some initial conclusions from this analysis appear in Section 5. Through retirement systems, social security, and insurance plans, annuities for the aged are a very significant matter, and worthy of inquiry from many view points. We hope our paper will stimulate such inquiry.

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1. INTRODUCTION

In reviewing Hans Gerber's Life Insurance Mathematics [4, Gerber, 1990], Cecil Nesbitt wrote the following paragraph concerning the whole life insurance reserve ${}_kV_x$:

"This can take a number of different forms, such as

$$ {}_kV_x = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} , \tag{1.1} $$

which the author regards as being of somewhat less importance. The right hand member, however, has an interesting interpretation in terms of survivorship amortization of a loan of 1 by annual payments of $\frac{1}{\ddot{a}_x}$ at the beginning of each year while (x) survives. By taking account of both interest and survivorship, one can see that the outstanding principal for a survivor at age $x + k$ is $\frac{\ddot{a}_{x+k}}{\ddot{a}_x}$, and its complement, $1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$ is the amount of principal considered to be repaid for a survivor at age $x + k$. For whole life insurances, these two quantities are the net amount at risk in the kth policy year, and the reserve at the end of k policy years, respectively. Survivorship amortization could have practical significance for the aged in the disposition of property."

That quotation provided the origin and set the initial direction for this paper. However, in the interest of brevity and relevance, the review editor omitted the whole paragraph!

To explore the duality between whole life insurance from age x , and survivorship amortization by payments $\frac{1}{\ddot{a}_x}$, we consider the random variable, J , the curtate future lifetime of a survivor aged $x + k$, and two loss variables, namely:

$$ {}_kL_1 = v^{J+1} - P_x \ddot{a}_{\overline{J+1|}} \tag{1.2} $$

and

$$ {}_kL_2 = \frac{\ddot{a}_{\overline{J+1|}}}{\ddot{a}_x} \tag{1.3} $$

$J = 0,1,2,\dots$. Here, ${}_kL_1$ expresses the random present value of the net future pay-out to a survivor at age $x + k$ under a whole life insurance of 1 , and ${}_kL_2$ similarly expresses the random present value of the future pay-out under a whole life annuity with payments of $\frac{1}{\ddot{a}_x}$, for a survivor who has attained age $x + k$.

We arranged (1.2) as

$$ {}_kL_1 = 1 - d\ddot{a}_{\overline{J+1|}} - \left(\frac{1}{\ddot{a}_x} - d\right) \ddot{a}_{\overline{J+1|}} $$

$$ = 1 - \frac{\ddot{a}_{\overline{J+1|}}}{\ddot{a}_x} , $$

$$ = 1 - {}_kL_2 . \tag{1.4} $$

Further, we have from $\left[1,(5.4.3)\right]$ that

$$ E\left[{}_kL_2\right] = \sum_{h=0}^{\infty} \frac{\ddot{a}_{\overline{h+1|}}}{\ddot{a}_x} \, {}_{h|}q_{x+k} , $$

$$ = \frac{\ddot{a}_{x+k}}{\ddot{a}_x} , $$

= the actuarial present value of the future payments of $\frac{1}{\ddot{a}_x}$ for a person aged $x + k$.

= the outstanding principal at age $x + k$ under the survivorship amortization.

Also,

$$E\left[{}_kL_1\right] = 1 - E\left[{}_kL_2\right]$$

$$= 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x} , \qquad (1.5)$$

$$= {}_kV_x \quad .$$

Finally,

$$\mathrm{Var}\left[{}_kL_1\right] = \mathrm{Var}\left[{}_kL_2\right]$$

$$= \mathrm{Var}\left[\frac{1 - v^{J+1}}{d\ddot{a}_x}\right],$$

$$= \left(\frac{1}{d\ddot{a}_x}\right)^2 \mathrm{Var}\left[v^{J+1}\right] ,$$

$$= \left(\frac{1}{d\ddot{a}_x}\right)^2 \left({}^2A_{x+k} - A^2_{x+k}\right) \qquad (1.6)$$

by [1, (7.4.3) and p. 99]. Note that $\frac{1}{d\ddot{a}_x} = 1 + \frac{P_x}{d}$ and ${}^2A_{x+k}$ is calculated at rate of interest $[(1+i)^2 - 1]$.

We were intrigued that the complementarity of the loss random variables, ${}_kL_1$ and ${}_kL_2$, carries through to complementarity of the functions ${}_kV_x$ and $\frac{\ddot{a}_{x+k}}{\ddot{a}_x}$, and their complements, $1 - {}_kV_x$ and $1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x}$, all of which have distinctive interpretations in their whole life insurance or survivorship amortization settings. The complementarity also implies equal variances of ${}_kL_1$ and ${}_kL_2$.

If we think of $\ddot{a}_y$ as providing an amortization annuity with payments of 1 during the survival of (y) , we have

$$\ddot{a}_y = 1 + vp_y\ddot{a}_{y+1}$$
$$1 = \ddot{a}_y - vp_y\ddot{a}_{y+1}$$
$$= (v + d)\,\ddot{a}_y - vp_y\ddot{a}_{y+1}$$
$$= d\ddot{a}_y + v(p_y + q_y)\,\ddot{a}_y - vp_y\ddot{a}_{y+1}$$
$$= d\ddot{a}_y + vq_y\ddot{a}_y + vp_y\left(\ddot{a}_y - \ddot{a}_{y+1}\right) \qquad (1.7a)$$

If we consider that the annuitant is the borrower of a loan of $\ddot{a}_x$ from an annuity organization, and is amortizing the loan by payments of 1 at the beginning of each year while the annuitant survives, we can interpret the payment at attained age y to consist of the sum of:

(1) Interest-in-advance, $d\ddot{a}_y$, on the remaining loan;

(2) The net single premium to provide, in case of death during the year, the discharge of the loan balance, $\ddot{a}_y$;

(3) The pure endowment single premium to provide, in case of survival over the year, for the reduction of the loan balance from $\ddot{a}_y$ down to $\ddot{a}_{y+1}$.

If we consider that the annuity organization has received a deposit of $\ddot{a}_x$ and at the beginning of each year pays 1 to (x) while (x) survives, then the payment of 1 at attained age y to the annuitant can be considered to be the sum of:

(1) Interest-in-advance, $d\ddot{a}_y$, on the remaining deposit,

(2) The present value of the expected discharge of the deposit in the case (y) dies in the year;

(3) The present value of the expected decrease in the deposit in case (y) survives the year.

Note that in (1.7a) interest-in-advance is paid on the outstanding principal at the beginning of the year, and that unlike in the reserve situation where the initial reserve is $\ddot{a}_y - 1$, the outstanding principal here remains at $\ddot{a}_y$ until the end of the year when it is either cancelled because of death or is reduced to $\ddot{a}_{y+1}$ in case of survival. Multiplying (1.7a) by $1 + i$, and using $\ddot{a}_z = 1 + a_z$, $z = y$, $y + 1$, yields

$$1 = i\, a_y + q_y\,(1 + a_y) + p_y(a_y - a_{y+1}) \qquad (1.7b)$$

which is interesting to interpret. It indicates again that survivorship amortization is different from annuity reserve theory.

Other arrangements of (1.7a), such as

$$1 = d\ddot{a}_y + v\left(\ddot{a}_y - \ddot{a}_{y+1}\right) + vq_y\ddot{a}_{y+1} \qquad (1.7c)$$

and

$$1 = v\left(i + q_y\right)\ddot{a}_y + vp_y\left(\ddot{a}_y - \ddot{a}_{y+1}\right) \qquad (1.7d)$$

give modified interpretations. Formula (1.7d) indicates on increasing yield rate $\left(i + q_y\right)$ on the decreasing annuity value $\ddot{a}_y$. In fact, for $y = \omega - 1$, where ω is the limiting age, the yield rate is $1 + i$, but on a balance that has decreased to 1 !

These ideas are more or less familiar. In the present setting they may provide new insights, since survivorship amortization for converting property into annual income may be of interest to seniors. For a contrasting approach to home equity conversion see [3, Diventi and Herzog]. We have also realized that all annuities can be interpreted as providing survivorship amortization over a term defined by the continued existence of some status. This is not the usual approach to life annuities, to which we now return. In the next section, we take a look at some of the probability functions that may be useful for valuing annuities for the aged.

2. PROBABILITY FUNCTIONS FOR THE AGED

Our purpose here is to illustrate the probabilities that are basic to the mathematical formulas to follow, rather than to present numerical probabilities judged to be realistic for a particular group. We first examine the probability function (p.f.)

$$ {}_{h|}q_x = \frac{l_{x+h} - l_{x+h+1}}{l_x} , \tag{2.1} $$

on the basis of the Blended 1983 α-D-Mortality Table [5, Johansen, 1987]. Figure 2.1 displays a graph of these functions for $x = 65$, 75, 85 and 95. Table A.1 in the Appendix exhibits these probability functions numerically.

We next consider the p.f., ${}_{h|}q_{\overline{xy}}$, where

$$ {}_{h|}q_{\overline{xy}} = {}_{h|}q_x + {}_{h|}q_y - {}_{h|}q_{xy} . \tag{2.2} $$

Formula (2.2) does not require an independence assumption. If independence is assumed, then

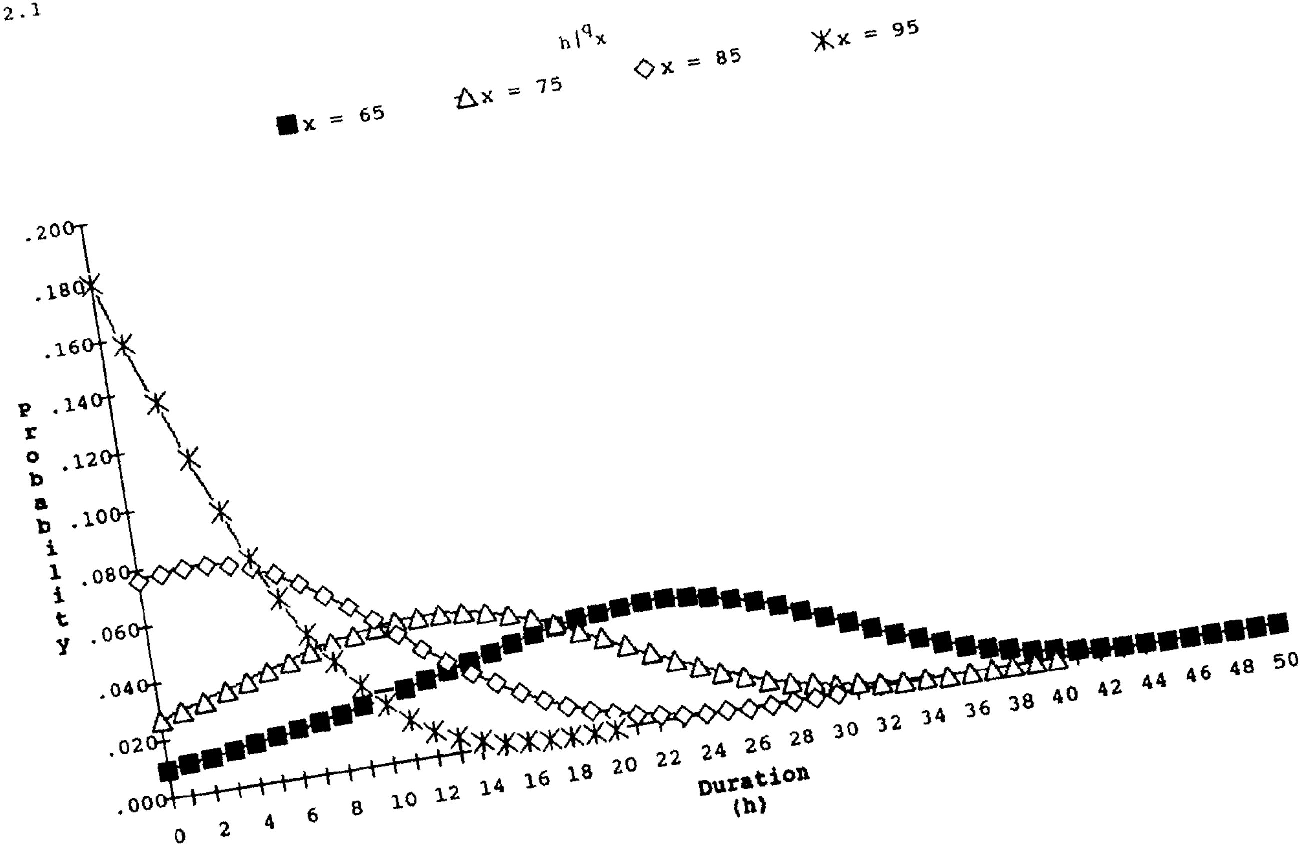
$$ \begin{aligned} {}_{h|}q_{\overline{xy}} &= {}_{h+1}q_x - {}_hq_x + {}_{h+1}q_y - {}_hq_y \\ &\quad - \left({}_hp_x \cdot {}_hp_y - {}_{h+1}p_x \cdot {}_{h+1}p_y\right) . \\ &= {}_{h+1}q_x - {}_hq_x + {}_{h+1}q_y - {}_hq_y - \left(1 - {}_hq_x\right)\left(1 - {}_hq_y\right) \\ &\quad + \left(1 - {}_{h+1}q_x\right)\left(1 - {}_{h+1}q_y\right) . \\ &= {}_{h+1}q_x \cdot {}_{h+1}q_y - {}_hq_x \cdot {}_hq_y , \end{aligned} \tag{2.3} $$

provides a simple means of computation.

The p.f.'s, ${}_{h|}q_x$ and ${}_{h|}q_{\overline{xx}}$ are compared graphically in Figure 2.2 for $x = 65$, and in Appendix Figures A.1, A.2, and A.3 for $x = 75$, 85, and 95. Figure 2.3 displays a graph of ${}_{h|}q_{\overline{xx}}$ for $x = 65$, 75, 85, and 95. Numerical values of ${}_{h|}q_{\overline{xx}}$ are tabulated in Table A.2.

Figure 2.1 is related to [1, Figure 3.2] and to [6, McCrory, Figure 1]. While McCrory examined the distributions of present values of life annuities for various groups of lives, we shall use the probability functions to study individual risk theory for various individual annuities. By both approaches, variance estimates can be made for individual and group annuities, but our emphasis will be on individual risks, especially at the higher ages.

FIGURE 2.1



$_{h|}q_x$
x = 65
x = 75
x = 85
x = 95
Probability
.200
.180
.160
.140
.120
.100
.080
.060
.040
.020
.000
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50
Duration
(h)

FIGURE 2.2

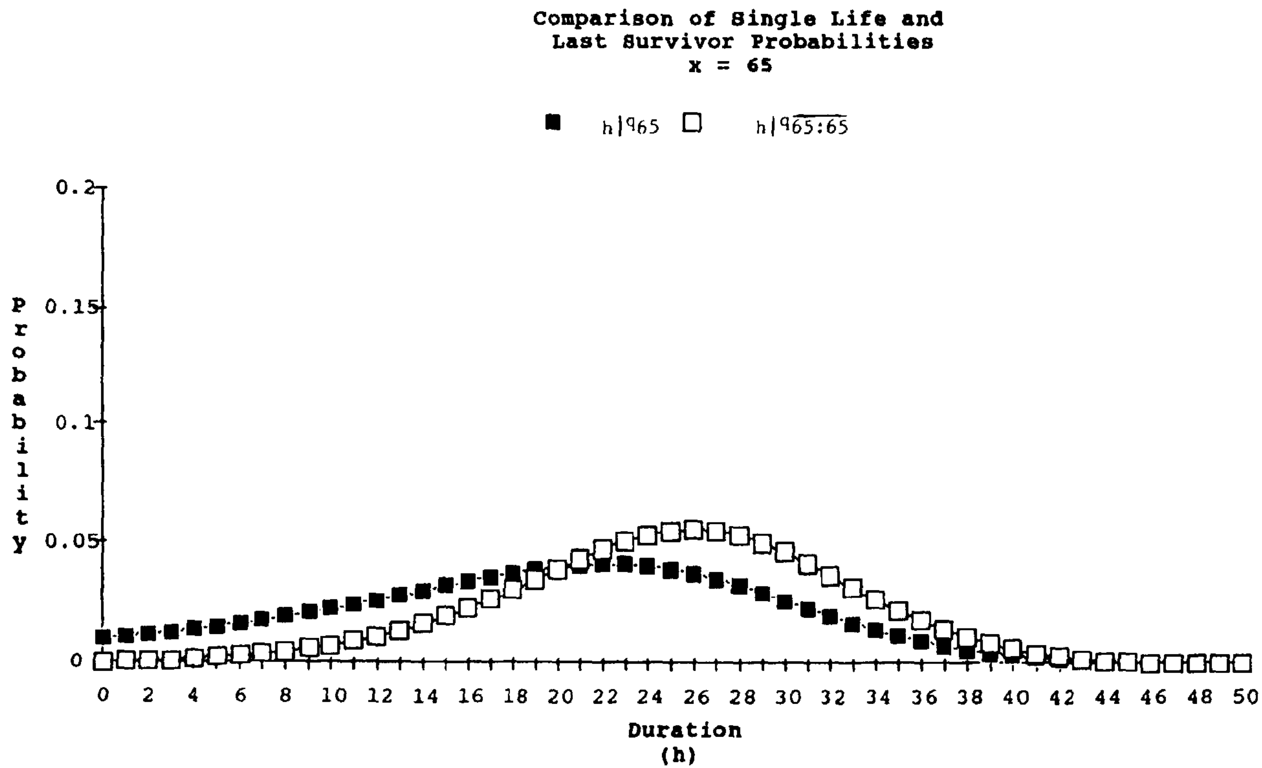
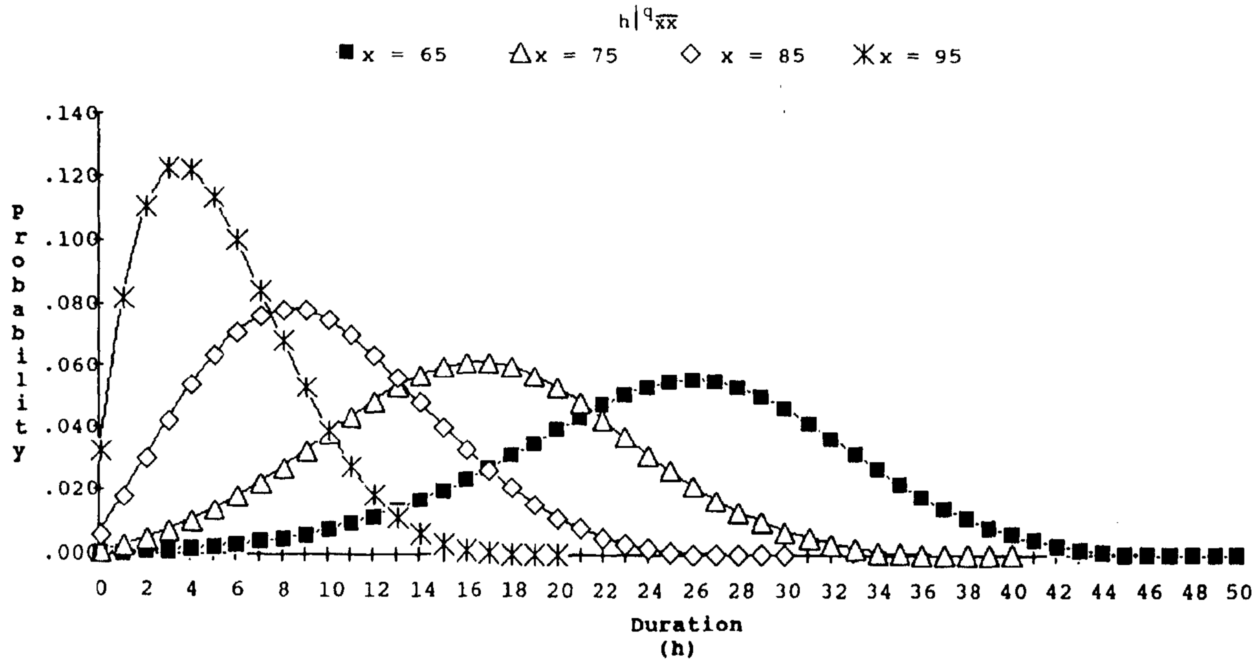

Comparison of Single Life and
Last Survivor Probabilities
x = 65
h|q65
h|q65:65
Probability
0
0.05
0.1
0.15
0.2
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50
Duration
(h)

FIGURE 2.3



$_{h|}q_{\overline{xx}}$
x = 65
x = 75
x = 85
x = 95
.140
.120
.100
.080
.060
.040
.020
.000
probability
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50
Duration
(h)

3. RISK THEORY FOR ANNUITIES DEPENDING ON LIFE STATUSES ONLY

With Section 7.10 of <u>Actuarial Mathematics</u> at hand, we review individual risk theory in regard to an annuity-due of 1 per year payable during the survivorship of (x). As overall loss, we have

$$L_3 = \ddot{a}_{\overline{K+1}|} - \ddot{a}_x \tag{3.1}$$

where K is the random variable denoting the curtate future lifetime of (x). We allocate to the $(h+1)$th year, the loss (gain) valued at time h, namely

$$\Lambda_h = \begin{cases} 0 & K \le h-1 \\ -\ddot{a}_{x+h} + 1 & K = h \\ v\ddot{a}_{x+h+1} - \ddot{a}_{x+h} + 1 & K \ge h+1 \end{cases} \tag{3.2}$$

Here Λ_h is based on the concepts that at the beginning of the year a payment reduces the reserve to $\ddot{a}_{x+h} - 1$; if (x) dies during the year, such initial reserve is released and provides a gain; and if (x) survives the year, the initial reserve must be built up to $\ddot{a}_{x+h+1}$. One calculates that

$$E\left[\Lambda_h\right] = {}_hp_x\left[\left(-\ddot{a}_{x+h} + 1\right)q_{x+h} + \left(v\ddot{a}_{x+h+1} - \ddot{a}_{x+h} + 1\right)p_{x+h}\right]$$

$$= {}_hp_x\left[-\ddot{a}_{x+h} + 1 + vp_{x+h}\,\ddot{a}_{x+h+1}\right]$$

$$= 0 \tag{3.3}$$

Also,

$$\operatorname{Var}\left[\Lambda_h\right] = {}_hp_x\left[\left(-\ddot{a}_{x+h} + 1\right)^2 q_{x+h} + \left(v\ddot{a}_{x+h+1} - \ddot{a}_{x+h} + 1\right)^2 p_{x+h}\right]$$

$$= {}_hp_x\left[\left(-vp_{x+h}\,\ddot{a}_{x+h+1}\right)^2 q_{x+h} + \left(vq_{x+h}\ddot{a}_{x+h+1}\right)^2 p_{x+h}\right]$$

$$= {}_hp_x\left[v^2\ddot{a}^{\,2}_{x+h+1}\,p_{x+h}\,q_{x+h}\right] \tag{3.4}$$

As in [1, Section 7.10], we consider the sum

$$\sum_{h=0}^{\infty} v^h \Lambda_h = \sum_{h=0}^{K-1} v^h \Lambda_h + v^K \Lambda_K + \sum_{h=K+1}^{\infty} v^h \Lambda_h$$

$$= \sum_{h=0}^{K-1} v^h \left(v\ddot{a}_{x+h+1} - \ddot{a}_{x+h} + 1 \right) + v^K \left(-\ddot{a}_{x+K} + 1 \right) + 0 \ ,$$

since in the first sum, $h \le K - 1$, or $K \ge h + 1$; and in the second sum, $h \ge K + 1$, or $K \le h - 1$, so that $\Lambda_h = 0$. Then,

$$\sum_{h=0}^{\infty} v^h \Lambda_h = \sum_{h=0}^{K-1} \left[\left(v^{h+1}\ddot{a}_{x+h+1} - v^h \ddot{a}_{x+h} \right) + v^h \right] - v^K \ddot{a}_{x+K} + v^K$$

$$= \sum_{h=0}^{K-1} \Delta \left(v^h \ddot{a}_{x+h} \right) + \ddot{a}_{\overline{K+1|}} - v^K \ddot{a}_{x+K}$$

$$= v^h \ddot{a}_{x+h} \Big|_0^K + \ddot{a}_{\overline{K+1|}} - v^K \ddot{a}_{x+K}$$

$$= \ddot{a}_{\overline{K+1|}} - \ddot{a}_x \ ,$$

that is,

$$\sum_{h=0}^{\infty} v^h \Lambda_h = L_3 \ . \tag{3.5}$$

Proceeding as in the proof of [1, Theorem 7.10], one can show that

a. $\mathrm{Cov}\left[\Lambda_h \ , \ \Lambda_j\right] = 0 \ , \ h \neq j$

b. $\mathrm{Var}\left[L_3\right] = \sum_{h=0}^{\infty} v^{2h} \ \mathrm{Var}\left[\Lambda_h\right]$

Then $\mathrm{Var}\left[L_3\right]$ can be calculated directly by the formula

$$\mathrm{Var}\left[L_3\right] = \sum_{h=0}^{\infty} \ddot{a}^{\,2}_{\overline{h+1}|}\ {}_{h|}q_x - \ddot{a}_x^2 , \tag{3.6}$$

or indirectly by the formula

$$\mathrm{Var}\left[L_3\right] = \mathrm{Var}\left[\frac{1-v^{K+1}}{d} - \ddot{a}_x\right]$$

$$= \mathrm{Var}\left[\frac{v^{K+1}}{d}\right]$$

$$= \frac{1}{d^2}\left[{}^2A_x - A_x^2\right] . \tag{3.7}$$

Alternatively, one can use

$$\mathrm{Var}\left[L_3\right] = \sum_{h=0}^{\infty} v^{2h}\ \mathrm{Var}\left[\Lambda_h\right]$$

$$= \sum_{h=0}^{\infty} v^{2h}\,{}_hp_x\left[v^2\ddot{a}^2_{x+h+1}\ p_{x+h}q_{x+h}\right] . \tag{3.8}$$

To interpret (3.8), let us consider the year of age $(y, y+1)$. Let M_y be a random variable denoting the present value at the beginning of the year of the annuity reserve required at the end of the year. Then

$$M_y = \begin{cases} 0 & \text{with probability } q_y \\ v\ddot{a}_{y+1} & \text{with probability } p_y . \end{cases}$$

It follows that

$$E\left[M_y\right] = v\ddot{a}_{y+1}p_y \tag{3.9}$$

and

$$\mathrm{Var}\left[M_y\right] = \left(v\ddot{a}_{y+1}\right)^2 p_yq_y \tag{3.10}$$

It is convenient to denote $\mathrm{Var}\left[M_y\right]$ by $V(y, y+1)$. Then formula (3.8) can be written as

$$\mathrm{Var}\left[L_3\right] = \sum_{h=0}^{\infty} v^{2h}\,{}_hp_x\; V(x+h,x+h+1) \tag{3.11}$$

Formula (3.11) indicates that the overall $\mathrm{Var}\left[L_3\right]$ is the sum of the one-year variances modified by appropriate interest and probability factors.

We can generalize (3.8) and (3.11) in various ways. As in Section 7.10 of [1], we can show that the variance of the loss, ${}_kL_3$, in respect to the years following attainment of age $x + k$ is given by

$$\mathrm{Var}\left[{}_kL_3\right] = \sum_{h=0}^{\infty} v^{2h}\,{}_hp_{x+k}\; V(x+k+h,x+k+h+1) \tag{3.12}$$

Let us denote the annuity variance, discounted to age y, in respect to the interval from age y to age $y + m$ by $V(y,y+m)$. Setting $y = x + k$ and $z = x + k + h$, we can rewrite (3.12) as

$$V(y,\infty) = \sum_{z=y}^{\infty} v^{2(z-y)}\,{}_{z-y}p_y V(z,z+1)\ . \tag{3.13}$$

Also, $V(y,y+m) = V(y,\infty) - v^{2m}\,{}_mp_y V(y+m,\infty)$

$$= \sum_{z=y}^{y+m-1} v^{2(z-y)}\,{}_{z-y}p_y V(z,z+1)\ . \tag{3.14}$$

[c.f. 1,(7.10.10)]. Formula (3.14) is a convenient means for estimating discounted variances for various intervals at various stages in the term of an annuity. Using the fact that $\mathrm{Cov}\left[\Lambda_h\ ,\ \Lambda_j\right] = 0$, $h \neq j$, the reader can verify from (3.4) and (3.10) that $V(y,y+m)$ is the variance of the loss variable $\sum_{h=0}^{m-1} v^h\,\Lambda_h$, where h is here duration from attainment of age y .

Figure 3.1 graphs $V(y,y+1)$ for $y = 65$ to $y = 115$, while Figure 3.2 graphs the cumulative variances for $x = 65, 75, 85, 95$.

FIGURE 3.1

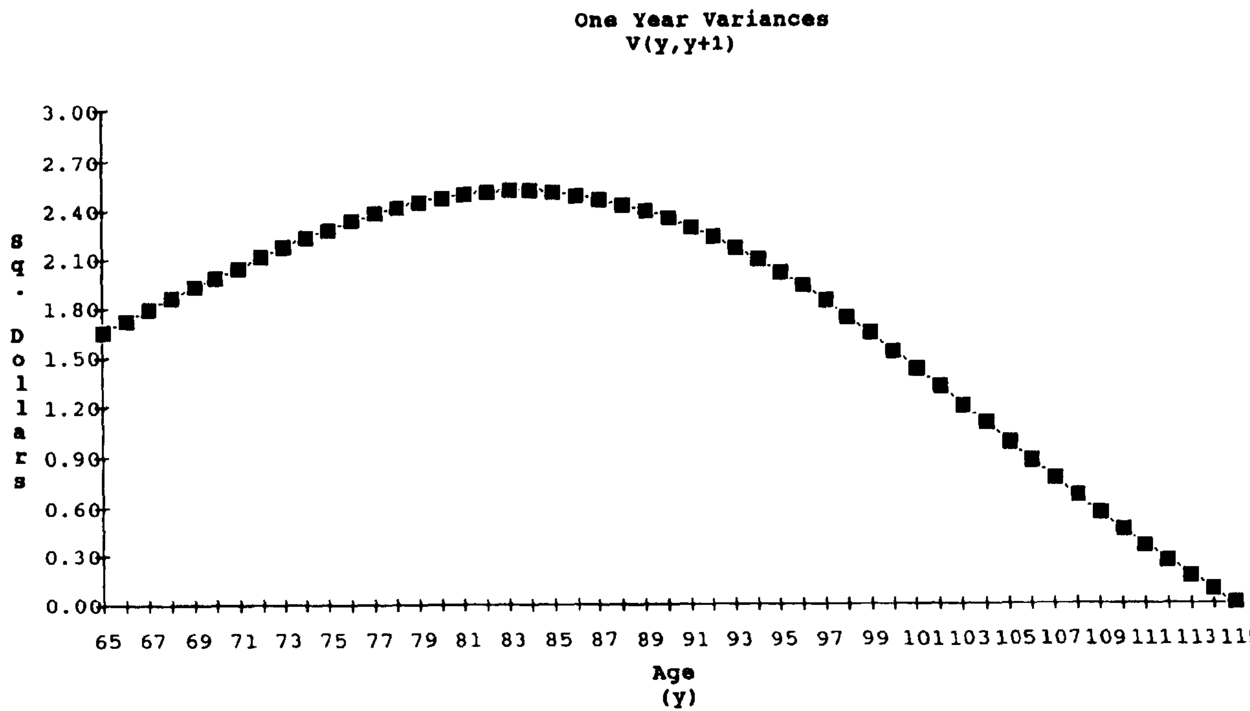
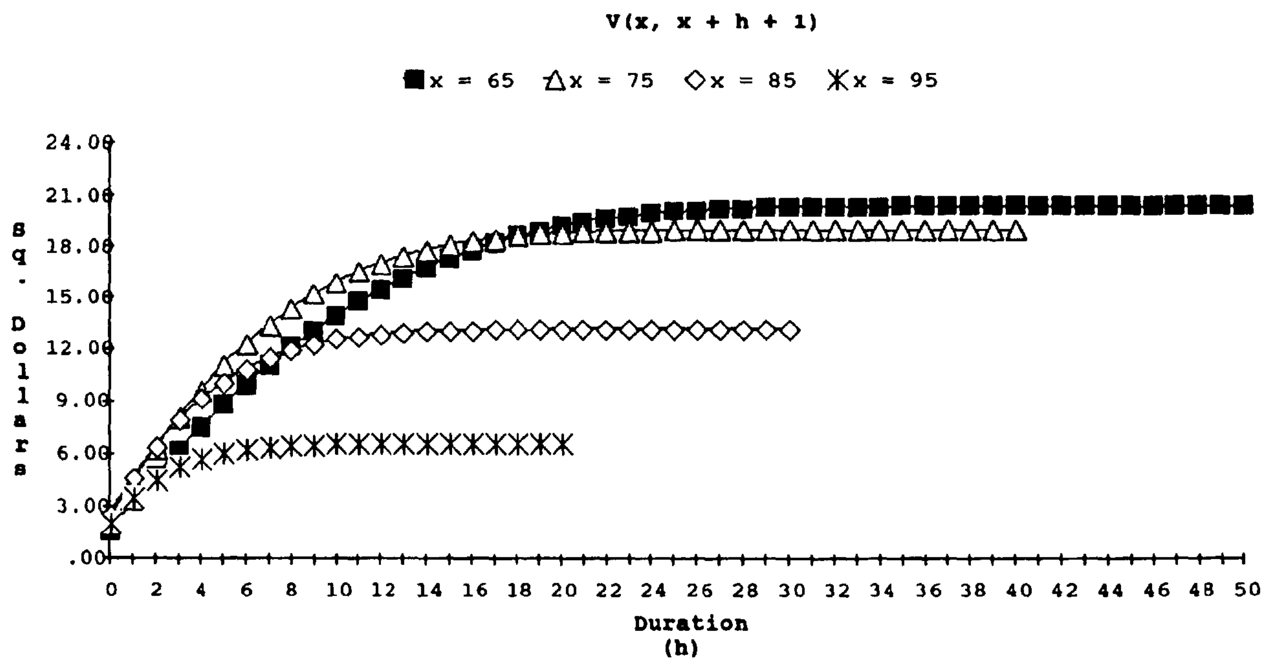

One Year Variances
V(y,y+1)
3.00
2.70
2.40
2.10
1.80
1.50
1.20
0.90
0.60
0.30
0.00
Sq. Dollars
65 67 69 71 73 75 77 79 81 83 85 87 89 91 93 95 97 99 101 103 105 107 109 111 113 115
Age
(y)

FIGURE 3.2



V(x, x + h + 1)
x = 65
x = 75
x = 85
x = 95
Sq. Dollars
24.00
21.00
18.00
15.00
12.00
9.00
6.00
3.00
.00
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40 42 44 46 48 50
Duration
(h)

Examples of the use of formula (3.14) are:

$V(65,75) = 13.058$	$V(65,85) = 18.948$	$V(65,95) = 20.309$	$V(65,\infty) = 20.403$
	$V(75,85) = 15.218$	$V(75,95) = 18.735$	$V(75,\infty) = 18.976$
		$V(85.95) = 12.273$	$V(85,\infty) = 13.114$
			$V(95,\infty) = 6.580$

How far is it reasonable to take this analysis if the annuity is payable while the status $\overline{xy}$ survives. The overall loss, L_4 , is given by

$$L_4 = \ddot{a}_{\overline{K+1}|} - \ddot{a}_{\overline{xy}} , \tag{3.15}$$

where now K is the curtate duration of $\overline{xy}$. For $x = y$, we can use Table A.2 to calculate $\mathrm{Var}\left[L_4\right]$ by means of the formula

$$\mathrm{Var}\left[L_4\right] = \sum_{h=0}^{\infty} \left(\ddot{a}_{\overline{h+1}|}\right)^2 {}_{h|}q_{\overline{xx}} - \left(\ddot{a}_{\overline{xx}}\right)^2 . \tag{3.16}$$

or, indirectly by the formula

$$\mathrm{Var}\left[L_4\right] = \left(\frac{1}{d}\right)^2 \left({}^2A_{\overline{xx}} - A_{\overline{xx}}^2\right) . \tag{3.17}$$

We have tested formulas (3.16) and (3.17) for $x = 65$, and on the basis of interest at 4 percent, find $\mathrm{Var}\left[L_4\right] = 8.575$.

The allocation of loss to individual annuity years is complicated by the fact that at the time of the first death, the annuity changes from one depending on the survivorship of $\overline{xx}$ to one depending on the survivorship of the remaining single life. This implies a discrete change in the annuity reserve, and also in the loss function. After these changes, the allocation of loss to future annuity years can proceed by the methods developed earlier in this section. At this point, it does not seem worthwhile to analyze the allocation of loss to the annuity years preceding the first death.

4. RISK THEORY FOR ANNUITIES INVOLVING A CERTAIN-PERIOD

Here we consider an annuity-due of 1 per annum payable during the survivorship of the status $\overline{x:\overline{n|}}$. This annuity can be considered as an annuity-certain with present value $\ddot{a}_{\overline{n|}}$, plus a life annuity-due deferred n years, with actuarial present value ${}_{n|}\ddot{a}_x = v^n\,{}_np_x\,\ddot{a}_{x+n}$. The only mortality risk pertains to this latter annuity, and we define

$$L_S = \begin{cases} 0 - {}_{n|}\ddot{a}_x & K < n \\ v^n\ddot{a}_{\overline{K-n+1|}} - {}_{n|}\ddot{a}_x & K \geq n \end{cases} \tag{4.1}$$

Then

$$E\left[L_S\right] = \sum_{h=0}^{\infty}\left(v^n\ddot{a}_{\overline{h+1|}}\;{}_np_x \cdot {}_{h|}q_{x+n}\right) - {}_{n|}\ddot{a}_x$$

$$= v^n\,{}_np_x\left(\sum_{h=0}^{\infty}\ddot{a}_{\overline{h+1|}} \cdot {}_{h|}q_{x+n} - \ddot{a}_{x+n}\right)$$

$$= 0 \tag{4.2}$$

$$\mathrm{Var}\left[L_S\right] = \sum_{h=0}^{\infty}\left[\left(v^n\ddot{a}_{\overline{h+1|}}\right)^2 {}_np_x \cdot {}_{h|}q_{x+n}\right] - \left({}_{n|}\ddot{a}_x\right)^2 \tag{4.3a}$$

$$= v^{2n}\,{}_np_x\sum_{h=0}^{\infty}\left(\ddot{a}_{\overline{h+1|}}\right)^2 {}_{h|}q_{x+n} - \left(v^n\,{}_np_x\,\ddot{a}_{x+n}\right)^2$$

$$= v^{2n}\,{}_np_x\left(\sum_{h=0}^{\infty}\left[\left(\ddot{a}_{\overline{h+1|}}\right)^2 {}_{h|}q_{x+n}\right] - \ddot{a}^2_{x+n}\right) + v^{2n}\,{}_np_x\,\ddot{a}^2_{x+n}\left(1 - {}_np_x\right)$$

$$= v^{2n}\ddot{a}^2_{x+n} \cdot {}_np_x\,{}_nq_x + v^{2n}\,{}_np_x\,V(x+n,\infty)\ , \tag{4.3b}$$

where $V(x+n,\infty)$ is defined by (3.13).

Formula (4.3b), which exhibits the variance in respect to the deferred annuity is made up from the variance arising from the uncertainty of (x) surviving to age $x+n$ with probability ${}_np_x$ and reward, $v^n\ddot{a}_{x+n}$, and the variance appropriately discounted under interest and survivorship, with respect to the annuity represented by $\ddot{a}_{x+n}$. For $V(x+n,\infty)$, we can use the analysis and formulas of the preceding section.

Thus, for ${}_{10|}\ddot{a}_{65}$, on the basis of the Blended 1983 α-D-Mortality Table with 4% interest, we have

$$\mathrm{Var}\left[L_5\right] = v^{20}\ddot{a}^2_{75}\ {}_{10}p_{65}\ {}_{10}q_{65} + v^{20}\ {}_{10}p_{65}\ V(75,\infty)$$

$$= 13.272\ .$$

For ${}_{20|}\ddot{a}_{65}$, we have

$$\mathrm{Var}\left[L_5\right] = v^{40}\ddot{a}^2_{85}\ {}_{20}p_{65}\ {}_{20}q_{65} + v^{40}\ {}_{20}p_{65}\ V(85,\infty)$$

$$= 3.656\ .$$

In Figure 4.1, these variances are compared with $\mathrm{Var}\left[L_3\right]$ for $x = 65$. Appendix Table A.3 gives numerical comparisons of $\mathrm{Var}\left[L_3\right]$ and $\mathrm{Var}\left[L_5\right]$ for $x = 65,\ 75,\ 85$ and 95.

A relation between $\mathrm{Var}\left[L_5\right]$ and $\mathrm{Var}\left[L_3\right]$ is easily obtained. We have

$$\mathrm{Var}\left[L_3\right] = \sum_{h=0}^{n-1} v^{2h}\,{}_hp_x V(x+h,x+h+1) + \sum_{h=n}^{\infty} v^{2h}\,{}_hp_x V(x+h,x+h+1)$$

$$= \sum_{h=0}^{n-1} v^{2h}\,{}_hp_x V(x+h,x+h+1) + v^{2n}\,{}_np_x \sum_{j=0}^{\infty} v^{2j}\,{}_jp_{x+n} V(x+n+j,x+n+j+1)$$

$$= \sum_{h=0}^{n-1} v^{2h}\,{}_hp_x V(x+h,x+h+1) + v^{2n}\,{}_np_x V(x+n,\infty)\ . \tag{4.4}$$

Comparing (4.4) and (4.3b), we see that

$$\mathrm{Var}\left[L_5\right] = \mathrm{Var}\left[L_3\right] - \left\{\sum_{h=0}^{n-1} v^{2h}\,{}_hp_x V(x+h,x+h+1) - v^{2n}\ddot{a}^2_{x+n}\;{}_np_x\;{}_nq_x\right\} \quad (4.5)$$

The expression in braces in the right member of (4.5) represents the reduction in annuity variance that results from specifying the annuity shall be certain for n-years.

For an annuity-due of 1 per annum payable during the survival of the status $\overline{\overline{xy}:\overline{n}|}$, the loss function is more complicated. Again, we can split off the annuity-certain portion, and consider that the mortality rate pertains to annuity payments from time n . These payments have actuarial present value, ${}_{n|}\ddot{a}_{\overline{xy}}$, where

$${}_{n|}\ddot{a}_{\overline{xy}} = v^n\,{}_np_x\,\ddot{a}_{x+n} + v^n\,{}_np_y\,\ddot{a}_{y+n} - v^n\,{}_np_{xy}\,\ddot{a}_{x+n:y+n} \; . \quad (4.6)$$

If the random variable K now denotes the curtate duration of the status $\overline{xy}$, the loss function, L_6 , can then be defined as

$$L_6 = \begin{cases} 0 - {}_{n|}\ddot{a}_{\overline{xy}} \; , & \text{for } K < n \; ; \\ v^n\ddot{a}_{\overline{K-n+1}|} - {}_{n|}\ddot{a}_{\overline{xy}} \; , & \text{for } K \geq n \; . \end{cases}$$

The probability that $K < n$ is ${}_nq_x \cdot {}_nq_y$ if an independence assumption is used. Also, without independence, we have for given n ,

$$\Pr[K=n+j] = {}_np_x\;{}_{j|}q_{x+n} + {}_np_y\;{}_{j|}q_{y+n} - {}_np_{xy}\;{}_{j|}q_{x+n:y+n} \quad (4.7)$$

but with independence this can be reduced to

$$\Pr[K=h] = {}_{h+1}q_x\;{}_{h+1}q_y - {}_hq_x\;{}_hq_y \quad (4.8)$$

where $h=n+j$. For $y=x$, this becomes

$$\Pr[K=h] = {}_{h+1}q_x^2 - {}_hq_x^2 \quad (4.9)$$

We can now verify that

$$E\left[L_6\right] = 0$$

For $\mathrm{Var}\left[L_6\right]$, we have

$$\mathrm{Var}\left[L_6\right] = \sum_{h=n}^{\infty} \left(v^n \ddot{a}_{\overline{h-n+1|}}\right)^2 \Pr[K=h] - \left({}_{n|}\ddot{a}_{\overline{xy}}\right)^2 \tag{4.10}$$

For $x = 65$, $\mathrm{Var}\left[L_6\right] = 7.848$ when $n = 10$, and $\mathrm{Var}\left[L_6\right] = 4.006$ for $n = 20$. For $x = 75$, these variances were 8.093 and 1.238 , respectively. Graphical comparisons of $\mathrm{Var}\left[L_4\right]$ and $\mathrm{Var}\left[L_6\right]$ for $n = 10$, 20 , are made in Figure 4.2 for $x = 65$. Appendix Table A.4 gives numerical comparisons of $\mathrm{Var}\left[L_4\right]$ and $\mathrm{Var}\left[L_6\right]$ for $x = 65$, 75, 85 and 95 .

In using formulas such as (4.8), we encounter the question as to whether the survival of (x) and of (y) can be considered as independent in the probability sense, or whether some allowance should be made for dependence. This is a whole topic in itself. For some insight into this question, the reader is referred to [2, Carriere and Chan, (1986)].

FIGURE 4.1

Comparison of $Var[L_3]$ and $Var[L_5]$

when n = 10 and n = 20

Age 65

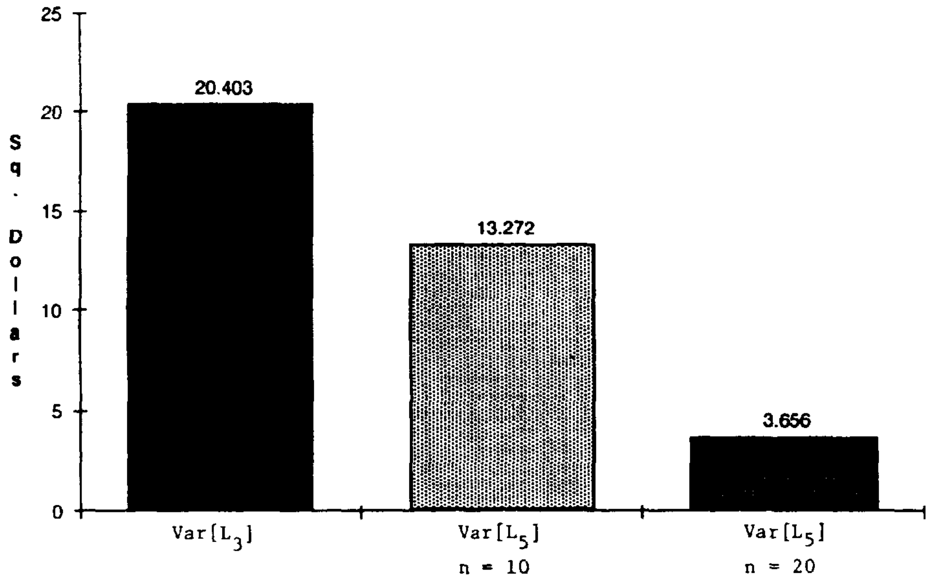
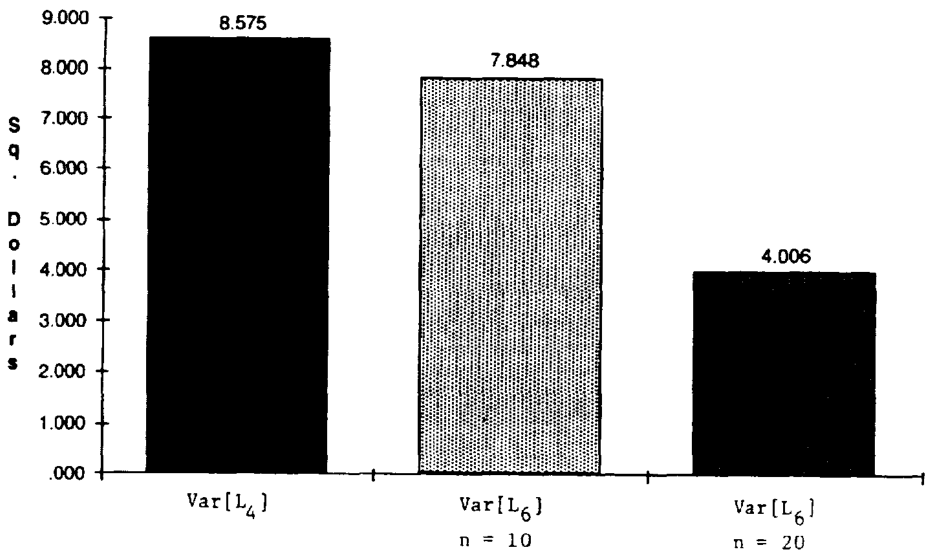


FIGURE 4.2

Comparison of Var[L_4] and Var[L_6]
when n = 10 and n = 20
Age 65



5. COMMENTS

5.1 Computations. The formulas in the preceding mathematical development have been illustrated and validated by extensive computations done by Beth Kirk. Beth is a University of Michigan actuarial student whose work on the paper was partially funded by a Research Experience for Undergraduates grant from the National Science Foundation. Such grants, from the National Science Foundation, or other sources, can help the growth of both actuarial students and actuarial knowledge.

The end products of the computations were $E\left[L_j\right]$ and $Var\left[L_j\right]$, $j = 1,2,\ldots 6$. Particularly, for single-life annuities, there was a choice of formulas and procedures, and by utilizing this choice fully, we were able to verify the results. The Appendix remarks briefly on the computations that were completed.

5.2 General Observations. The paper began with an exploration of survivorship amortization. This was indicated for single-life annuities but the concept could be extended. The survivorship amortization theory appears to be different from the more usual approach to annuity theory in Sections 2-4. At this stage, we are not sure how much the survivorship amortization concepts should be developed, but there appears to be considerable possibilities for further exploration.

A second observation is that Hattendorf Theory presents difficulties for annuities with initial certain-periods or annuities payable during the existence of a last-survivor status. For such annuities, we used only basic formulas, and not the more complex analysis of Hattendorf Theory.

A third observation is that our theory can be used to assess the risk undertaken by an individual purchaser of an annuity, as well as to estimate the risks borne by an annuity organization covering groups of annuitants.

It is noteworthy that a loss for an individual annuitant is a gain for the annuity organization, and vice-versa a gain for the individual is a loss for the organization. In the case of an individual purchaser, there is the possibility of restricting the risk by a suitable choice of annuity-form.

5.3 Standard Deviations and Coefficients of Variation.

The coefficient of variation, the ratio of the standard deviation of a random variable to the mean value of the variable, is a useful summarizing index of the variation that one may experience for that variable. Before tabulating such coefficients, we display the more complete analysis of annuity risk that can be made for whole life annuities. This is given in the folloiwng Table 5.1 of standard deviations for risk-periods running from issue to various attained ages.

TABLE 5.1

Standard Deviations for Whole Life Annuities with Risk-Periods from Age at Issue to Attained Age*,**

Age at Issue	Attained Age			
	75	85	95	ω
65	3.61 (80.0%)	4.35 (96.4%)	4.51 (99.8%)	4.52 (100%)
75		3.90 (89.6%)	4.33 (99.4%)	4.36 (100%)
85			3.50 (96.7%)	3.62 (100%)
95				2.57 (100%)

* The standard deviation equals $\sqrt{V(x,x+k)}$.

** The percent figures are relative to the full-range standard deviation.

To show the impact on annuity risk of initial certain-periods or of last survivor-status in place of single-life status, we present

TABLE 5.2

Coefficients of Variation $\left(\frac{\sigma}{\mu}\right)$

Initial Age	Whole Life Annuity	Annuity for 10 years Certain and Life	Annuity for 20 years Certain and Life	Annuity for Last Survivor of pair with Equal Ages, and having a Certain-Period of n years.		
				n=0	n=10	n=20
65	32.9%	25.7%	12.2%	18.0%	17.2%	12.0%
75	43.4%	25.7%	5.9%	26.5%	22.3%	7.5%
85	55.6%	16.6%	1.1%	37.3%	19.2%	1.5%
95	63.7%	4.9%	0%	45.9%	6.7%	0%

For whole life and last survivor annuities, both without an initial certain-period, the coefficients of variation increase with initial age. With certain-periods, the coefficients generally decrease with advancing initial age. For a given initial age, the coefficients of variation decrease with lengthening of the certain-period.

5.4 *Variable Annuities.* The TIAA-CREF organizations have been leaders in providing variable annuities for retirees from academic institutions. The CREF-type variable annuity is based on common stock investments, and is expressed as a fixed number of annuity units, the dollar value of which is determined on March 31 of each year. Since 1980, the annuity unit has increased from \$26.27 to \$99.44 with only two set-backs (in 1982 and in 1988). This is a strong argument for a portion of an individual's retirement income to be based on broadly diversified equities.

For the discussion immediately following, it is assumed that mortality and expense assumptions remain unchanged. The TIAA-type variable annuity (called a graded benefit annuity) is based on fixed income securities and an Assumed Investment Return (AIR) of 4 percent per year. The additional investment income (in excess of the AIR) is used at the end of the calendar year $\left(k,k+1\right)$ to adjust the annual annuity income, from the level, b_k , to b_{k+1} for the next calendar year $(k+1,k+2)$. Whether one considers doing this by the purchase of an incremental annuity at time $k+1$, or proceeds as in [1,(16.5.4)] by the adjustment formula

$$b_{k+1} = b_k \frac{1+i'_{k+1}}{1.04} \tag{5.4.1}$$

the result on the $(k+1,k+2)$-year annuity income is the same. Provided the investment return exceeds the AIR, the graded benefit annuity income will steadily increase from its initial value. The TIAA variable annuity, as for the CREF variable annuity, is based on an AIR of 4 percent. Its progress will be less dramatic than for the CREF variable annuity. A 1982 TIAA graded benefit annuity is predicted to have an increase of 114 percent by 1992, for an annual compound rate of increase of 7.9% . What rate of increase will be maintained for the 1992-2002 decade?

How do coefficients of variation behave for varying annuities. Our preliminary conclusion is that for attained age $x+k+1$, the expressions $b_{k+1}\sqrt{V(x+k+1,\infty)}$ and $b_{k+1}\ddot{a}_{x+k+1}$ would be in the same ratio, namely

$$\frac{\sigma}{\mu} = \frac{\sqrt{V(x+k+1,\infty)}}{\ddot{a}_{x+k+1}} \tag{5.4.2}$$

as they would have been before the annuity income was adjusted. The equality of the ratios holds provided that the mortality basis has remained unchanged.

Through [6, McCrory, 1990] and this paper, annuity risk for individuals and groups has been explored from several viewpoints. In one form or another annuities are becoming an increasingly significant part of retirement income. Recent concepts such as variable annuities and survivorship amortization are still immature, and promise more future development. We hope ideas in this paper may lead to further contributions to our understanding and operation of annuity systems.

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APPENDIX

The purpose of the computations was to illustrate the application of the various formulas. The main mathematical formulas were all tested by this process. For straight life annuities for a single individual, extensive analysis was available through application of Hattendorf theory. For annuities involving a last-survivor status or an initial-certain period, only basic formulas were developed.

The mortality table used was the Blended 1983 a-D-Mortality Table [5, Johannsen, 1987]. This is a life table blended from the 1983 Table a individual annuity mortality tables by requiring at pivotal age 65 the male l_x to be 50% of the total l_x. We were concerned with individual annuitant risk but our calculations are only illustrative and not necessarily applicable to a given situation. The calculations are focused on ages 65 and greater, and we did not project mortality improvements that may emerge in the next 20 to 50 years. The effective annual interest rate was taken at 4 percent to allow considerable margin for variable annuities. Some variance calculations involved net single premiums such as ${}^2A_x$, with effective interest at $(1.04)^2 - 1 = 0.0816$, or equivalently, at force of interest $2\delta = 2\log(1.04)$.

A "basic" spreadsheet was employed for interest functions, commutation columns, and net single premiums. The last were checked by relations such as $1 = d\ddot{a}_x + A_x$.

An "Attained Age" spreadsheet, with durations measured from initial ages 65, 75, 85, and 95, was used to compute means and variances of the loss functions developed for various annuity forms, as described in the preceding text. Much cross-checking was available by comparing computational results for different formulas or different loss functions.

The spreadsheets that were utilized did not lend themselves to interest rates varying by duration. Considerable programming would be encountered to allow such variation, and it did not seem justified for our illustrative purposes.

APPENDIX TABLES AND FIGURES

TABLE A.1

$_{h|}q_x$

h	x = 65	x = 75	x = 85	x = 95
0	.010	.027	.076	.180
1	.011	.029	.078	.158
2	.012	.032	.078	.136
3	.013	.034	.078	.115
4	.014	.036	.077	.096
5	.015	.039	.075	.078
6	.017	.041	.071	.063
7	.018	.043	.067	.050
8	.020	.045	.062	.038
9	.021	.047	.057	.029
10	.023	.048	.051	.021
11	.025	.049	.044	.014
12	.027	.049	.038	9.36E-03
13	.029	.049	.032	5.76E-03
14	.031	.048	.027	3.27E-03
15	.033	.047	.022	1.68E-03
16	.035	.045	.018	7.55E-04
17	.036	.042	.014	2.86E-04
18	.038	.039	.011	8.57E-05
19	.040	.036	8.00E-03	1.80E-05
20	.041	.032	5.78E-03	1.99E-06
21	.041	.028	4.00E-03	
22	.042	.024	2.62E-03	
23	.042	.020	1.61E-03	
24	.041	.017	9.16E-04	
25	.040	.014	4.70E-04	
26	.038	.011	2.12E-04	
27	.036	8.75E-03	8.02E-05	
28	.033	6.73E-03	2.40E-05	
29	.030	5.03E-03	5.05E-06	
30	.027	3.63E-03	5.57E-07	
31	.024	2.51E-03		
32	.020	1.65E-03		
33	.017	1.01E-03		
34	.014	5.75E-04		
35	.012	2.95E-04		
36	9.42E-03	1.33E-04		
37	7.42E-03	5.04E-05		
38	5.70E-03	1.51E-05		
39	4.26E-03	3.17E-06		
40	3.08E-03	3.50E-07		
41	2.13E-03			
42	1.40E-03			
43	8.59E-04			
44	4.88E-04			
45	2.50E-04			
46	1.13E-04			
47	4.27E-05			
48	1.28E-05			
49	2.69E-06			
50	2.97E-07			

TABLE A.2

$$h|q_{\overline{xx}}$$

h	x = 65	x = 75	x = 85	x = 95
0	.000	.001	.006	.033
1	.000	.002	.018	.082
2	.001	.005	.030	.111
3	.001	.007	.042	.123
4	.002	.010	.054	.122
5	.002	.014	.063	.114
6	.003	.018	.071	.100
7	.004	.022	.076	.085
8	.005	.027	.079	.068
9	.006	.032	.078	.053
10	.008	.038	.075	.039
11	.009	.043	.070	.028
12	.011	.049	.064	.018
13	.014	.053	.056	.011
14	.017	.057	.048	6.51E-03
15	.020	.060	.041	3.34E-03
16	.023	.061	.033	1.51E-03
17	.027	.061	.027	5.72E-04
18	.031	.060	.021	1.71E-04
19	.035	.057	.016	3.60E-05
20	.040	.053	.011	3.97E-06
21	.044	.048	7.93E-03	
22	.048	.043	5.22E-03	
23	.051	.037	3.22E-03	
24	.054	.032	1.83E-03	
25	.055	.026	9.39E-04	
26	.056	.021	4.23E-04	
27	.055	.017	1.60E-04	
28	.053	.013	4.81E-05	
29	.060	.010	1.01E-05	
30	.047	7.20E-03	1.11E-06	
31	.042	4.99E-03		
32	.037	3.28E-03		
33	.032	2.02E-03		
34	.027	1.15E-03		
35	.022	5.90E-04		
36	.018	2.66E-04		
37	.015	1.01E-04		
38	.011	3.02E-05		
39	8.44E-03	6.34E-06		
40	6.11E-03	6.99E-07		
41	4.24E-03			
42	2.79E-03			
43	1.72E-03			
44	9.75E-04			
45	5.00E-04			
46	2.25E-04			
47	8.55E-05			
48	2.56E-05			
49	5.37E-06			
50	5.93E-07			

FIGURE A.1

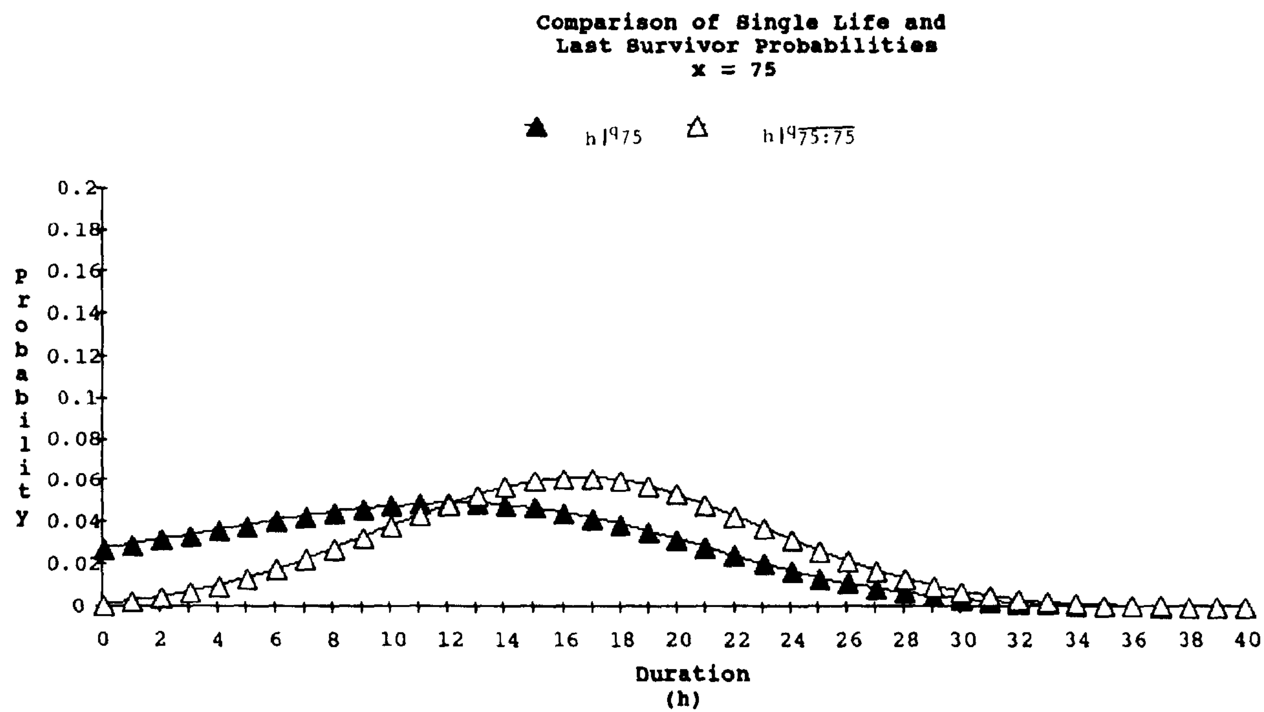
Comparison of Single Life and
Last Survivor Probabilities
x = 75
$_{h|}q_{75}$
$_{h|}q_{\overline{75:75}}$
Probability
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
Duration
(h)

FIGURE A.2

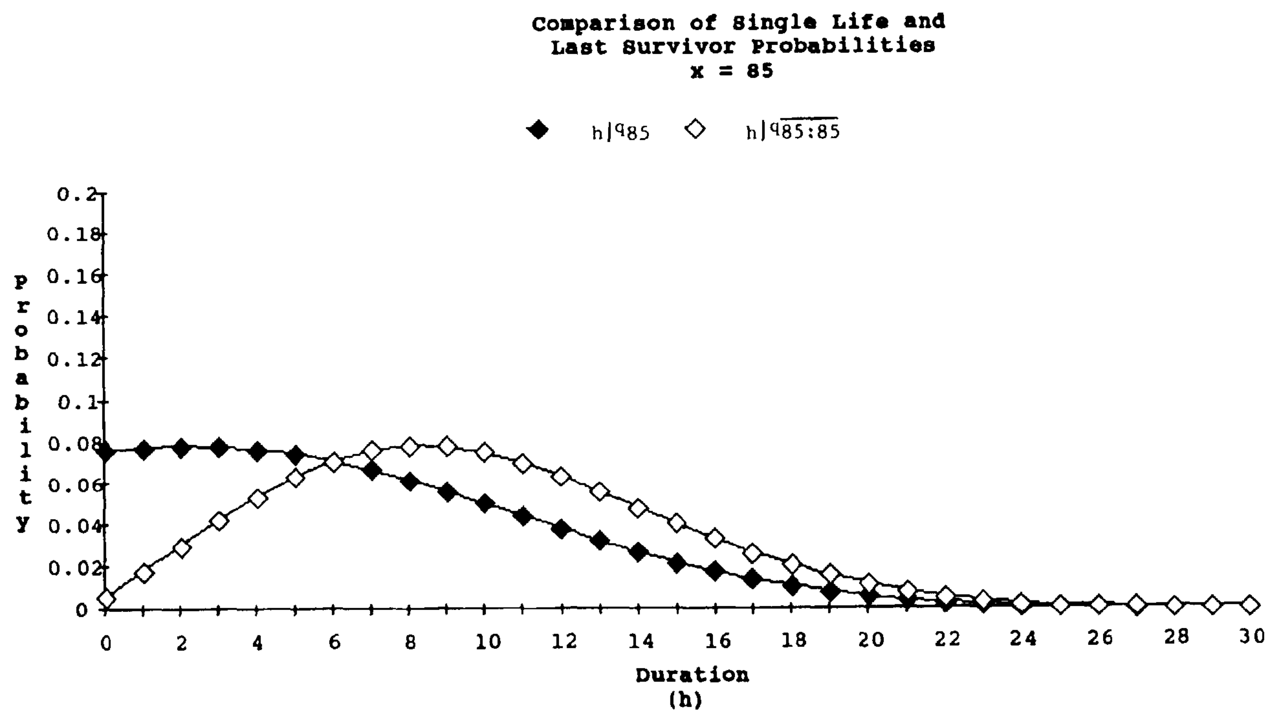
Comparison of Single Life and
Last Survivor Probabilities
x = 85
h|q85
h|q85:85
Probability
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30
Duration
(h)

FIGURE A.3

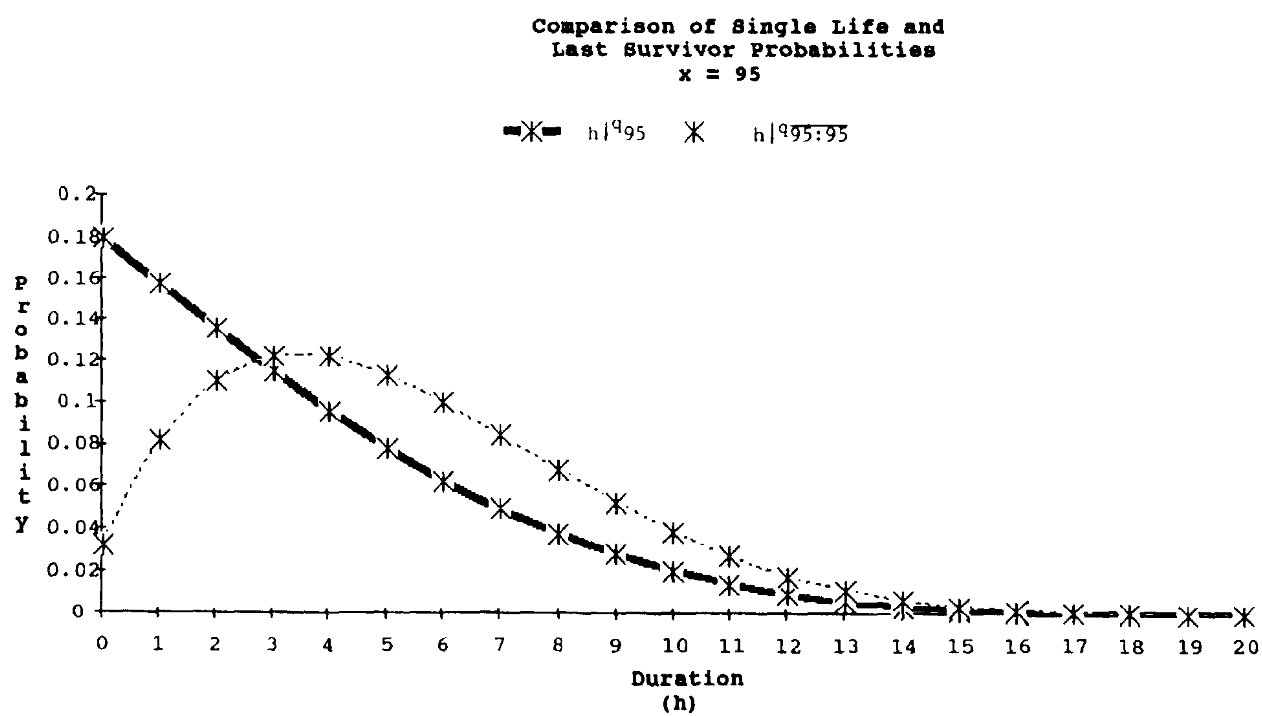
Comparison of Single Life and
Last Survivor Probabilities
x = 95
$_{h|}q_{95}$
$_{h|}q_{\overline{95:95}}$
0.2
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
Probability
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
Duration
(h)

TABLE A.3

Comparison of $Var\left[L_3\right]$ and $Var\left[L_5\right]$

Age	L_3	L_5			
	(1)	n=10		n=20	
	Var	Var	%(1)	Var	%(1)
65	20.403	13.272	65.0%	3.656	17.9%
75	18.976	8.285	43.7%	0.732	3.9%
85	13.114	2.336	17.8%	0.023	0.2%
95	6.580	0.177	2.7%	0.000	0.0%

TABLE A.4

Comparison of $Var\left[L_4\right]$ and $Var\left[L_6\right]$

Joint Age	L_4	L_6			
	(1)	n=10		n=20	
	Var	Var	%(1)	Var	%(1)
65	8.575	7.848	91.5%	4.006	46.7%
75	11.051	8.093	73.2%	1.238	11.2%
85	10.277	3.565	34.7%	0.046	0.4%
95	6.255	0.336	5.4%	0.000	0.0%