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THE PRACTICAL USE OF RECURSIVE FUNCTIONS

L. Timothy Giles, F.S.A.

Actuary, Farm Family Insurance, Glenmont, New York

The Practical Use of Recursive Functions

I learned about recursive functions from Elias Shiu 's discussion of "Life Insurance Transformations" and "Financial Accounting Standards No. 87", both in TSA XXXIX (1987). The practical applications discussed here are:

1. Universal life, both for the calculation of target premiums and reserves.
2. Paid-up Rider. Recursive functions can directly determine a premium that precisely matures the rider.
3. Traditional Whole Life. Reserves can be efficiently calculated when needed instead of referring to a table of stored factors

Recursive functions are more like a process than a function. If the accumulation rule for advancing a fund to the next duration is known, a series of difference equations can be reduced to a simple formula involving the first and last fund values.

The APL programming Language is particularly amenable to recursive functions as Shiu points out.

Key Words: APL, Transformation functions, Universal Life, Paid-Up Rider

Introduction

My discovery of recursive functions was the end of an often frustrating journey that began with calculating target premiums for universal life by trial and error. The next stage was using transformation functions. They were a delight until I tried to use them for accumulations when the TEFRA corridor was in effect. The necessary algebra was more than I could muster. The recursive function worked wherever it was needed.

Universal Life changed the rules for actuarial calculations. Account values are calculated by an iterative process of crediting interest and deducting expense and mortality charges. The iterative process becomes inadequate when a target premium needs to be calculated. What is the premium that will exactly mature the policy? A trial and error routine is typically involved. However, the result is sensitive to small changes in premium that makes this a tenuous process.

The first improvement was the development of transformation functions. Commutation functions served well for years and were even embraced by APL experts. The idiosyncrasies of Universal Life (the monthly processing, mortality charges at the beginning of the month, and the net amount at risk based on a preceding accumulation value) could be overcome by modifying the traditional commutation functions. The modified commutation functions were called transformation functions. They did enable the direct calculation of Universal Life premiums.

For me, the bête noire of the transformation function was the TEFRA corridor. The iterative process could handle the increased face amounts of overfunded policies with relative ease. Presumably, there is a transformation function that could too, but I never found it.

The recursive function appeared in discussions by Elias Shiu in two adjacent TSA papers. It was a breakthrough. A recursive function is a solution to a series of linear difference equations, which when added condense down to the first and last values of the function. If you know the beginning value (usually zero), the ending value (usually the face amount) and the succession rule, you can solve for the premium and then all the intermediate values.

Actuaries who learned life contingencies from Jordan are familiar with this technique. Recall his presentation of the payment of the reserve in addition to the face amount. Everything canceled except the first and last values. Notably, however, the additional benefit ended at a specific age to stabilize the premium .

Convenience of APL

The APL programming language facilitates recursive functions. A fundamental task in actuarial practice is to accumulate non - equal amounts; for example, accumulating dividends on deposit. Whereas equal payments can easily be accumulated by +\ (1+i)*ι n, non - level payments seem to call for a loop. However,

{1} (+\DIV÷(1+i)*ι n)×(1+i)*ι n works well. Discount all the dividends to the present, then accumulate the cumulative pieces. The same process is used with recursive functions.

Practical Example

Here is the procedure for deriving a recursive function illustrated with a traditional whole life reserve:

1.) ***Establish the succession rule.*** **You must know precisely how to get from t to t+1.**

For example, for traditional whole life reserves, the succession rule is

{2} $({}_{t}V+P)\times(1+i) = {}_{t+1}V\times(1-Q)+Q$

2.) ***Cast the succession rule into linear form.*** **In order for the cancellations to occur, there must be a linear form aX+b.**

Continuing the example,

{3} ${}_{t+1}V = {}_{t}V\times((1+i)\div(1-Q)) + P\times((1+i)\div(1-Q)) - Q\div(1-Q)$

(3) ***Compute the compounding element from beginning to end.*** **The compounding element is the a in aX+b. The compounding element is (1+i)÷(1-Q). Call it Q1 and let V1=×\Q1, a vector of expanding products.**

(4) ***Discount the ending value to the beginning.*** **The ending value is the face amount, 1 in this case. Its present value is 1÷V1**

(5) ***Discount the vector of b's to the beginning.***

$P \div +/(^{-}1 \downarrow 1,V1) - +/(Q\div(1-Q))\div V1$

(6) ***Solve for P.*** **Use Last value - First value = Premiums - Claims**

{4} $P = ((1\div \times/Q1)++/(Q\div(1-Q)\div V1))\div(^{-}1\downarrow 1,V1)$

(7) ***Generate the intermediate values from the succession rule and the premium.*** **Here we use the non - level dividend accumulation technique described above.**

{5} ${}_{t}V = V1 \times (+\backslash P\div{}^{-}1\downarrow 1,V1) - +\backslash(Q\div(1-Q))\div V1$

The calculations can be done for all premiums and/or reserves at once, or by limiting the mortality rates to the particular case, singular values are obtained.

Universal Life

The succession rule for universal life is

{6} ${}_{t+1}V = ({}_{t}V + P)\times(1+i) - Q\times(\div(1+i)-{}_{t}V-P)\times(1+i)$ **and for the TEFRA corridor T,**

{7} ${}_{t+1}V = ({}_{t}V + P)(\times 1+i) - Q\times({}_{t}V+P)\times(T-1)$

Monthly interest and mortality charges can be used directly or an algorithmic adjustment to annual processing can slightly shorten the computation time. There is, of course, a different succession rule when the TEFRA corridor is in effect. In some cases the calculation has to jump between the two rules, but usually not more than once.

Paid-Up Insurance Rider

Here's a problem similar to finding the universal life premium that will mature the policy. What is the single premium for a participating whole life product with a level death benefit, provided dividends are paid as projected. The face amount is composed of an annual term piece and a paid-up piece. Only the paid-up piece gets a dividend. At some pre-defined age, the entire face amount is paid-up. The computational procedure is: 1. Total the cash available: the dividend plus the cash value of the prior paid-up amount. 2. Deduct the cost of the one year term. 3. Divide the remaining balance by the single premium attained age cost of paid-up. The linear form for a typical application of unit face amount would be:

{7} $PU_{t+1}=PU_t\times[A_{x+t}\div(A_{x+t}-C_{x+t})]+(DIV-C_{x+t})\div(A_{x+t}-C_{x+t})$ where DIV is the cash dividend. The compounding element is $A_{x+t}\div(A_{x+t}-C_{x+t})$.

We know the ending value; it's 1. The beginning value must be found. Then the premium equals $(PU_1\times A_x)+C_x\times(1-PU_1)$.

Conclusion

Commutation functions have served well. However, the more complex products of today call for new techniques. The very idea of valuations being done by accessing a table of stored reserve factors can be improved upon. The reserves can be calculated on an as needed basis which is the way universal life reserves must be calculated. Future actuaries will be as literate with recursive functions as present actuaries are with commutation functions.

References

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