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Solution to the Risk Load Problem of Effect on Variability

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According to the Capital Asset Pricing Model (CAPM), from the point of view of a stockholder it is the "non-diversifiable" variance of an insurance company which affects its value.

From the point of view of someone who is insured by the company, or someone who works for the company, it is preferable, for any given rate of return on surplus, for the company to have lower variance and not just lower non-diversifiable variance. Also, empirical studies have shown that variance, in addition to non-diversifiable variance, affects the stock price of a company.

One of the assumptions used in the derivation of CAPM is that given any desired rate of return, stockholders use Markowitz diversification to attempt to minimize the variance of the return of their portfolio. This paper assumes the same desire for minimizing the variance of an insurer's return, at a given expected rate or return, on the part of the insurer's management.

The method of this paper is to allocate surplus to each category of business and reserves, in proportion to its estimated effect on an insurer's surplus variation. Risk load for each category of business is then selected so that the return on allocated surplus for each category is equal to the overall return on surplus. Depending on the goal, or the acceptable standard, for overall return on surplus, risk load for each category is determined. Risk load can also be estimated for a single contract, and this risk load may be different from the average risk load for its category.

The term "risk load" is sometimes used with a different meaning than it is given in this paper. Other meanings of the term include:

1. The risk load that a customer is willing to pay. This may be based on the market, on the effect of a contract on the customer's variance, or on the customer's beliefs about the effect of the contract on variability.

2. The risk load that an underwriter desires, based on the possible effect that a contract may have on the total results of the contracts he or she has underwritten.

The explanation of the phrase "effect on surplus variation" will be given below. A well-known problem in surplus allocation is that the effect on the standard deviation of the probability distribution of surplus one year in the future (this will be called "the standard deviation of surplus") of a portion of the book of business can not be estimated by simply estimating the standard deviation of surplus with and without the portion of business, and then taking the difference.

The standard deviation of surplus equals the standard deviation of the sum of the effects on surplus of the portions of the business. Suppose those portions are arranged in a list. Suppose the effect of each portion on the total standard deviation is defined as the difference between the standard deviation of the sum of the portions up to and including that portion on the list and the standard deviation of the sum of the portions prior to it on the list. The sum of all these "effects" equals the total standard deviation, but the effect of a particular portion depends on where it appears in the list. This has been considered a barrier to using the method of effect on variability to estimate required risk loads. For example, the following quotation is from "Quantifying Riskiness for Insurers" by Gary Venter in the October/November 1992 Actuarial Digest.

> It's tempting for actuaries to invent (or re-invent) the Mean-Variance Pricing Model (MVPM). The price for a risk with random outcome X is EX+VX. where EX denotes the expectation of X, VX the variance, and b is a constant which would be negative if pricing a security or positive if pricing insurance.

Preference ordering would presumably be based on the differential between offered and model prices. Nonetheless, all the goals raise difficulties for this method.

Presumably the change in variance of your whole portfolio of risks or securities is more important than that of the new entrant by itself. If independence of units can be assumed, there is no problem because both the mean and variance of the units will sum to that of the portfolio. Otherwise, all sorts of impracticalities arise. MVPM could be applied to the portfolio with and without the new entrant, whose price then becomes the difference. But then the order of entry will influence the price, which it should not. Or you could estimate in advance the make-up of the portfolio and then pro-rate to each unit a credit based on the reduction in variance achieved by the combination. The mind boggles. Besides needing a fair way to allocate credits, which this theory does not provide, any difference from the predicted result will give the wrong price overall. Because of co-variance, MVPM does not seem usable for pricing individual risks in a portfolio.

There is a way around this problem which has not previously been presented anywhere, and it is used in this study. If a pro-rata share of (1/N)th of each portion is added to the list, and this is done N times, the limit as N approaches infinity of the total effect on the standard deviation of surplus of a portion of business turns out to be the covariance of the portion with surplus, no matter what order the portions are added in. The proof of this Theorem 1 is given at the end of the paper. It will be assumed in the proof of Theorem 1, and of Theorem 2 (stated below), that the covariance of the portion with surplus is not zero. The case in which the covariance equals zero will be left to the reader.

When surplus is allocated to each portion in proportion to cov (portion, surplus), and risk loads for each portion of business are produced in such a way that each portion produces the same rate of return on its allocated surplus, it follows that:

(1) The expected profit on each portion is proportional to its "effect on the standard deviation of surplus."

(2) The risk load for a portion of business, as a percentage of the portion's premium, is proportional to the ratio of cov (portion, surplus) to the portion's premium. (This has an interesting similarity to the CAPM formula relating expected rate of return to the covariance of a stock with the stock market, but the proof of the CAPM formula requires the assumption that a stockholder is a Markowitz diversifier. Covariance has the useful property that the covariance of a sum of portions of business always equals the sum of the covariances. Therefore, the surplus allocated to the sum of portions of business is the same whether the surplus is allocated based on the total covariance of the sum of the portions, or allocated to each individual portion based on its covariance.)

(3) Suppose that a contract, or category of business, is written with less then the indicated risk load. Then the contract, or category of business, will lower the rate of return if the following is assumed: Its premium is increased, its correlation with surplus is unchanged, its standard deviation increases proportionately, and, also, the rest of the insurer's premium is reduced, its correlation with surplus is unchanged, and its standard deviation decreases proportionately so as to maintain the same total amount of surplus variance. Conversely, a

contract written at greater than the indicated risk load will have the reserve effect. See the end of the paper for the proof of this Theorem 2.

The variance considered in this study is the variance of the probability distribution of a type of adjusted surplus, representing the value of an insurer one year in the future. The value (negative) of the loss reserve is considered to be greater than the discounted value at an available "risk-free" interest rate. This is because it would be necessary to pay an insurer more than this amount, as a reward for risk, in order for them to be willing to assume this liability. In this study, the loss reserves are discounted at a rate lower than the interest rate on supporting assets. By always using a lower discount rate to determine the value of loss reserves, and "setting aside" an amount of assets equal to this value, the following occurs: In the course of a year, the assets grow at a greater rate of interest than the liability, providing a reward for the risk of having the liability. This idea is explained at length in "Determining the Proper Interest Rate for Loss Reserve Discounting: An Economic Approach" by R.P. Butsic in Evaluating Insurance Company Liabilities (CAS, 1988).

In this study, the "value" of an insurer means the value of the surplus computed by using a risk-based discount rate for the loss reserves. The method of determining this discount rate will be explained later.

In the course of a year, this value changes due to:

1. The value of the assets matching the discounted loss reserves changes during the year, some of those loss reserves are paid during the year, and the remainder of those reserves is re-estimated and has a certain discounted value (negative).

2. The change in value of an amount of assets which equals the "value" (surplus) of the insurer at the beginning of the year (using discounted loss reserves).

3. The value of the premium earned in a year minus the effect of accident year losses paid during the year and the value of accident year loss reserves at the end of the year.

The way that the appropriate discount rate for loss reserves is selected is as follows. First, the surplus (referred to in 2 above) is allocated to the sources of variance referred to in 1 and 3 above, which we will call "reserve runoff" and "underwriting results." The surplus is allocated to each in proportion to its "effect on the standard deviation" of the probability distribution of the surplus one year later. In order to estimate these effects, it is necessary to select a discount rate for loss reserves. It is not known at this point what the appropriate discount rate is, but if one is selected then surplus can be allocated and the expected return on equity from 1 and 3 above can be computed. It is then possible, by a process of iteration, to find the discount rate which equalizes the rates of return on equity from 1 and 3. This discount rate gives the value of the reserves in such a way that the return on reserves and matching assets, and the return on underwriting, are proportional to the effect of each on the standard deviation of surplus.

The appropriate risk load for each portion of underwriting (e.g. a category of business, a contract, etc.) can then be determined. The risk load is an amount such that the resulting premium equals the present value of expenses, expected losses paid during the coming year, expected discounted reserves at the end of the year (using the discount rate described above), and "appropriate profit" at the end of the year. The "appropriate profit" divided by the allocated surplus is equal for all portions of business.

The proof of the theorems stated previously are as follows.

Proof of Theorem 1

If a $1/n$ pro-rata share of each category is added in any order, then when a pro-rata share of category C is added for the k^{th} time, let

$X_{1,k,n}$ =the random variable of the effect on surplus one year in the future of the sum of the pro-rata shares of categories other than C which have already been added

$X_{2,k,n}$ = the random variable of the effect on surplus one year in the future of the sum of the pro-rata shares of C which have already been added

$X_{3,k,n}$ = the random variable of the effect on surplus one year in the future of the pro-rata share of C which has just been added

Then, $\mathrm{var}(X_{1,k,n} + X_{2,k,n} + X_{3,k,n}) = \mathrm{var}(X_{1,k,n} + X_{2,k,n}) + \mathrm{var}(X_{3,k,n}) + 2\mathrm{cov}(X_{1,k,n}, X_{3,k,n}) + 2\mathrm{cov}(X_{2,k,n}, X_{3,k,n})$. So, if $\Delta_n\mathrm{var}$ is the change in variance when the pro-rata share of C is added, then $\Delta_k\mathrm{var} = \mathrm{var}(X_{3,k,n}) + 2\mathrm{cov}(X_{1,k,n}, X_{3,k,n}) + 2\mathrm{cov}(X_{2,k,n}, X_{3,k,n})$. Also, $(\mathrm{var}(X_{3,k,n})/(2\mathrm{cov}(X_{1,k,n}, X_{3,k,n}) + 2\mathrm{cov}(X_{2,k,n}, X_{3,k,n})))$ approaches 0 as $k \to \infty$ and $n \to \infty$.

For the above random variables $X_{1,k,n}$, $X_{2,k,n}$, $X_{3,k,n}$, and $X_{1,k,n} + X_{2,k,n}$, let the standard deviations, respectively, be denoted $\sigma_{1,k,n}$, $\sigma_{2,k,n}$, $\sigma_{3,k,n}$, and $\sigma_{1,k,n+2,k,n}$. The correlation between $X_{1,k,n}$ and $X_{3,k,n}$ is the same for any k and n. Call it ρ.

Then, $\left(\sigma_{1,k,n+2,k,n} + \left(\frac{\rho\sigma_{1,k,n} + \sigma_{2,k,n}}{\sigma_{1,k,n+2,k,n}}\right)\sigma_{3,k,n}\right)^2 - \sigma^2_{1,k,n+2,k,n} =$
$2\rho\sigma_{1,k,n}\sigma_{3,k,n} + 2\sigma_{2,k,n}\sigma_{3,k,n} + ((\rho\sigma_{1,k,n} + \sigma_{3,k,n})/\sigma_{1,k,n+2,k,n})^2\sigma^2_{3,k,n}$
and $(((\rho\sigma_{1,k,n} + \sigma_{2,k,n})/\sigma_{1,k,n+2,k,n})^2\sigma^2_{3,k,n})/(2\rho\sigma_{1,k,n}\sigma_{3,k,n} + 2\sigma_{2,k,n}\sigma_{3,k,n})$
approaches 0 as $k \to \infty$ and $n \to \infty$.

Therefore, as $k \to \infty$ and $n \to \infty$,
$(\Delta_k\mathrm{var})/((\sigma_{1,k,n+2,k,n} + ((\rho\sigma_{1,k,n} + \sigma_{2,k,n})/\sigma_{1,k,n+2,k,n})\sigma_{3,k,n})^2 - \sigma_{1,k,n+2,k,n}) \to 1$
Let $\Delta_k\mathrm{sd}$ be change in standard deviation corresponding to $\Delta_k\mathrm{var}$.
Then
$(\Delta_k\mathrm{sd})/(((\rho\sigma_{1,k,n} + \sigma_{2,k,n})/\sigma_{1,k,n+2,k,n})\sigma_{3,k,n})$
approaches 1 as $k \to \infty$ and $n \to \infty$.

Also, as $k \to \infty$ and $n \to \infty$,
$(\rho\sigma_{1,k,n} + \sigma_{2,k,n})/(\sigma_{1,k,n} + {}_{2,k,n})$ approaches $(\rho\sigma_1 + \sigma_2)/\sigma_s$ where σ_1 is the standard deviation of the effect on surplus of the sum of all categories other than C, σ_2 is the standard deviation of the effect on surplus of C, and σ_s is the standard deviation of surplus. Therefore, as $n \to \infty$, $\sum_{k=1}^{n} \Delta_k sd$ approaches
$((\rho\sigma_1 + \sigma_2)/\sigma_s)(\sum_{k=1}^{n} \sigma_{2,k,n})$
which equals $((\rho\sigma_1 + \sigma_2)/\sigma_s)(\sigma_2)$. Therefore, the limit as $n \to \infty$ of
$(\sum_{k=1}^{n} \Delta_k sd)/\sigma_s = (\rho\sigma_1\sigma_2 + \sigma_2^2)/\sigma_s^2 = (\text{cov}(C,S))/\sigma_s^2$.

<u>Proof of Theorem 2</u>

Suppose that a category C of underwriting or loss reserves is increased and that its expected effect on the surplus in a one year period increases proportionately.

Let X_c and X_s be random variables which represent respectively the effect on surplus of C, and the change in the surplus S, in a one year period. Suppose that when C is increased a small enough amount, the standard deviation of X_c increases proportionately and the correlation of X_c with X_s does not change.

Suppose C is increased. Call the increase ΔC and let $X_{\Delta c}$ and $X_{s+\Delta c}$ be defined similarly to X_c and X_s. The variance of $X_{s+\Delta c}$ equals var (X_s) + 2cov$(X_{\Delta c}, X_s)$+ + var$(X_{\Delta c})$. As ΔC approaches 0, var $(X_{\Delta c})$ becomes negligible in proportion to 2cov$(X_{\Delta c}, X_s)$ so the standard deviation of $X_{s+\Delta c}$ equals st.dev. of (X_s) + (cov$(X_{\Delta c}, X_s)$)/st.dev. (X_s) + a term which approaches 0 in proportion to cov$(X_{\Delta c}, X_s)$. Therefore, (st.dev.$(X_{s+\Delta c})$)/st dev. (X_s) equals 1 + (cov$(X_{\Delta c}, X_s)$)/var (X_s). However, $E(X_{s+\Delta c})/E(X_s)$ also equals 1+(cov$(X_{\Delta c}, X_s)$)/var(X_s) by the following reasoning.

$E(X_c) = E(X_s)(\text{cov}(X_c, X_s)/\text{var}(X_s))$, and it follows from previous assumptions that
$E(X_{\Delta c}) = E(X_s)(\text{cov}(X_{\Delta c}, X_s)/\text{var}(X_s))$ and that
$E(X_{s+\Delta c}) = E(X_s) + E(X_{\Delta c})$.

A similar result follows if another category C is reduced, with corresponding assumptions. The theorem follows immediately.