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Immunizing Stochastic Cash Flows

Elias S.W. Shiu

1. Introduction

Consider a block of insurance business and the associated assets. If the terms of the assets are shorter than those of the liabilities, *reinvestment risk* arises because interest rates can fall. On the other hand, if the assets are invested longer than the liabilities, then *disinvestment risk* or *price risk* exists as interest rates can rise. Shifts in the term structure of interest rates are major sources of risk for the insurance industry.

A classical actuarial tool for dealing with interest rate risk is Redington's theory of immunization. The British actuary Frank M. Redington suggested that, to protect a portfolio against interest rate fluctuations, one should equate the "mean term" of the assets to that of the liabilities while requiring the assets to be more dispersed than the liabilities.

A basic assumption underlying Redington's model is that the asset and liability cash flows are independent of interest rate fluctuations. However, many assets and liabilities contain built-in interest-sensitive options (which can be exercised against the insurance company) giving rise to *disintermediation risk*. Insurance products, such as Single Premium Deferred Annuities (SPDA), Universal Life (UL), etc., have put options, i.e., customers have the option to *put* the policy back to the insurance company when interest rates go up and reinvest the proceeds elsewhere for a higher rate. On the other hand, many assets held by the insurance company have call features, i.e., when interest rates go down, the issuer of the assets may *call* the assets and refinance at a lower interest rate.

This paper reviews Redington's theory of immunization and sketches how it may be extended to the general case of stochastic cash flows by means of modern option-pricing theory.

2. Redington's Theory of Immunization

Consider a block of insurance business and the associated assets at time $t = 0$. For $t \geq 0$, let A_t denote the *asset cash flow* (or *investment cash flow*) expected to occur at time t, i.e., the interest income, dividends, rent, capital maturities, repayments and prepayments expected to occur at that time. Let L_t denote the *liability cash flow* (or *insurance cash flow*) expected to occur at time t, i.e., the policy claims, policy surrenders, policy loan payments, policyholder dividends, expenses and taxes, less premium income, less policy loan repayments, less policy loan interest expected at that time. Define the *net cash flow* as the difference between the asset cash flow and liability cash flow,

$$N_t = A_t - L_t.$$

A positive net cash flow means that the asset cash flow exceeds the liability cash flow, generating excess cash for (re)investment. Losses may occur, if interest rates are below the current ($t = 0$) level when the net cash flows are positive. On the other hand, negative net cash flows mean cash shortages for meeting liability obligations. At such times it would involve the liquidation of assets or borrowing within the company. Losses may occur, if interest rates are above the current level when the net cash flows are negative.

Let $S(\delta)$ denote the surplus (the difference between asset and liability values) of the block of business, evaluated at a force of interest δ,

$$S(\delta) = \sum_t N_t e^{-\delta t}. \tag{2.1}$$

If the force of interest changes from δ to $\delta + \varepsilon$, the surplus value changes from $S(\delta)$ to $S(\delta + \varepsilon)$, which, under the assumption that the cash flows are fixed and certain, is given by the formula

$$S(\delta + \varepsilon) = \sum_t N_t e^{-(\delta + \varepsilon)t}.$$

What are the conditions on the cash flows such that

$$S(\delta + \varepsilon) \geq S(\delta)? \tag{2.2}$$

By Taylor's expansion,

$$S(\delta + \varepsilon) - S(\delta) = \sum_t N_t e^{-\delta t}(e^{-\varepsilon t} - 1)$$

$$= \sum_t N_t e^{-\delta t}(-\varepsilon t + \varepsilon^2 t^2/2 - \dots). \tag{2.3}$$

Thus, if

$$\sum_t t N_t e^{-\delta t} = 0 \tag{2.4}$$

and

$$\sum_t t^2 N_t e^{-\delta t} > 0, \tag{2.5}$$

we have

$$S(\delta + \varepsilon) > S(\delta) \tag{2.6}$$

for $|\varepsilon|$ sufficiently small.

Inequality (2.6) implies a deficiency in Redington's model, because it means that, as the interest rate changes, the surplus would increase automatically. However, an efficient market would not allow such automatic "free lunches." This inconsistency in the model arises from using the same force of interest δ to discount cash flows of all terms, i.e., there is no distinction between short-term and long-term interest rates and the yield curves are always assumed to be flat. A remedy for this problem is to replace the discount factor $e^{-\delta t}$ in (2.1) with

$$e^{-\delta_t t},$$

i.e., we allow the force of interest to be a function of time. The graph of $\{\delta_t, t \geq 0\}$ is called the (current) yield curve. Formulas (2.4) and (2.5) then become

$$\sum_t t N_t e^{-\delta_t t} = 0 \tag{2.7}$$

and

$$\sum_t t^2 N_t e^{-\delta_t t} > 0, \tag{2.8}$$

respectively. These two conditions guarantee that the surplus would increase for a small instantaneous *parallel* yield-curve shift.

We remark that conditions for immunizing a portfolio against larger classes of yield curve shifts can be found in the papers by Fong and Vasicek (1983, 1984), Shiu (1988, 1990) and Montrucchio and Peccati (1991). Another approach can be found in Bierwag (1987, pp. 282-285, Appendix 11B) and Reitano (1990).

In deriving the immunization conditions (2.4), (2.5), (2.7) and (2.8), we made the assumption that the asset and liability cash flows, $\{A_t\}$ and $\{L_t\}$, are fixed and certain. This assumption certainly does not hold for assets such as callable bonds, mortgage-backed securities (MBS) and interest rate futures, and for liabilities such as single premium deferred annuities (SPDA), single premium whole life insurance (SPWL) and universal life insurance (UL). These assets and liabilities cannot be valued by means of a simple formula such as (2.1). What we need is a general method for valuing stochastic cash flows.

3. Arbitrage Valuation Theory

The option-pricing theory of Fischer Black and Myron Scholes (1973) has been described as the most important single advance in the theory of financial economics in the 1970's. These authors derived a formula for valuing a European call option on a non-dividend paying stock by showing that the option and stock could be combined linearly to form a riskless hedge, which, by the *principle of no arbitrage,* must earn interest exactly at the risk-free rate. The theory for pricing stock options has been generalized and extended to include the pricing of debt options. In general, a debt security can be viewed as a risk-free asset plus or minus various contingent claims, which usually can be modeled as options. The option-pricing methodology can be applied to value stochastic cash flows.

Let $\delta(t)$ denote the one-period risk-free force of interest at time t, i.e., if one invests \$1 at time t, one will receive $\$e^{\delta(t)}$ at time t + 1. We also assume that there is a finite number of primitive securities. Let $V_j(t)$ denote the value of the j-th primitive security at time t and let $D_j(t)$ denote the dividend or interest payment for the j-th security at time t. (We assume that $V_j(t)$ is the value of the security after $D_j(t)$ has been paid.) Note that, for $s < t$, $\delta(t)$, $V_j(t)$ and $D_j(t)$ are random variables as seen from time s. It can be shown that the assumption of no arbitrage is *equivalent* to the existence of a probability measure under which the conditional expectation

$$\mathbf{E}_t[V_j(t+1) + D_j(t+1)]$$

equals $e^{\delta(t)}S_j(t)$, t = 0, 1, 2, ... , and j = 1, 2, 3, ... , i.e.,

$$V_j(t) = \mathbf{E}_t\{e^{-\delta(t)}[V_j(t+1) + D_j(t+1)]\}. \tag{3.1}$$

The subscript t following the expectation operator **E** signifies that the expectation is taken with the knowledge of all information up to time t. In particular, the one-period force of interest $\delta(t)$ is a constant with respect to $\mathbf{E}_t$ (at time t, $\delta(t)$ is known and no more a random variable). Note that, as seen from time s, s < t, the two sides of equation (3.1) are random variables.

This probability measure, normally not the same as the actual probability measure, is called a *risk-neutral* probability measure. That the absence of arbitrage is equivalent to the existence of a consistent positive linear pricing rule is called (the first half of) the *Fundamental Theorem of Asset Pricing* by Dybvig and Ross (1987); see also Back and Pliska (1991). Positive linear pricing rules can be found in the actuarial literature in the context of equilibrium reinsurance markets [Borch (1960, 1990), Bühlmann (1980, 1984)]. See also De Finetti (1974, Chapter 3). It follows from (3.1) that, for each j and n,

$$V_j(0) = \mathbf{E}\left[\sum_{t=0}^{n} e^{-\sum_{k=0}^{t}\delta(k)} D_j(t+1) + e^{-\sum_{k=0}^{n}\delta(k)} V_j(n+1)\right]. \tag{3.2}$$

In general, the value at time 0 of a (stochastic) cash flow stream, {D(t); t = 1, 2, 3, ... }, which can be replicated by the primitive securities, is given by

$$\mathbf{E}\left[\sum_{t=0}^{\infty} e^{-\sum_{k=0}^{t}\delta(k)} D(t+1)\right]. \tag{3.3}$$

See Dothan (1990, Section 3.5), Harrison and Kreps (1979), Huang and Litzenberger (1988, Chapter 8) and Dalang, Morton and Willinger (1990, Theorem 3.3) for technical details.

It should be noted that the expectation

$$\mathbf{E}\left[\exp\left(-\sum_{k=0}^{n-1}\delta(k)\right)\right]$$

gives the price, at time 0, of an n-period default-free zero-coupon bond. In terms of the notation in the last section, this value is equal to $\exp(-n\,\delta_n)$. The zero-coupon bond yields $\{\delta_n\}$ are derived from the current prices of noncallable government Treasury bonds and are variables exogenous to the model.

4. Valuation of Interest-Insensitive Cash Flows

Let us consider (3.3) for the special case in which the (stochastic) cash flows $\{D(t)\}$ do not "depend" on interest rates. In this case we might expect that

$$\mathbf{E}[D(t) \mid \delta(0), \delta(1), \ldots, \delta(t-1)] = \mathbf{E}[D(t)].$$

(It should be noted that this equation may not be true, because the expectation is taken with respect to a risk-neutral probability measure, not the actual probability measure. The concept of stochastic independence is measure-specific. Stochastic independence is not necessarily preserved under a change of probability measure. See Delbaen and Haezendonck (1989) for some related results.) By the iterative rule of expectations

$$\begin{aligned}
\mathbf{E}\left[e^{-\sum_{k=0}^{t}\delta(k)} D(t+1)\right] &= \mathbf{E}\left[\mathbf{E}\left[e^{-\sum_{k=0}^{t}\delta(k)} D(t+1) \,\middle|\, \delta(0), \delta(1), \ldots, \delta(t)\right]\right] \\
&= \mathbf{E}\left[e^{-\sum_{k=0}^{t}\delta(k)} \mathbf{E}\left[D(t+1) \mid \delta(0), \delta(1), \ldots, \delta(t)\right]\right] \\
&= \mathbf{E}\left[e^{-\sum_{k=0}^{t}\delta(k)} \mathbf{E}[D(t+1)]\right] \\
&= \mathbf{E}\left[e^{-\sum_{k=0}^{t}\delta(k)}\right] \mathbf{E}[D(t+1)] \\
&= e^{-(t+1)\delta_{t+1}} \mathbf{E}[D(t+1)]
\end{aligned}$$

and (3.3) becomes

$$\sum_{t=1}^{\infty} e^{-t\,\delta_t} \mathbf{E}[D(t)]. \tag{4.1}$$

As an illustration, let us derive a formula for the "market" value of a whole life insurance, with death benefit of 1 payable at the end of the year of death, issued to a life aged x. Let $T(x)$ denote the time-until-death random variable (Bowers et al, 1986, p. 46). Let $\lceil \cdot \rceil$ denote the ceiling function (Shiu, 1982). In the theory of Life Contingencies (Bowers et al, 1986, Chapter 4), the **net** single premium for the whole life insurance, for a given force of interest δ, is

$$\bar{A}_x = E\left(e^{-\delta \lceil T(x) \rceil}\right). \tag{4.2}$$

Note the expectation operator E in (4.2) is not in bold face, because the expectation is taken with respect to the actual probability measure, not with respect to a risk-neutral probability measure. To apply (3.3), we need to define the cash flow stream {D(t)} for the whole life insurance. Let I(·) denote the indicator function, i.e.,

$$I(\text{event is true}) = 1$$

and

$$I(\text{event is false}) = 0.$$

Then $D(t) = I(\lceil T(x) \rceil = t)$, t = 1, 2, 3, ... Since the random variable T(x) should be independent of interest rates, it follows from (4.1) that the "market" value of the whole life insurance is

$$\sum_{t=1}^{\infty} e^{-t\delta_t} E[I(\lceil T(x) \rceil = t)] = \sum_{t=1}^{\infty} e^{-t\delta_t} {}_{t-1|}\mathbf{q}_x . \qquad (4.3)$$

Note that the symbol **q** in (4.3) is in bold face, because the probability is calculated with respect to a risk-neutral probability measure. For instance, the "market" consensus may be that the single premium for the whole life insurance should be priced by setting

$${}_{t|}\mathbf{q}_x = {}_{t|}q_{x+3}.$$

For a second example, let us derive a formula for the "market" value of an n-year life annuity-immediate. Here, the cash flow at time t, D(t), is given by

$$D(t) = I\{[t < T(x)] \text{ and } (t \le n)\}.$$

It follows from (4.1) that the "market" value of the temporary life annuity is

$$\sum_{t=1}^{\infty} e^{-t\delta_t} E\{I[(t < T(x)) \text{ and } (t \le n)]\} = \sum_{t=1}^{n} e^{-t\delta_t} {}_t\mathbf{p}_x. \qquad (4.4)$$

Another application of (4.1) is the determination of default probabilities. Let {C(t) | t = 1, 2, ... } be the contractual cash-flow stream of a non-callable fixed-income security. For simplicity we assume that, once default occurs, there are no more payments nor any salvage value. Let τ denote the time-until-default random variable; here, τ is integer-valued. Then the market value of the security can be determined by (3.3) with

$$D(t) = C(t)I(\tau > t).$$

If we make the assumption that default and interest rates are not related and apply (4.1), the market value of the security is given by

$$\sum_{t=1}^{\infty} e^{-\delta_t t} C(t)\, \mathbf{E}[I(\tau > t)], \qquad (4.5)$$

with

$$E[I(\tau > t)] = Pr(\tau > t)$$

$$= \prod_{j=0}^{t-1} Pr(\tau > j + 1 \mid \tau > j). \tag{4.6}$$

If we further assume that the conditional probabilities in (4.6) are identical, i.e., there exists a positive number s such that, for each j,

$$Pr(\tau > j + 1 \mid \tau > j) = e^{-s},$$

then

$$E[I(\tau > t)] = e^{-st},$$

and the market price of the fixed-income security is

$$\sum_{t=1}^{\infty} e^{-(\delta_1 + s)t} C(t). \tag{4.7}$$

The positive number s is the spread that investors require for investing in the security which may go into default. We remark that Bierman (1990) has presented a lucid discussion on the annual default rate and the probability of default over the life of the security. Also, Section 4 of Artzner and Delbaen (1990) is on credit risk; their model allows for dependency between default and the interest rate process.

5. Implementation

To get a better understanding of (3.3), we rewrite it as

$$\sum_{\text{all } \omega} Pr(\omega) \left[\sum_{t=0}^{\infty} e^{-\sum_{k=0}^{t} \delta(k, \omega)} D(t + 1, \omega) \right]. \tag{5.1}$$

Here, each event ω can be identified as an interest-rate path or scenario path; $\{\delta(0), \delta(1, \omega), \delta(2, \omega), \dots\}$ and $\{D(1, \omega), D(2, \omega), D(3, \omega), \dots\}$ are the one-period forces of interest and cash flows along the path. Formulas (3.3) and (5.1), in different notation, can be found in Tilley (1988).

We now digress to elaborate on the information structure of the model. In technical terms, the forces of interest and cash flows have to be "adapted" to the underlying "filtration." For each s, the cash flow $D(s, \nu)$ is determined *only* with all the information up to time s. At time-state (s, ν), the interest rates $\{\delta(0), \delta(1, \nu), \delta(2, \nu), \dots, \delta(s, \nu)\}$ are known and the future interest rates $\{\delta(s + 1), \delta(s + 2), \dots\}$ are unknown, i.e., they are random variables. The cash flow $D(s, \nu)$ should not be determined with the knowledge that the forces of interest $\{\delta(s+1, v), \delta(s+2, v), \ldots\}$ are to occur; for a simple example illustrating this error, see Finnerty and Rose (1991, p. 71).

Let $V(\{\delta_n\})$ denote the expected value (3.3) or (5.1). Applying the method of differential calculus or finite differences to V as a function of $\{\delta_n\}$ (or other factors), we can derive various price-sensitivity measures, based upon which we can design strategies to manage the assets and liabilities. For example, suppose that we wish to study how V varies with respect to a parallel shift of the initial yield curve. For a fixed small value h, we calculate the first- and second-order differences

$$\Phi = V(\{\delta_n + h\}) - V(\{\delta_n\})$$

and

$$\Psi = V(\{\delta_n + h\}) - 2V(\{\delta_n\}) + V(\{\delta_n - h\}),$$

and then apply the approximate formula

$$V(\{\delta_n + \varepsilon\}) \approx V(\{\delta_n\}) + \frac{\varepsilon}{h}\Phi + \frac{1}{2}\frac{\varepsilon}{h}\left(\frac{\varepsilon}{h} - 1\right)\Psi.$$

The quantities $-\Phi/[hV(\{\delta_n\})]$ and $\Psi/[h^2V(\{\delta_n\})]$ may be called generalized (or option-adjusted) duration and convexity, respectively. To insulate the surplus of a block of business against small parallel yield-curve shifts, one would match the Φ of the assets to that of the liabilities and try to make the Ψ of the assets larger than that of the liabilities.

To apply formula (3.3) or (5.1) we need to specify the risk-neutral probability measure and the one-period risk-free force-of-interest process; the probability measure and the one-period forces of interest should be such that the model can reproduce a set of exogenously prescribed forces of interest $\{\delta_n\}$, i.e., the condition

$$e^{-n\delta_n} = \sum_{\text{all } \omega} \Pr(\omega)\, e^{-\sum_{k=0}^{n-1} \delta(k, \omega)}$$

holds for all n. The model is usually implemented as a binomial lattice. A simple way for constructing a binomial lattice with an exogenously prescribed initial yield curve is the method of Ho and Lee (1986) or its generalization given by Pedersen, Shiu and Thorlacius (1989). Another efficient method is the technique of forward induction as explained in Jamshidian (1991). The lognormal binomial lattice described in Black, Derman and Toy (1990) is quite popular; see also Black and Karasinski (1991). We remark that Tilley (1991) has criticized the binomial lattices and presented a different model.

In practice, it may be difficult to apply formula (5.1) to value a path-dependent cash flow stream. For example, to value an MBS pool, we would probably need 360 time-periods, each time-period corresponding to one month. If the one-period force-of-interest process is generated by a binomial model, there are 2^{360} paths. As the mortgage prepayment rate is usually modeled as a function of interest rate history, the method of backward induction cannot be applied to value MBS. Thus one needs to estimate (5.1) by means of simulation, i.e., one (randomly) picks a subset Ω' of all paths and calculates

$$\frac{1}{\sum_{\omega \in \Omega'} \Pr(\omega)} \sum_{\omega \in \Omega'} \Pr(\omega) \left[\sum_{t \geq 0} e^{-\sum_{k=0}^{t} \delta(k, \omega)} D(t+1, \omega) \right]. \tag{5.2}$$

Many investment banking firms on Wall Street have constructed valuation models similar to the one described here. See Tilley, Noris, Buff and Lord (1985), Noris and Epstein (1989) and Griffin (1990) for an analysis on SPDA. For applications to MBS, see Jacob, Lord and Tilley (1987) and Richard (1991).

6. Concluding Remarks

Except for reinsurance, there is really no market for the exchange of insurance liabilities. It is difficult to actually come up with the *market value* of each liability. However, by constructing an arbitrage-free valuation model such as the one described in the last section, one can compute *relative* market values and price-sensitivity indexes for both assets and liabilities. Based on such information, one can design appropriate strategies for managing assets and liabilities. Indeed, without such a model, it would be hard even to estimate the values of the various options in the assets and liabilities.

Finally, it should be pointed out that insurance may not be priced linearly, i.e., some insurance products may not follow any linear pricing rule. For example, see the various formulas in Section 14.4 of Bowers et al (1986). Discussions on principles of premium calculation can be found in Gerber (1979, Chapter 5) and in Goovaerts, de Vylder and Haezendonck (1984). See also Delbaen and Haezendonck (1989) and Albrecht (1991).

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Department of Actuarial & Management Sciences
The University of Manitoba
Winnipeg, Manitoba R3T 2N2

Investment Division
The Great-West Life Assurance Company
Winnipeg, Manitoba R3C 3A5

Current Address:
Department of Statistics and Actuarial Science
The University of Iowa
Iowa City, Iowa 52242