# Two Explicit Formulae for Yield

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## Abstract

Historically, it has always been a challenging mathematical task to get an explicit formula for the solution of a polynomial equation of the degree n, when n>2. As we know the yield rate of a portfolio is usually an implicit solution of the algebraic equation of degree greater than 2, i.e., it usually cannot be calculated explicitly by means of a finite number of fractions and radicals. Therefore, the yield of a portfolio of fixed income securities may be estimated by means of iterative methods such as Newton's method, for example.

In this paper we give two explicit formulae that estimate the yield of a portfolio of fixed income securities that are not interest-sensitive. The yield is explicitly estimated in terms of the discounting interest rates of the securities that comprise the portfolio.

## 1. Introduction, Definitions, Ideas

Let A denote a vector of cash flows: A = ($A_0$, $A_1$, … , $A_n$), at the times 0= $t_0$, $t_1$, $t_2$, …, $t_n$, where **$A_i$ can be positive, negative or zero**. Every transaction and portfolio can be represented as a series of cash flows. Therefore, we will use the words "transaction" and "portfolio" interchangeably. The rate of interest "y" is called a *yield rate* if it satisfies the following equation:

$$P(y) := \sum_{i=0}^{n} v^{t_i} A_i = 0 , \qquad (1)$$

where $v = \frac{1}{1+y}$, (see (5.3) in Kellison (1991)).

The yield rate is crucial for correct assessment of transactions.

Let us mention that we will **not** restrict ourselves on polynomials in the equation (1) in this paper, i.e., **times $t_i$ , i = 1, 2, …, n, can be any real numbers, including the most frequently used, fractions of the year**.

A well known result from algebra is that in the general situation there is no explicit formula to solve the equation (1), even if it is only a polynomial. The only available methods in general are iterative. Hence, no explicit formula for yield exists that works in the general situation**.** We should be **motivated** to find an explicit formula to estimate yield, even if there is very sophisticated software that estimates yield by iterative methods. People struggle in all areas of science to find explicit solutions of algebraic equations even though numerical solutions are available. The reason for that is that an explicit formula explains relations between the terms, which contribute to our understanding of a given problem. In particular, from the formulae that we present in this paper, we learn about relations between yield and discount rates of the given portfolio, which is impossible to see from any software based on numerical methods. The difference between knowing a numerical result and knowing a formula is the same as the difference between knowing a fact and deducting (deriving) a fact. We need both.

The nonexistence of a formula for explicit solution of the equation (1) is just the first problem with the yield rate.

The second major problem "inherited" from algebra arises from the fact that sometimes there is more than one real number solution to equation (1), and sometimes there are no real number solutions at all. Worse yet, we can construct transactions for both cases—multiple real number solutions, and no real number solutions. In Kellison (1991), Promislow (1980), Teichroew, et al. (1965) and Jean (1968), we can find some examples of such transactions, and various attempts to correctly assess the transactions with either multiple yield rates, or with no yield rates.

**Example 1.** This is Example 5.3 from Kellison (1991). A transaction is observed where a person borrows one unit for one year at rate i=8 percent and lends the same amount for one year at rate j=10%. The yield rate, as a solution of the equation (1) does not exist in this transaction.

**Example 2.** This is Example 5.12 in Kellison (1991). Consider a transaction in which a person makes payments of $100 immediately and $132 at the end of 2 years in exchange for a payment in return of $230 at the end of one year. Equation (1) for this portfolio is $100-230v+132v^2=0$ which transforms into $100y^2-30y+2=0$ , where y denotes the unknown yield rate.

These quadratic equations have the following (two) solutions:
$v_1=\frac{1}{1.1}$ and $v_2=\frac{1}{1.2}$, i.e., $y_1=0.1$ and $y_2=0.2$

**Example 3.** This is Example 5.4. in Kellison (1991). The cash flow is ($A_0$, $A_1$, $A_n$) **=** (100, -200, 101). The equation (1) is $100-200v+101v^2=0$ which transforms to $100y^2+1=0$. This equation does not have a real solution for y.

Various authors tried to identify necessary and sufficient conditions that the coefficients of (1) should satisfy that would guarantee a unique real number solution of (1), a unique yield of the corresponding transaction. However, every condition on coefficients of (1) reduces the set of available transactions. For example, a concept of *simple project* is defined as a transaction in which the signs of all $A_i$'s are different from that of $A_0$. The existence of the unique yield rate has been proven for simple projects in Teichroew, et al. (1965). The difficulties emerge with the nonsimple projects. In order to study nonsimple projects, Teichroew, et al. (1965) introduce two different interest rates, lending and borrowing rates.

Jean (1968) proved the existence of a unique yield rate for the transactions when either first or last inflows are negative. The difficulties of determining yield emerge for transactions with negative middle inflows.

In Chapter 5 of Kellison (1991) and in Becker (1988) some more general sufficient conditions that cash flows A must satisfy were given in order to have a unique solution of (1). One such sufficient condition is that the outstanding investment balance is positive at all times t=1, 2, … , n.

This was a very brief illustration of the main ideas and useful classical results. Those results are very important, but in order to be achieved, very strong restrictions on the set of available transactions had to be made, including that the times $t_i$ are assumed to be natural numbers, i.e., that the equation (1) is a polynomial. There is no need to go into details of any of the above results, because we approached the problem of yield very differently by setting assumption of **arbitrage free** market rather than making assumptions on the coefficients of the equation (1) as was the case in all the above papers.

An interesting idea came from Promislow (1980). Promislow says: "Individual must assess a transaction by using quantities that depend on his particular circumstances and that are independent of the transaction itself, as opposed to using yield rates, which depend solely on the transaction and are independent of any particular individual."

In our paper, as a "particular circumstance" of an investor, we see the current discount rates of the portfolio. No assumptions about the coefficients of the equation (1) are made in advance in this paper, unlike most of the other research about the yield, e.g., Becker (1988), Kellison (1991), Promislow (1980), Teichroew, et al. (1965) and Jean (1968). The idea is to assume other, in many situations more natural, restrictions on the series of transactions, the restrictions consistent with the "**no arbitrage**" principle. The assumptions that the discount rates are given and that we are in the "arbitrage free" market restrict the set of available transactions of the equation (1) in a natural way. In fact, we restrict our research to the set of transactions such that have the coefficient $-A_0$ equal to the value of the portfolio, i.e., it is the present value of the cash flows $A_i$, where each $A_i$ is discounted with the corresponding discount rate $r_i$ of the portfolio. This means that we follow the advice of Promislow (1980) not to look at the transactions as isolated events. The fact that the yield rate is affected by current market rates is utilized. The goal is to find an explicit estimate of yield that depends on current interest rates, i.e., on the discount rates of the "zeros" in the portfolio of interest.

For that purpose we will limit ourselves to **portfolios of fixed income securities which are not interest-sensitive.** This means that all securities in any observed portfolio P have fixed cash flows. Let us denote the set of discount rates of the portfolio P by $R = (r_1, r_2, \ldots, r_n)$. These may be discount rates of any strips or zero coupon bonds, **adjusted for any spread** over spot rates. **These discount rates do not need to be spot rates**, even though they are spot rates if we deal with Treasury securities and if we assume holding them until maturity. **Let me also repeat that we assume that the portfolio P is a part of an "arbitrage free" market M.**

Under these assumptions, Example 1 shows that yield "y" is infinite, if a risk spread is not included into consideration, i.e. in that case it is a "free lunch."

In this paper, two formulae, (10) and (17), which estimate yield of a *legitimate* (available in arbitrage free market M) transaction P, in terms of the given discount rates $R=(r_1, r_2, \ldots, r_n)$ of P will be proven. These formulae are important results because they are explicit and therefore enable us to avoid computation by iterative techniques, and in addition, give us a new relation between yield and discount rates of the given portfolio. We will be able to estimate yield of legitimate transactions manually, and to create a simple worksheet to calculate yield, as demonstrated in the numerical examples later. The formulae (10) and (17) estimate yield rate of the arbitrage free transaction defined by the cash flow $A_1, A_2, \ldots, A_n$, at times $t_1, t_2 \ldots, t_n$, and the corresponding discount rates $r_1, r_2, \ldots, r_n$. We do not need to know the cash flow $-A_0$ at t=0, the "value of the portfolio," as an input for the formulae (10) and (17). This is in sync with the fact that $A_0$ is uniquely determined in the arbitrage free market M.

A *zero-coupon bond maturing at time t* is a bond that pays its face value at t and no coupon prior. Every portfolio of fixed income securities that are not interest-sensitive can be considered as a collection of zero-coupon bonds ("zeros"). *Spot rates* give us the yield to maturity of the associated riskless zero-coupon bonds.

The formula (10) is a linear approximation of yield, and the formula (17) takes into account convexity as well.

## 2. Derivation of the Results

### 2.1 Linear Approximation

Let us consider a set $Z = (Z_1, Z_2, \ldots, Z_n)$, n >1, of zero coupon bonds with face amounts of \$1, with maturities at times: $T=( t_1, t_2, \ldots, t_n)$, and a vector of corresponding discount rates: $R = (r_1, r_2, \ldots, r_n)$. In order to simplify notation we will sometimes identify a *unit zero coupon bond* $Z_i$ with its price at t=0. In this notation

$$Z_i = \frac{1}{(1+r_i)^{t_i}}, i = 1,2,\ldots n \qquad (2)$$

Let us denote vectors in $R^n$, as usually by: $X=(x_1, x_2, \ldots, x_n) \in R^n$
Let us consider the portfolio P consisting of zero coupon bonds with cash flows $A_i$ at times $t_i$, i.e., $P=\{-A_0, A_1Z_1, A_2Z_2, \ldots, A_nZ_n\}$. Then the *present value* of portfolio P is given by the following formula

$$-A_0 = PV(P) = \sum_{i=1}^{n} A_i Z_i$$

Notice that every bond is a special case of such a portfolio, where the first (n-1) cash flows $A_i$ are the coupon payments, and $\mathbf{A}_n$=Face value + coupon payment. This means that every portfolio of fixed income securities can be broken down to a set of "zeros" P={$-A_0$, $A_1Z_1$, $A_2Z_2$, …, $A_nZ_n$ }.

**Example 4.** In order to demonstrate the above notation, let us consider portfolio P that consists of two bonds. The bond A matures in 2 years. Coupon rate is 7 percent, and spread is 0.1 percent. The bond B matures in 3 years, has coupon rate 8 percent, and we assume no spread for B. Notice that we do not need to know the spot rates in the market M; we only need to know the final, adjusted discount rates $r_1$, $r_2$, $r_3$, $r_4$, $r_5$. Then portfolio P, broken into building blocks (2), has the following cash flows A = (7, 107, 8, 8, 108), adjusted discount rates R= (0.071, 0.071, 0.08. 0.08, 0.08) and the corresponding times T= (1, 2, 1, 2, 3). ///

In the above notation yield "y**"** satisfies the equation:

$$\sum_{i=1}^{n}\frac{A_i}{(1+y)^{t_i}}=\sum_{i=1}^{n}\frac{A_i}{(1+r_i)^{t_i}} \qquad (3)$$

The following holds

$$-A_0=\sum_{i=1}^{n}A_iZ_i=\sum_{i=1}^{n}\frac{A_i}{(1+r_i)^{t_i}} \qquad (4)$$

Let us introduce a function A of n variables $x_i$:

$$A(X)=A(x_1,x_2,...,x_n)=\sum_{i=1}^{n}\frac{A_i}{(1+x_i)^{t_i}} \qquad (5)$$

Let us assume here that $r_1$, $r_2$, …, $r_n$ are **sufficiently** close to each other**.** This is a realistic assumption because usually the discount rates of similar durations are close to each other. The assumption that the discount rates are "sufficiently close" is not very restrictive. For example, no matter how steep the yield curve is, the point ($r_1$, $r_2$, …, $r_n$) is just one point in $R^n$. Whether or not it a large neighborhood in some absolute units is sufficiently small for our purpose depends on the shape of the multivariable surface of the function $A(x_1,x_2, ..., x_n)$ around the point R = ($r_1$, $r_2$, …, $r_n$) in $R^n$.

Let us apply the multivariable Taylor formula of the first order on the function A(X) in the neighborhood of the given point of discount rates R = ($r_1$, $r_2$, …, $r_n$). Then it follows that

$$A(x_1, x_2, ..., x_n) \approx \sum_{i=1}^{n} \frac{A_i}{(1+r_i)^{t_i}} - \sum_{i=1}^{n} \frac{t_i A_i (x_i - r_i)}{(1+r_i)^{t_i+1}} \qquad (6)$$

By the definition of the yield of portfolio P and by definition of the function A given by equation (5) it follows:

$$-A_0 = \sum_{i=1}^{n} \frac{A_i}{(1+y)^{t_i}} = A(y, y, ..., y) \qquad (7)$$

If we substitute Y=(y, y, …, y) from the neighborhood of R for X=($x_1$, $x_2$, …, $x_n$) in the equation (6) we get:

$$A(y, y, ..., y) \approx \sum_{i=1}^{n} \frac{A_i}{(1+r_i)^{t_i}} - \sum_{i=1}^{n} \frac{t_i A_i (y - r_i)}{(1+r_i)^{t_i+1}} \qquad (8)$$

From (3), (7), and (8) we get:

$$\sum_{i=1}^{n} \frac{t_i A_i (y - r_i)}{(1+r_i)^{t_i+1}} = 0 \qquad (9)$$

Solving this equation for y gives:

$$y \cong \left. \sum_{i=1}^{n} \frac{t_i r_i A_i}{(1+r_i)^{t_i+1}} \middle/ \sum_{i=1}^{n} \frac{t_i A_i}{(1+r_i)^{t_i+1}} \right. \qquad (10)$$

This is the linear formula that estimates yield explicitly in terms of given discount rates. The above results can be summarized in the following proposition:

**Proposition 1.** Assume we have a portfolio of fixed income securities P in an arbitrage free market M. As we know we can consider it as a set of "zeros", i.e.,

$P = \{-A_0, A_1Z_1, A_2Z_2, \ldots, A_nZ_n\}$, where $Z_i$ are unit zeros given by (2), with the current discount rates $R = (r_1, r_2, \ldots, r_n)$ of the portfolio P, at times $0 = t_0, t_1, t_2, \ldots, t_n$, respectively. Then the yield of the portfolio can be estimated by the formula (10).

Let us notice that the "arbitrage free" assumption transforms to an assumption on the coefficients of the equation (1), i.e., it is assumed that the cash flow at the time t=0, $-A_0$ is present value of the cash flows $A_1, A_2, \ldots, A_n$, with respect to the current discount rates $R=(r_1, r_2, \ldots, r_n)$. No classical result uses such assumptions regarding the coefficients of the equation (1).

**Corollary 1.** The yield of portfolio P with discount rates $R = (r_1, r_2, \ldots, r_n)$ and cash flows $A = (A_0, A_1, \ldots, A_n)$, at the times $0 = t_0, t_1, t_2, \ldots, t_n$, in an arbitrage free market M is the weighted average of the discount rates $r_i$. The weights of the discount rates $r_i$ are equal to

$$w_i = \frac{\dfrac{t_i A_i}{(1+r_i)^{t_i+1}}}{\sum_{i=1}^{n} \dfrac{t_i A_i}{(1+r_i)^{t_i+1}}} \qquad (11)$$

**Remark:** In Babbel and Merrill (1996), on pages 22-24, it has been proven that **YTM can be approximated as a weighted average of the spot rates and that the weights applied to spot rates are the dollar durations of their respective zeros relative to the dollar duration of the entire portfolio.** In the proof of that statement the portfolio function P(y) defined by (A3) in Babbel and Merrill (1996) was considered as a function of one variable "y" (yield) and its derivative with respect to "y" was denoted by $\frac{\partial P}{\partial y}$. Also, both zero coupon bonds $C_n(y)$ and $C_m(y)$ were defined by the relation (A4) as functions of one variable, y, and Taylor formula for functions of one variable was used around the yield "y" for both functions $C_n(y)$ and $C_m(y)$. The formula (A8) was proven for portfolio of only two zero coupon bonds.

$$y \cong \frac{s_n(\partial C_n / \partial y) + s_m(\partial C_m / \partial y)}{\partial P / \partial y} \qquad \text{(A8)}$$

If we expand that formula to n "zeros" and if we use our notation, than the formula (A8) becomes

$y \cong \sum_{i=1}^{n} w_i s_i$ , where weights are given as ratios of dollar durations by formulae

$$w_i = \frac{\dfrac{t_i A_i}{(1+y)^{t_i+1}}}{\sum_{i=1}^{n} \dfrac{t_i A_i}{(1+y)^{t_i+1}}} ,$$

and S = ($s_1$, $s_2$, …, $s_n$) is the set of the spot rates of the portfolio of risk-free fixed income securities.

This means that our formula (10) is a refining of the formula (A8) from Babbel and Merrill (1996) because our weights are calculated by means of the exact discount rate in each time point, while weights of (A8) are all calculated at the yield rate "y" that has yet to be determined. The other advantage of formula (10) over (A8) is that the yield "y" is on one side of (10), left side, and all discount rates are on the right side, while the formula (A8) has yield on both sides. **Our multidimensional approach, and Taylor development around the point R = ($r_1$, $r_2$, …, $r_n$), not around "y", enabled that separation which then enabled us to get a second order approximation of yield, our formulae (17), in terms of the discount rates.** If we wanted to use formula (A8) to estimate yield, we would have again to use an iterative method, because "y" is on both sides of (A8) and cannot be solved explicitly in terms of the spot rates $s_i$, while the formula (10) is an explicit expression of yield in terms of the discount rates of the portfolio.

**2.2 Second Order Approximation**

If, instead of (6), we use the multivariable Taylor formula of the second order, we will get a better estimate of yield than (10).

In order to simplify further derivations, let us introduce the following notation for first and second derivatives of the functions $Z_i(r_i)$:

$$DZ_i = -\frac{t_i}{(1+r_i)^{t_i+1}} , \quad D^2 Z_i = \frac{t_i(t_i+1)}{(1+r_i)^{t_i+2}} , \qquad i=1,2,\dots,n.$$

The second order Taylor formula of the multivariable function A(X) in the neighborhood of R can be written as

$$A(X) \approx \sum_{i=1}^{n} Z_i A_i + \sum_{i=1}^{n} DZ_i A_i (x_i - r_i) + \frac{1}{2}\sum_{i=1}^{n} D^2 Z_i A_i (x_i - r_i)^2 \qquad (12)$$

From (2), and (3) we have:

$$\mathrm{A(Y)} = \sum_{i=1}^{n} Z_i A_i \qquad (13)$$

Substituting Y for X in (12), and applying (13) we get

$$\sum_{i=1}^{n} DZ_i A_i (y - r_i) + \frac{1}{2}\sum_{i=1}^{n} D^2 Z_i A_i (y - r_i)^2 \approx 0 \qquad (14)$$

After some derivations we get:

$$\left(\frac{1}{2}\sum_{i=1}^{n} D^2 Z_i A_i\right) y^2 + \left(\sum_{i=1}^{n} DZ_i A_i - \sum_{i=1}^{n} D^2 Z_i A_i r_i\right) y + \frac{1}{2}\sum_{i=1}^{n} D^2 Z_i A_i r_i^{\,2} - \sum_{i=1}^{n} DZ_i A_i r_i = 0 \qquad (15)$$

Let us introduce notation for coefficients of the equation (15):

$$E = \frac{1}{2}\sum_{i=1}^{n} D^2 Z_i A_i = \frac{1}{2}\sum_{i=1}^{n} \frac{t_i (t_i + 1) A_i}{(1 + r_i)^{t_i + 2}} \qquad (16.a)$$

$$F = \sum_{i=1}^{n} DZ_i A_i - \sum_{i=1}^{n} D^2 Z_i A_i r_i = -\sum_{i=1}^{n} \frac{t_i A_i}{(1 + r_i)^{t_i + 1}} - \sum_{i=1}^{n} \frac{t_i (t_i + 1) A_i r_i}{(1 + r_i)^{t_i + 2}} \qquad (16.b)$$

$$G = \frac{1}{2}\sum_{i=1}^{n} D^2 Z_i A_i r_i^{\,2} - \sum_{i=1}^{n} DZ_i A_i r_i =$$

$$= \frac{1}{2}\sum_{i=1}^{n} \frac{t_i (t_i + 1) A_i r_i^{\,2}}{(1 + r_i)^{t_i + 2}} + \sum_{i=1}^{n} \frac{t_i A_i r_i}{(1 + r_i)^{t_i + 1}}, \qquad (16.c)$$

**If** $F^2 - 4EG \geq 0$, then the formula

$$y = \frac{-F \pm \sqrt{F^2 - 4EG}}{2E}, \tag{17}$$

gives a better estimate of yield than the linear approximation (10). We have to pick up the solution "$y$" of (17) which is close to the discount rates, because the derivations of the formulae were done under the assumption that that Y=(y, y, …, y) is in the neighborhood of R = ($r_1$, $r_2$, …, $r_n$).

**Example 4 continued:** The formula (10) gives

$$y = \frac{\frac{0.071*7}{1.071^2} + \frac{2*0.071*107}{1.071^3} + \frac{0.08*8}{1.08^2} + \frac{2*0.08*8}{1.08^3} + \frac{3*0.08*108}{1.08^4}}{\frac{7}{1.071^2} + \frac{2*107}{1.071^3} + \frac{8}{1.08^2} + \frac{2*8}{1.08^3} + \frac{3*108}{1.08^4}} = 0.07630$$

It is easy to verify that the present value function (1) takes value P(0.0763) = 0.013 which is very close to zero. This means that even the linear approximation of yield is very good.

The formula (17) gives y = 0.076326, and P(0.076326) = -0.000011, practically zero. This calculation is done by a spreadsheet, see the results below.

If we approximate yield of the portfolio by the weighted average of yields (0.071, 0.08), where weights are values of bonds; A = 99.81945, and B = 100, then we have Avg(y) = 0.0755, and P(0.0755) = 0.361, which is a worse result even than our linear approximation (10).

| **i** | **Maturity Times in Years $t_i$** | **Spot Rates $s_i$** | **Cash Flows Ai** | **Zi** | **Ai*Zi** | **Ai*DZ** | **Ai*DZi*S** | **Ai*D²Zi** | **Ai*D²Zi*S** | **Ai*D²Zi*S** |
|---|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | | -199.81945 | | | | | | | |
| 1 | 1 | 0.0710 | 7 | 0.9337 | 7 | -6 | 0 | 11 | 1 | 0 |
| 2 | 2 | 0.0710 | 107 | 0.8718 | 93 | -174 | -12 | 488 | 35 | 2 |
| 3 | 1 | 0.0800 | 8 | 0.9259 | 7 | -7 | -1 | 13 | 1 | 0 |
| 4 | 2 | 0.0800 | 8 | 0.8573 | 7 | -13 | -1 | 35 | 3 | 0 |
| 5 | 3 | 0.0800 | 108 | 0.7938 | 86 | -238 | -19 | 882 | 71 | 6 |
| **Sums** | | | | | **200** | **-438** | **-33** | **1,429** | **110** | **8** |
| | **Formula (10)** | | | | | | | | | |
| **y** | **0.07630** | **P(y)=** | 0.013374 | | | | | | | |
| | | | | | | | | | | |
| | **Formula (17)** | | | | | | | | | |
| E | 715 | | | | | | | | | |
| F | -548 | | | | | | | | | |
| G | 38 | | | | | | | | | |
| **$y_1$ =** | **0.076326** | **P($y_1$)=** | -0.000011 | | | | | | | |
| **$y_2$ =** | **0.690260** | **P($y_2$)=** | -128.33 | | | | | | | |
| | | | | | | | | | | |
| **Avg(y)** | **0.075504** | **P(Avg(y))=** | 0.361057 | | | | | | | |

The solution $y_2$ of (17) does not need to be yield because it is not in the neighborhood of the discount rates 0.071, and 0.08, and it is not yield, as the polynomial (1) takes value P(0.69026) = -128.33, not close to zero.

## 3. Numerical Examples

**Example 2 continued:** Let us assume some realistic spot rates in the market M, e.g., $s_1$=0.03=3%, $s_2$=0.0325=3.25% and as before $A_1$=-230, $A_2$ =132, then

$$-A_0=A_1Z_1+A_2Z_2 = \frac{-230}{1.03}+\frac{132}{1.0325^2} = -99.480112 \neq -100.$$

This means that the transaction from the Example 2 is not allowed in the arbitrage free market M in which $s_1$=0.03, and $s_2$=0.0325. The only allowed transaction with given cash flows $A_1$=-230, $A_2$ =132 is described by $99.48-230v+132v^2 = 0$, rather than by $100-230v+132v^2 = 0$. The corresponding equation in terms of yield became $99.48y^2-31.04y+1.48=0$, rather than $100y^2-30y+2=0$.

The formula (10) gives the unique estimate for yield.

$$y=\frac{\frac{-230*0.03}{1.03^2}+\frac{132*2*0.0325}{1.0325^3}}{\frac{-230}{1.03^2}+\frac{132*2}{1.0325^3}}=0.056014,$$

which is a linear estimate. The formula (17) gives much better estimate $y_1$ = 0.058932. The second solution of (17), $y_2$ = 0.180767 is not a solution because it does not satisfy the condition to be close to the discount rates R = ($r_1$, $r_2$).

The equation that describes the allowed transaction, $99.48y^2-31.04y+1.48=0$, has two exact solutions, and they are $y_1$ = 0.058738, $y_2$ = 0.253848. The formula (17) nicely approximated $y_1$, the solution that is close to spot rates of the market M.

**Example 3 continued:** In order to be able to use our formulae for the cash flow ($A_1$, $A_2$) **=** (-200, 101), the discount rates need to be very close (as assumed before the proof of the formulae), for example $r_1$=0.06, $r_2$=0.0612. Let us show what happens in this case.

$$-A_0= A_1Z_1+A_2Z_2 = \frac{-200}{1.06}+\frac{101}{1.0612^2} = -98.992783 \neq 100.$$

This means that the transaction from Example 3 is not legitimate in the arbitrage free market M with discount rates (0.06, 0.0612) at durations of 1 year, and 2 years, respectively. The only allowed transaction with given cash flows $A_1$=-200, $A_2$ =101 is described by $98.99278347 - 200v + 101v^2 = 0$, rather than by $100 - 200v + 101v^2 = 0$. The equation in terms of yield becomes

$98.99278347y^2 - 2.01443306y - 0.00721653 = 0$, rater than $100y^2 + 1 = 0$.

The equation (17) gives the yield y = 0.026074. The other solution from (17) y = -0.024354 need not to be yield because it is far from the discount rates (0.06, 0.0612). Then polynomial (1) takes value P(0.026074) = 0.00718, which is pretty close to zero, even though this example represents an extreme situation.

Because there is NO real solution of the equation with such cash flows, everybody would have to adjust some of the coefficients of the equation (1) in order to get some estimate of the yield. The question arises: How to adjust the coefficients in order to get some estimate of yield in such case? The formulae of this paper offer one answer. They estimate the yield without any adjustments of the cash flows $A_1$, $A_2$,…, $A_n$, which means that only the coefficient **-$A_0$ has to be (implicitly) adjusted**. In this example, as well as in Example 2, the assumption that the discount rates have to be sufficiently close plays a big role. If we are not happy with the estimate, that is because the discount rates $r_1$, $r_2$ are not close enough. The estimates would be better if the discount rates were closer.

**Corollary 2 (of formulae (10) and (17).** In case of a transaction that does not have a real number solution for yield we might be able to find an estimate of yield for the portfolio that has the same cash flows $A_1$, $A_2$, … , $A_n$. The estimate of yield will depend on the current discount rates $r_1$, $r_2$, …, $r_n$ . Then the cash flow **-$A_0$** is adjusted to be the present value of cash flows

$A_1$, $A_2$, … , $A_n$, discounted by $r_1$, $r_2$, …, $r_n$ , respectively. The final result is the legitimate transaction, an approximation of the original transaction, with the yield rate which is approximated by (10) and (17).

**Example 5**: Calculate yield of the portfolio that consists of n=10 zero coupon bonds, with Maturity Times, T = (0.5; 1; 1.5; 2; 4; 6; 7; 8; 9; 10),

Discount Rates, R = (0.03; 0.0325; 0.035; 0.0375; 0.04; 0.0425; 0.045; 0.0475; 0.05; 0.055),
Cash Flows, A = (4000; 1,000,000; 250,000; 400,000; 700,000; 85,000; -1,000,000; 100,000; 100,000; 1,000,000) Vectors T, R, and A are given in columns 2, 3,and 4, respectively, of the following table:

| **i** | **Maturity Times in Years $t_i$** | **Spot Rates $r_i$** | **Cash Flows Ai** | **Zi** | **Ai*Zi** | **Ai*DZi** | **Ai*DZi*r** | **Ai*D$^2$Zi** | **Ai*D$^2$Zi*ri** | **Ai*D$^2$Zi*ri$^2$** |
|---|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **0** | -2,230,127 | | | | | | | |
| 1 | 0.5 | 0.0300 | 4,000 | 0.9853 | 3,941 | -1,913 | -57 | 2,786 | 84 | 3 |
| 2 | 1 | 0.0325 | 1,000,000 | 0.9685 | 968,523 | -938,037 | -30,486 | 1,817,020 | 59,053 | 1,919 |
| 3 | 1.5 | 0.0350 | 250,000 | 0.9497 | 237,427 | -344,097 | -12,043 | 831,151 | 29,090 | 1,018 |
| 4 | 2 | 0.0375 | 400,000 | 0.9290 | 371,607 | -716,351 | -26,863 | 2,071,375 | 77,677 | 2,913 |
| 5 | 4 | 0.0400 | 700,000 | 0.8548 | 598,363 | -2,301,396 | -92,056 | 11,064,403 | 442,576 | 17,703 |
| 6 | 6 | 0.0425 | 85,000 | 0.7790 | 66,216 | -381,099 | -16,197 | 2,558,938 | 108,755 | 4,622 |
| 7 | 7 | 0.0450 | -1,000,000 | 0.7348 | -734,828 | 4,922,296 | 221,503 | -37,682,648 | -1,695,719 | -76,307 |
| 8 | 8 | 0.0475 | 100,000 | 0.6899 | 68,987 | -526,870 | -25,026 | 4,526,809 | 215,023 | 10,214 |
| 9 | 9 | 0.0500 | 100,000 | 0.6446 | 64,461 | -552,522 | -27,626 | 5,262,114 | 263,106 | 13,155 |
| 10 | 10 | 0.0550 | 1,000,000 | 0.5854 | 585,431 | -5,549,105 | -305,201 | 57,857,967 | 3,182,188 | 175,020 |
| **Sums** | | | | | **2,230,127** | **-6,389,094** | **-314,053** | **48,309,916** | **2,681,833** | **150,260** |
| | **Formula (10)** | | | | | | | | | |
| **y** | **0.049154** | | | | | | | | | |
| | **Formula (17)** | | | | | | | | | |
| E | 24,154,958 | | | | | | | | | |
| F | -9,070,926 | | | | | | | | | |
| G | 389,182 | | | | | | | | | |
| **$y_1$** | **0.049404** | | | | | | | | | |
| **$y_2$** | **0.326127** | | | | | | | | | |

This was an example of a "nonsimple transaction," as defined in Teichroew, et al. (1965) and a transaction with a middle "negative inflow" as defined in Jean (1968); not an easy transaction to determine yield according to Teichroew, et al. (1965) and Jean (1968). The formula (10) gives y = 0.049154, while formula (17) gives better estimate, $y_1$ = 0.049404. Note that we have fractional times $t_i$ (could be any real numbers) and that there are three changes of the signs in the sequence A of cash flows. The second solution of (17) $y_2$ = 0.326127 is not yield. It is easy to verify that $y_2$ does not satisfy equation (1) of this example. We did not expect that to be the case, because $y_2$ is too far from discount rates of the market in this example.

## 4. Conclusion

Yield rate is important to investors for assessing transactions. In this paper we estimate the yield rate by means of quantities that depend on the market (the current interest discount rates of the "zeros" that comprise the portfolio), as opposed to calculating yield rates solely by means of the cash flows, independently of any particular environment. We deal with interest rates to calculate yield, rather than dealing only with the coefficients of the equation (1). We limit ourselves to the portfolio P, which consists of fixed income securities that are not interest-sensitive in an arbitrage free market M. A natural assumption for the discount rates of the portfolio is utilized; that they are sufficiently close to each other. Under these assumptions we were able to prove two explicit estimates for yield in terms of discount rates in the portfolio; formula (10) is a linear approximation of yield, and formula (17) uses second derivatives (convexity) as well. The approximations (10) and (17) approximate the yield that is closest to the discount rates of the portfolio P.

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