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ON UNIQUENESS OF INTEREST RATES IN A BORROWING/LENDING MODEL

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Note: The enclosed proof is given in some detail for possible student use in undergraduate classes in the theory of interest.

With a pure mathematics background but lately interested in actuarial science, I share a grievance, perhaps, with other "switchers" to applied areas: the literature is not always as precise as we would like. Sometimes we don't know just what is being claimed. In this note we close a small gap that may have existed in the actuarial literature for years. More importantly, we illustrate how some simple mathematics is used in "unusual" ways to solve an actuarial/business question. (This author has worked as an actuary, and this is but one example among many.)

The notation follows [2] since that text is required for the actuarial examinations, whereas research papers are not. Here is the situation:

An investor or businessman undergoes a project for n years. The outstanding monetary balance, B_t, at the end of year t may be positive, negative or zero, for $t = 1,2,\ldots, n-1$. The initial amount (at beginning of year 1) is B_0. If $B_t \geq 0$, the investor's interest preference rate as a lender, or "project return rate", as a decimal is r for the next year; if $B_t < 0$, the investor is in borrower status, and his interest preference rate, or "project financing rate", is f for next year. Ordinarily r exceeds f but this is not required. At the end of the n years, when the project is complete and all funds are accounted for, the outstanding balance is zero: $B_n = 0$. The actual cash flows, positive or negative, at time $t = 0,1,2,\ldots,n-1$, are denoted by the constants C_t. So we have:

$$B_0 = C_0$$

$$\left.\begin{array}{l} B_t = B_{t-1}(1+r) + C_t \text{ if } B_{t-1} \geq 0 \\ B_t = B_{t-1}(1+f) + C_t \text{ if } B_{t-1} < 0 \end{array}\right\} \quad t = 1,2,\ldots, n$$

$$B_n = 0$$

Clearly B_n is a polynomial in r and f. This question arises:

Assuming solution pairs (r,f) exist for $B_n = 0$, are r and f unique functions of each other?

For this discussion, we assume both r and f exceed -1. (Interest rates of -100% or less rarely arise in practice.) In [3] and [4] the authors may claim the answer to our query is "yes". However, their "function" may mean merely "relation" (e.g., in [4, p. 169] they reference a "set of functions k = k(r) or r = r(k)" where k is our f (italics supplied). Kellison [2, p. 158] references no other papers but [3] and [4]; he uses the term "functional relationship" without proof. His one example gives a linear relationship between r and f which is clearly a function, but higher degree polynomials $B_n = 0$ are not addressed. The other recent text in interest theory for actuarial students, [1] , does not deal with the question.

We now show that, if solution pairs (r,f) solve $B_n = 0$, then r is a unique function of f if at least one $B_t > 0$ for $t = 0,1,...,n-1$. (A similar proof shows f is a unique function of r if at least one $B_t < 0$.) We argue by contradiction.

Suppose (r,f) and (r',f) are distinct solutions to $B_n = 0$, where $r > r'$. Let B'_t denote project balances associated with r'. Clearly $B_0 = B'_0$ since both equal C_0. As long as $B_{t-1} = B'_{t-1}$ is nonpositive, $B_t = B'_t$ since (1) for $B_{t-1} = B'_{t-1} < 0$, we have $B_t = B_{t-1}(1+f) + C_t$ and $B'_t = B'_{t-1}(1+f) + C_t$, or (2) if $B_{t-1} = B'_{t-1} = 0$, then $B_t = 0\cdot(1+r) + C_t = C_t$ and likewise if we prime B_t and r.

By assumption, there is a least index i such that $B_i > 0$ (where i may be zero). By the above reasoning $B_i = B'_i$. However,

$B_{i+1} = B_i(1+r) + C_{i+1}$ while $B'_{i+1} = B'_i(1+r') + C_{i+1}$. Since $B_i = B'_i > 0$ and $r > r' > -1$, thus $B_{i+1} > B'_{i+1}$. To see what happens next we consider four, exhaustive, possibilities: (1) $B_{i+1} > B'_{i+1} \geq 0$, (2) $0 = B_{i+1} > B'_{i+1}$, (3) $0 > B_{i+1} > B'_{i+1}$, (4) $B_{i+1} > 0 > B'_{i+1}$.

In the case:

(1) $B_{i+2} = B_{i+1}(1+r) + C_{i+2}$ and similarly if we prime the B's and the r. Clearly $B_{i+2} > B'_{i+2}$ since $B_{i+1} > B'_{i+1}$ and $r > r' > -1$

(2) $B_{i+2} = B_{i+1}(1+r) + C_{i+2}$, while $B'_{i+2} = B'_{i+1}(1+f) + C_{i+2}$. Again $B_{i+2} > B'_{i+2}$ since $B_{i+1}(1+r) = 0$ but $B'_{i+1}(1+f) < 0$.

(3) $B_{i+2} = B_{i+1}(1+f) + C_{i+2} > B'_{i+1}(1+f) + C_{i+2} = B'_{i+2}$ since $B_{i+1} > B'_{i+1}$ and $f > -1$.

(4) $B_{i+2} = B_{i+1}(1+r) + C_{i+2} > B'_{i+1}(1+f) + C_{i+2} = B'_{i+2}$ since $B_{i+1}(1+r) > 0$ and $B'_{i+1}(1+f) < 0$.

Thus in all four possible cases $B_{i+2} > B'_{i+2}$. Continuing in this way we ultimately get $B_n > B'_n$. But B_n and B'_n are both supposed to be zero. This completes the proof.

It not only suffices that at least one B_t exceed zero; it is also necessary (in almost all practical examples we would expect one or more positive, and one or more negative, B_t's):

Let $f = 1$ (admittedly unrealistic but it simplifies the work). Let $C_0 = C_2 = -1$, $C_1 = C_3 = 2$, and $n = 3$. Since both B_1 and B'_1 are zero, $B_2 = B_1(1+r) - 1$ and $B'_2 = B'_1(1+r') - 1$ are both -1 for <u>any</u> $r > r'$. Yet $B_3 = B'_3 = 0$. Likewise, if, given r, f is to be unique, at least one B_t must be less than zero. So a unique one-to-one functional relationship between r and f requires a positive B_t and a negative one.

In like fashion one can show (1) B_n is an increasing function of $r > -1$, for fixed $f > -1$, if at least one B_t exceeds zero (of course we here remove the restriction that $B_n = 0$): raise r' to r and proceed with B_t' associated with r' as above, and (2) B_n is a decreasing function of $f > -1$, for fixed $r > -1$, if at least one B_t is negative. Hence for B_n to remain zero, r must be an increasing function of f: $dr/df > 0$ on "segments" (see below).

Also (1) and (2) show the partial derivative of B_n with respect to r (respectively, f) is positive (negative) on each "segment" as defined below. All this establishes some key results in [3] , and the remaining results in [3] and [4] follow as shown there.

[Regarding our main proof, the fact that r and f are unique functions of each other also follows from the work in [3] from (1) the mean value theorem for those r (call them a "segment") for which B_n does not change degree in r or f, for fixed f, (and vice versa), (2) the fact that on each segment r and f are increasing functions of each other, and (3) the fact that the segments "join" (continuity arguments). However, statements equally "simple" are explicitly proved in [3], and there is the above quote from [4], a later paper referencing [3].]

Finally, if we set $r = f = i$, a lack of uniqueness problem for i may exist if the polynomial B_n has more than one sign change in its coefficients (the C_t); that is, $i > i' > -1$ may both solve $B_n = 0$, a fact which has complicated financial analysis for a long time. Our work exhibits at least one situation where uniqueness is guaranteed!

In any event, our proof differs from [3], using only simple algebra.

References

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