Multivariate Operational Risk: Dependence Modelling with Lévy Copulas

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Abstract

Simultaneous modelling of operational risks occurring in different event type/business line cells poses *the* challenge for operational risk quantification. Invoking the new concept of Lévy copulas for dependence modelling yields simple approximations of high quality for multivariate operational VAR.

1 Introduction

A required feature of any advanced measurement approach (AMA) of Basel II for measuring operational risk is that it allows for explicit correlations between different operational risk events. The core problem here is multivariate modelling encompassing all different event type/business line cells, and thus the question how their dependence structure affects a bank's total operational risk. The prototypical loss distribution approach (LDA) assumes that, for each cell $i = 1, \ldots, d$, the cumulated operational loss $S_i(t)$ up to time t is described by an aggregate loss process

$$S_i(t) = \sum_{k=1}^{N_i(t)} X_k^i \,, \quad t \geq 0 \,, \tag{1.1}$$

where for each i the sequence $(X_k^i)_{k\in\mathbb{N}}$ are independent and identically distributed (iid) positive random variables with distribution function F_i describing the magnitude of each loss event (loss severity), and $(N_i(t))_{t\geq 0}$ counts the number of losses in the time interval

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$[0,t]$ (called frequency), independent of $(X_k^i)_{k\in\mathbb{N}}$. The bank's total operational risk is then given as

$$S^+(t) := S_1(t) + S_2(t) + \cdots + S_d(t)\,, \quad t \geq 0\,. \tag{1.2}$$

The present literature suggests to model dependence between different operational risk cells by means of different concepts, which basically split into models for frequency dependence on the one hand and for severity dependence on the other hand.

Here we suggest a model based on the new concept of Lévy copulas (see e.g. Cont & Tankov (2004)), which models dependence in frequency and severity simultaneously, yielding a model with comparably few parameters. Moreover, our model has the same advantage as a distributional copula: the dependence structure between different cells can be separated from the marginal processes S_i for $i = 1,\dots,d$. This approach allows for closed-form approximations for operational VAR (OpVAR).

2 Dependent Operational Risks and Lévy Copulas

In accordance with a recent survey of the Basel Committee on Banking Supervision about AMA practices at financial services firms, we assume that the loss frequency processes N_i in (1.1) follows a homogeneous Poisson process with rate $\lambda_i > 0$. Then the aggregate loss (1.1) constitutes a compound Poisson process and is therefore a Lévy process (actually, the compound Poisson process is the only Lévy process with piecewise constant sample paths).

A key element in the theory of Lévy processes is the notion of the so-called Lévy measure. A Lévy measure controls the jump behaviour of a Lévy process and, therefore, has an intuitive interpretation, in particular in the context of operational risk. The Lévy measure of a single operational risk cell measures the expected number of losses per unit time with a loss amount in a prespecified interval. For our compound Poisson model, the Lévy measure Π_i of the cell process S_i is completely determined by the frequency parameter $\lambda_i > 0$ and the distribution function F_i of the cell's severity: $\Pi_i([0,x)) := \lambda_i P(X^i \leq x) = \lambda_i F_i(x)$ for $x \in [0,\infty)$. The corresponding one-dimensional tail integral is defined as

$$\overline{\Pi}_i(x) := \Pi_i([x,\infty)) = \lambda_i P(X^i > x) = \lambda_i \overline{F}_i(x)\,. \tag{2.1}$$

Our goal is modelling multivariate operational risk. Hence, the question is how different one-dimensional compound Poisson processes $S_i(\cdot) = \sum_{k=1}^{N_i(\cdot)} X_k^i$ can be used to construct a d-dimensional compound Poisson process $S = (S_1, S_2, \dots, S_d)$ with in general dependent components. It is worthwhile to recall the similar situation in the case of the more restrictive setting of static random variables. It is well-known that the dependence structure of a random vector can be disentangled from its marginals by introducing a distributional copula. Similarly, a multivariate tail integral

$$\overline{\Pi}(x_1, \dots, x_d) = \Pi([x_1, \infty) \times \dots \times [x_d, \infty)), \quad x \in [0, \infty]^d, \tag{2.2}$$

can be constructed from the marginal tail integrals (2.1) by means of a Lévy copula. This representation is the content of Sklar's theorem for Lévy processes with positive jumps, which basically says that every multivariate tail integral $\overline{\Pi}$ can be decomposed into its marginal tail integrals and a Lévy copula $\widehat{C}$ according to

$$\overline{\Pi}(x_1, \dots, x_d) = \widehat{C}(\overline{\Pi}_1(x_1), \dots, \overline{\Pi}_d(x_d)), \qquad x \in [0, \infty]^d. \tag{2.3}$$

For a precise formulation of this Theorem we refer to Cont & Tankov (2004), Theorem 5.6. Now we can define the following prototypical LDA model.

Definition 2.1. [Multivariate Compound Poisson model]
(1) All aggregate loss processes S_i for $i = 1, \dots, d$ are compound Poisson processes with tail integral $\overline{\Pi}_i(\cdot) = \lambda_i F_i(\cdot)$.
(2) The dependence between different cells is modelled by a Lévy copula $\widehat{C} : [0, \infty)^d \to [0, \infty)$, i.e., the tail integral of the d-dimensional compound Poisson process $S = (S_1, \dots, S_d)$ is defined by

$$\overline{\Pi}(x_1, \dots, x_d) = \widehat{C}(\overline{\Pi}_1(x_1), \dots, \overline{\Pi}_d(x_d)).$$

3 The Bivariate Clayton Model

A bivariate model is particularly useful to illustrate how dependence modelling via Lévy copulas works. Therefore, we now focus on two operational risk cells as in Definition 2.1(1). The dependence structure is modelled by a Clayton Lévy copula, which is similar to the well-known Clayton copula for distribution functions and parameterized by $\vartheta > 0$ (see Cont & Tankov (2004), Example 5.5):

$$\widehat{C}_\vartheta(u, v) = (u^{-\vartheta} + v^{-\vartheta})^{-1/\vartheta}, \quad u, v \geq 0.$$

This copula covers the whole range of positive dependence. For $\vartheta \to 0$ we obtain independence and then, as we will see below, losses in different cells never occur at the same time. For $\vartheta \to \infty$ we get the complete positive dependence Lévy copula given by $\widehat{C}_\parallel(u, v) = \min(u, v)$. We decompose now the two cells' aggregate loss processes into different components (where the time parameter t is dropped for simplicity):

$$\begin{aligned} S_1 &= S_{\perp 1} + S_{\| 1} = \sum_{k=1}^{N_{\perp 1}} X^1_{\perp k} + \sum_{l=1}^{N_{\|}} X^1_{\| l}\,, \\ S_2 &= S_{\perp 2} + S_{\| 2} = \sum_{m=1}^{N_{\perp 2}} X^2_{\perp m} + \sum_{l=1}^{N_{\|}} X^2_{\| l}\,, \end{aligned} \tag{3.1}$$

where $S_{\|1}$ and $S_{\|2}$ describe the aggregate losses of cell 1 and 2 that is generated by "common shocks", and $S_{\perp 1}$ and $S_{\perp 2}$ describe aggregate losses of one cell only. Note that apart from $S_{\|1}$ and $S_{\|2}$, all compound Poisson processes on the right-hand side of (3.1) are mutually independent. The frequency of simultaneous losses is given by

$$\widehat{C}_\vartheta(\lambda_1, \lambda_2) = \lim_{x \downarrow 0} \overline{\Pi}_{\|2}(x) = \lim_{x \downarrow 0} \overline{\Pi}_{\|1}(x) = (\lambda_1^{-\theta} + \lambda_2^{-\theta})^{-1/\theta} =: \lambda_{\|}\,,$$

which shows that the number of simultaneous loss events is controlled by the Lévy copula. Obviously, $0 \le \lambda_{\|} \le \min(\lambda_1, \lambda_2)$,where the left and right bounds refer to $\vartheta \to 0$ and $\vartheta \to \infty$, respectively. Consequently, in the case of independence, losses never happen at the same instant of time.

Also the severity distributions of $X^1_{\|}$ and $X^2_{\|}$ as well as their dependence structure are determined by the Lévy copula. To see this, define the joint survival function as

$$\overline{F}_{\|}(x_1, x_2) := P(X^1_{\|} > x_1, X^2_{\|} > x_2) = \frac{1}{\lambda_{\|}} \widehat{C}_\vartheta(\overline{\Pi}_1(x_1), \overline{\Pi}_2(x_2)) \tag{3.2}$$

with marginals

$$\overline{F}_{\|1}(x_1) = \lim_{x_2 \downarrow 0} \overline{F}_{\|}(x_1, x_2) = \frac{1}{\lambda_{\|}} \widehat{C}_\vartheta(\overline{\Pi}_1(x_1), \lambda_2) \tag{3.3}$$

$$\overline{F}_{\|2}(x_2) = \lim_{x_1 \downarrow 0} \overline{F}_{\|}(x_1, x_2) = \frac{1}{\lambda_{\|}} \widehat{C}_\vartheta(\lambda_1, \overline{\Pi}_2(x_2))\,. \tag{3.4}$$

In particular, it follows that $F_{\|1}$ and $F_{\|2}$ are different from F_1 and F_2, respectively. To explicitly extract the dependence structure between the severities of simultaneous losses $X^1_{\|}$ and $X^2_{\|}$ we use the concept of a distributional survival copula. Using (3.2)–(3.4) we see that the survival copula S_ϑ for the tail severity distributions $\overline{F}_{\|1}(\cdot)$ and $\overline{F}_{\|2}(\cdot)$ is the well-known distributional Clayton copula; i.e.

$$S_\vartheta(u, v) = (u^{-\vartheta} + v^{-\vartheta} - 1)^{-1/\vartheta}, \qquad 0 \le u, v \le 1\,.$$

For the tail integrals of the independent loss processes $S_{\perp 1}$ and $S_{\perp 2}$. we obtain for $x_1, x_2 \ge 0$

$$\begin{aligned} \overline{\Pi}_{\perp 1}(x_1) &= \overline{\Pi}_1(x_1) - \overline{\Pi}_{\|1}(x_1) = \overline{\Pi}_1(x_1) - \widehat{C}_\vartheta(\overline{\Pi}_1(x_1), \lambda_2)\,, \\ \overline{\Pi}_{\perp 2}(x_2) &= \overline{\Pi}_2(x_2) - \overline{\Pi}_{\|2}(x_2) = \overline{\Pi}_2(x_2) - \widehat{C}_\vartheta(\lambda_1, \overline{\Pi}_2(x_2))\,, \end{aligned}$$

so that $\lambda_{\perp 1} = \lambda_1 - \lambda_{\|}\,, \qquad \lambda_{\perp 2} = \lambda_2 - \lambda_{\|}$.

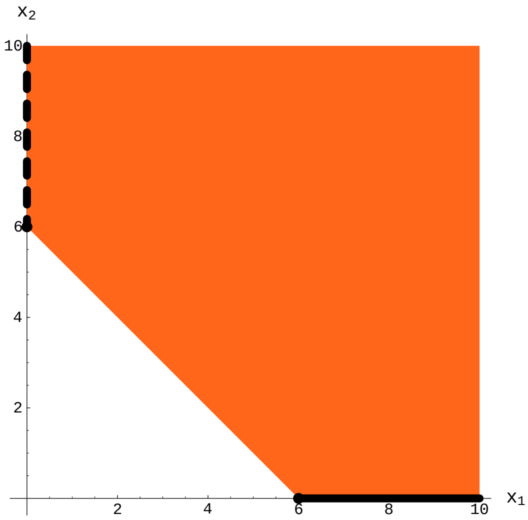

Figure 4.1. *Decomposition of the domain of the tail integral $\overline{\Pi}^+(z)$ for $z = 6$ into a simultaneous loss part $\overline{\Pi}_\|^+(z)$ (orange area) and independent parts $\overline{\Pi}_{\perp 1}(z)$ (solid black line) and $\overline{\Pi}_{\perp 2}(z)$ (dashed black line).*

4 Analytical Approximations for Operational VAR

In this section we turn to the quantification of total operational loss encompassing all operational risk cells and, therefore, we focus on the total aggregate loss process S^+ defined in (1.2). Our goal is to provide some general insight to multivariate operational risk and to find out, how different dependence structures (modelled by Lévy copulas) affect OpVAR, which is the standard metric in operational risk measurement. The tail integral associated with S^+ is given by

$$\overline{\Pi}^+(z) = \Pi(\{(x_1, \dots, x_d) \in [0,\infty)^d : \sum_{i=1}^d x_i \geq z\}), \quad z \geq 0. \tag{4.1}$$

For $d = 2$ we can write

$$\overline{\Pi}^+(z) \quad = \quad \overline{\Pi}_{\perp 1}(z) + \overline{\Pi}_{\perp 2}(z) + \overline{\Pi}_\|^+(z), \qquad z \geq 0, \tag{4.2}$$

where $\overline{\Pi}_{\perp 1}(\cdot)$ and $\overline{\Pi}_{\perp 2}(\cdot)$ are the independent jump parts and

$$\overline{\Pi}_\|^+(z) \quad = \quad \Pi(\{(x_1, x_2) \in (0,\infty)^2 : x_1 + x_2 \geq z\}), \qquad z \geq 0,$$

describes the dependent part due to simultaneous loss events; the situation is depicted in Figure 4.1.

Since for every compound Poisson process with intensity $\lambda > 0$ and positive jumps with distribution function F, the tail integral is given by $\overline{\Pi}(\cdot) = \lambda \overline{F}(\cdot)$, it follows from

(4.2) that the total aggregate loss process S^+ is again compound Poisson with frequency parameter and severity distribution

$$\lambda^+ = \lim_{z \downarrow 0} \overline{\Pi}^+(z) \quad \text{and} \quad F^+(z) = 1 - \overline{F}^+(z) = 1 - \frac{\overline{\Pi}^+(z)}{\lambda^+}, \quad z \geq 0. \tag{4.3}$$

This result proves now useful to determine a bank's total operational risk consisting of several cells. Before doing that, recall the definition of OpVAR for a single operational risk cell (henceforth called stand-alone OpVAR.) For each cell, stand-alone OpVAR at confidence level $\kappa \in (0,1)$ and time horizon t is the κ-quantile of the aggregate loss distribution, i.e. $\mathrm{VAR}_t(\kappa) = G_t^{\leftarrow}(\kappa)$ with $G_t^{\leftarrow}(\kappa) = \inf\{x \in \mathbb{R} : \; P(S(t) \leq x) \geq \kappa\}$.

In Böcker & Klüppelberg (2005, 2006, 2007) it was shown that OpVAR at high confidence level can be approximated by a closed-form expression, if the loss severity is subexponential, i.e. heavy-tailed. As this is common believe we consider in the sequel this approximation, which can be written as

$$\mathrm{VAR}_t(\kappa) \sim F^{\leftarrow}\left(1 - \frac{1-\kappa}{EN(t)}\right), \quad \kappa \uparrow 1, \tag{4.4}$$

where the symbol "$\sim$" means that the ratio of left and right hand side converges to 1. Moreover, $EN(t)$ is the cell's expected number of losses in the time interval $[0,t]$. Important examples for subexponential distributions are lognormal, Weibull, and Pareto. Here, we extend the idea of an asymptotic OpVAR approximation to the multivariate problem. In doing so, we exploit the fact that S^+ is a compound Poisson process with parameters as in (4.3). In particular, if F^+ is subexponential, we can apply (4.4) to estimate total OpVAR. Consequently, if we are able to specify the asymptotic behaviour of $\overline{F}^+(x)$ as $x \to \infty$ we have automatically an approximation of $\mathrm{VAR}_t(\kappa)$ as $\kappa \uparrow 1$.

To make more precise statements about OpVAR, we focus our analysis on Pareto distributed severities with distribution function

$$\overline{F}(x) = \left(1 + \frac{x}{\theta}\right)^{-\alpha}, \quad x > 0,$$

with shape parameters $\theta > 0$ and tail parameter $\alpha > 0$. Pareto's law is the prototypical parametric example for a heavy-tailed distribution and suitable for operational risk modelling. As a simple consequence of (4.4), in the case of a compound Poisson model with Pareto severities (Pareto-Poisson model) the analytic OpVAR is given by

$$\mathrm{VAR}_t(\kappa) \sim \theta \left[\left(\frac{\lambda\, t}{1-\kappa}\right)^{1/\alpha} - 1\right] \sim \theta \left(\frac{\lambda\, t}{1-\kappa}\right)^{1/\alpha}, \quad \kappa \uparrow 1. \tag{4.5}$$

To demonstrate the kind of results we obtain by such approximation methods we consider a Pareto-Poisson model, where the severity distributions F_i of the first (say)

$b \le d$ cells are tail equivalent with tail parameter $\alpha > 0$ and dominant to all other cells, i.e.

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \left(\frac{\theta_i}{\theta_1}\right)^{\alpha}, \; i = 1, \dots, b, \qquad \lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = 0\,, \; i = b+1, \dots, d\,. \tag{4.6}$$

In the important cases of complete positive dependence and independence, closed-form results can be found and may serve as extreme cases concerning the dependence structure of the model.

Theorem 4.2. *Consider a compound Poisson model with cell processes $S_1, \dots, S_d$ with Pareto distributed severities satisfying* (4.6)*. Let $\mathrm{VAR}_t^i(\cdot)$ be the stand-alone* OpVAR *of cell i.*

(1) *If all cells are completely dependent with the same frequency λ for all cells, then S^+ is compound Poisson with parameters*

$$\lambda^+ = \lambda \qquad \text{and} \qquad \overline{F}^+(z) \sim \left(\sum_{i=1}^{b} \theta_i\right)^{\alpha} z^{-\alpha}, \quad z \to \infty\,,$$

and total OpVAR *is asymptotically given by*

$$\mathrm{VAR}_{\|t}^{+}(\kappa) \sim \sum_{i=1}^{b} \mathrm{VAR}_t^i(\kappa), \qquad \kappa \uparrow 1\,. \tag{4.7}$$

(2) *If all cells are independent, then S^+ is compound Poisson with parameters*

$$\lambda^+ = \lambda_1 + \dots + \lambda_d \quad \text{and} \quad \overline{F}^+(z) \sim \frac{1}{\lambda^+} \sum_{i=1}^{b} \left(\frac{\theta_i}{z}\right)^{\alpha} \lambda_i\,, \quad z \to \infty\,, \tag{4.8}$$

and total OpVAR *is asymptotically given by*

$$\mathrm{VAR}_{\perp t}^{+}(\kappa) \sim \left[\sum_{i=1}^{b} \left(\mathrm{VAR}_t^i(\kappa)\right)^{\alpha}\right]^{1/\alpha}, \qquad \kappa \uparrow 1\,. \tag{4.9}$$

On the one hand, Theorem 4.2 states that for the completely dependent Pareto-Poisson model, total asymptotic OpVAR is simply the sum of the dominating cell's asymptotic stand-alone OpVARs. Recall that this is similar to the new proposals of Basel II, where the standard procedure for calculating capital charges for operational risk is just the simple-sum VAR. To put it another way, regulators implicitly assume complete dependence between different cells, meaning that losses within different business lines or risk categories always happen at the same instants of time.

Very often, the simple-sum OpVAR (4.7) is considered to be the worst case scenario and, hence, as an upper bound for total OpVAR in general, which in the heavy-tailed case can be grossly misleading. To see this, assume the same frequency λ in all cells also for the independent model, and denote by $\mathrm{VAR}_{\|}^{+}(\kappa)$ and $\mathrm{VAR}_{\perp}^{+}(\kappa)$ completely dependent and independent total OpVAR, respectively. In this case we obtain from (4.9) in the situation (4.6) from Theorem 4.2(2)

$$\mathrm{VAR}_{\perp}^{+}(\kappa) \sim \left(\frac{\lambda\, t}{1-\kappa}\right)^{1/\alpha} \left(\sum_{i=1}^{b} \theta_i^{\alpha}\right)^{1/\alpha}, \quad \kappa \uparrow 1\,,$$

whereas $\mathrm{VAR}_{\|}^{+}(\kappa)$ is given by (4.7). Then, as a consequence of convexity ($\alpha > 1$) and concavity ($\alpha < 1$) of the function $x \mapsto x^{\alpha}$,

$$\frac{\mathrm{VAR}_{\perp}^{+}(\kappa)}{\mathrm{VAR}_{\|}^{+}(\kappa)} = \frac{\left(\sum_{i=1}^{b} \theta_i^{\alpha}\right)^{1/\alpha}}{\sum_{i=1}^{b} \theta_i} \begin{cases} < 1\,, & \alpha > 1 \\ = 1\,, & \alpha = 1 \\ > 1\,, & \alpha < 1\,. \end{cases} \tag{4.10}$$

This result says that for heavy-tailed severity data with $\overline{F}_i(x_i) \sim (x_i/\theta_i)^{-\alpha}$ as $x_i \to \infty$, subadditivity of OpVAR is violated because the sum of stand-alone OpVARs is smaller than independent total OpVAR.

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