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A TWELVE PARAMETER MODEL OF SELECT MORTALITY RATES

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ABSTRACT

This paper presents a parsimonious twelve parameter model that explains the pattern of mortality for the combined mortality rates of the 1965-70 Select and Ultimate Basic Tables. The parameters in this model provide insightful demographic and statistical information about the effects of selection. This parametric model is useful because it can predict the select mortality rates beyond the fifteen year select period and it can predict the effects of selection beyond seventy years old.

1. INTRODUCTION

The purpose of this paper is to present a parsimonious parametric model that explains the pattern of mortality for select and ultimate mortality tables. Specifically, we will model the combined rates found in the 1965-70 Age Last Birthday Basic Tables produced by the Committee on Ordinary Insurance and Annuities [2]. Before proceeding, it is instructive to plot the rates and examine the pattern of mortality in the Basic Tables. All the graphs in this paper were produced with the statistical computing language GAUSS.

Examining Figure 1, the reader will find plots of the logarithm of the select and ultimate combined rates from the Basic Tables. Let $q_{[x]+k}$ denote a graduated select mortality rate for an issue age $x \geq 0$, at the last birthday, and for policy year $k+1 \geq 1$. We will denote the attained age as $y = x + k$. Note that the select period for these tables is fifteen years and so $q_{[x]+k}$ is given only for $k = 0, \ldots, 14$. Next, let q_y denote a graduated ultimate rate for a person aged y. Now, consider the graph in Figure 1. This graph plots $\log_e(q_y)$ for $y = 15, \ldots, 90$ and it plots fifteen curves for each of the policy years. That is, for each k, the graph plots $\log_e(q_{[y-k]+k})$ for the attained ages $y = k, \ldots, k+67$. Many of the values for $q_{[y-k]+k}$ are not given in the Basic Tables because of grouping, therefore the function $\log_e(q_{[y-k]+k})$, with respect to y, was approximated linearly. Examining the graph, we find that the pattern exhibited by $\log_e(q_{[y-k]+k})$, shows a decrease in the childhood years, a hump at about age 20 and a linear pattern at the adult ages. A necessary condition for select tables is the *monotonicity condition*

$$q_{[y-k]+k} \leq q_{[y-k-1]+k+1}. \tag{1.1}$$

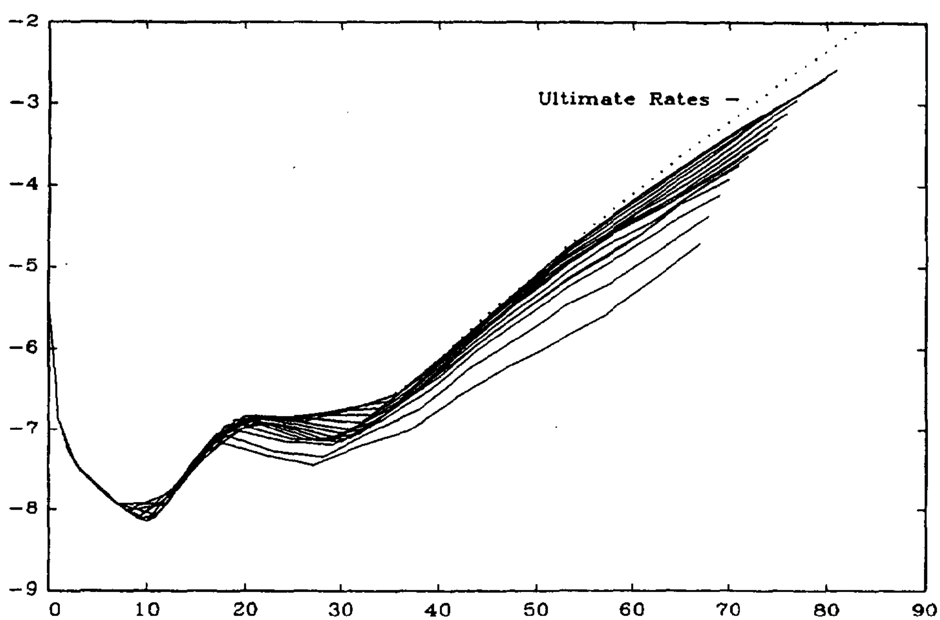
Examining Figure 1 we find that the rates exhibit considerable monotonicity, although the monotonicity condition does not hold strictly for all $y = 1,\ldots,67$ and

$k = 0, \ldots, \min(y-1,\ 14)$, because of the linear approximation.

The Basic Tables present select rates in five-year age groupings, which may be inconvenient to practitioners because the rates must be interpolated, which is a strength of mathematical formulas. Another strength of our mathematical model is the ability to predict or estimate the select rates at issue ages above seventy, which is impossible with the current tabulated rates. Still another strength of our parametric model is the ability to extend the select period beyond 15 years. Finally, the parameters in our model will provide insightful demographic and statistical information about the effects of selection. Therefore, in our opinion, a mathematical model is the most convenient way for practitioners to calculate select rates.

FIGURE 1

Plots of $\log_e(q_y)$ and $\log_e(q_{[y-k]+k})$ from the Combined 1965-70 Basic Tables



A review of the literature reveals that very little research has been done on the fitting of parametric formulas to select rates. Using Canadian data, Panjer and Giuseppe [3] did a graduation of select and ultimate rates that they refer to as "parametric." In fact, a true parametric formula was only developed at the higher ages. This is also true of the laws of select and ultimate mortality developed by Tenenbein and Vanderhoof [4]. In both cases the formulas are based on Gompertz's law or generalizations thereof and in neither case were they able to develop formulas that fit the pattern of mortality from childhood to early adulthood. In contrast, this paper will present a parametric formula that will reflect the fall in mortality at the childhood years, the hump at about age 20 and the exponential pattern at the adult ages.

2. A PARAMETRIC MODEL

In this section, we present our mathematical law of select and ultimate mortality and discuss some of its features. A general mathematical law of select and ultimate mortality can be defined as follows. Let $x \geq 0$ be the issue age at the nearest birthday, let $k+1 \geq 1$ be the policy year and let $y = x + k$ be the attained age. Also, let $s(y \mid \boldsymbol{\theta}_k)$ denote a parametric survival function with a parameter vector $\boldsymbol{\theta}_k$ that converges to $\boldsymbol{\theta}_\infty$ as $k \rightarrow \infty$. Then, the select mortality rates can be defined as

$$q_{[y-k]+k} = 1 - \frac{s(y+1 \mid \boldsymbol{\theta}_k)}{s(y \mid \boldsymbol{\theta}_k)}, \tag{2.1}$$

while the ultimate rates can be defined as

$$q_y = 1 - \frac{s(y+1 \mid \boldsymbol{\theta}_\infty)}{s(y \mid \boldsymbol{\theta}_\infty)}. \tag{2.2}$$

Note that the monotonicity condition in (1.1) is a necessary condition to ensure that the rates calculated according to (2.1) are actually select rates.

Now, let us specify the formula for $s(y \mid \theta)$. Note that the pattern of mortality exhibited in Figure 1 is very similar to that of the total population of the United States. One of the models developed by Carriere[1] proved successful in modeling the pattern of mortality of the United States population. Therefore, we propose to use Carriere's model as the basic formula $s(y \mid \theta)$ in (2.1) and (2.2). The model given by Carriere [1] is a mixture of a Weibull survival function, an Inverse-Weibull survival function and a Gompertz survival function. In this eight-parameter model, the probability of surviving to age $y > 0$ is

$$s(y \mid \theta) = \psi_1 s_1(y) + \psi_2 s_2(y) + \psi_3 s_3(y) \tag{2.3a}$$

where

$$s_1(y) = \exp\left\{-\left(\frac{y}{m_1}\right)^{m_1/\sigma_1}\right\}, \tag{2.3b}$$

$$s_2(y) = 1 - \exp\left\{-\left(\frac{y}{m_2}\right)^{-m_2/\sigma_2}\right\}, \tag{2.3c}$$

$$s_3(y) = \exp\left\{e^{-m_3/\sigma_3} - e^{(y-m_3)/\sigma_3}\right\}, \tag{2.3d}$$

$$\psi_3 = 1 - \psi_1 - \psi_2, \tag{2.3e}$$

$$\theta = (\psi_1, \psi_2, \psi_3, m_1, m_2, m_3, \sigma_1, \sigma_2, \sigma_3)'. \tag{2.3f}$$

The parameters in this model are $\psi_i \in [0, 1]$, $m_i > 0$ and $\sigma_i > 0$ for $i = 1, 2, 3$ and they are summarized with the vector θ. These parameters provide demographic and statistical information. For instance, ψ_1 is the probability that a new life will die from childhood causes, ψ_2 is the probability of dying from teenage causes and ψ_3 is the probability of dying from adult causes. The values m_1, m_2 and m_3 can be interpreted as location parameters, while σ_1, σ_2 and σ_3 can be interpreted as scale parameters. For instance, m_3 is the mode of the probability density function of the Gompertz survival function $s_3(t)$, while σ_3 is a scale parameter because all the mass concentrates around m_3

when σ_3 is small. See Carriere [1] for more details.

Now, let us model $\boldsymbol{\theta}_k = (\psi_{1k}, \psi_{2k}, \psi_{3k}, m_{1k}, m_{2k}, m_{3k}, \sigma_{1k}, \sigma_{2k}, \sigma_{3k})'$ for $k \geq 0$. With this notation, $\boldsymbol{\theta}_0$ denotes the parameter values at issue while $\boldsymbol{\theta}_\infty$ denotes the ultimate parameter values. Using $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_\infty$ we could define $\boldsymbol{\theta}_k$ as a weighted average

$$\boldsymbol{\theta}_k = \boldsymbol{\theta}_0 + (\boldsymbol{\theta}_\infty - \boldsymbol{\theta}_0)(1 - \exp\{-a k^b\}), \tag{2.4}$$

where $a > 0$ and $b > 0$. If we use (2.4), then the resulting model will have eighteen parameters. Obviously, other ways of modeling $\boldsymbol{\theta}_k$ are possible but we found that using (2.4) leads to a good fit of the data. An idea similar to (2.4) was used by Panjer and Giuseppe [3], where a weighted average was taken of q_y and $q_{[y]}$.

3. PARAMETER ESTIMATION

In this section, we will estimate the parameters a, b, $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_\infty$ that yield a good fit to the combined select rates given in the 1965-70 Basic Tables. All parameter estimates were calculated by the NONLIN module of the statistical computer software called SYSTAT. This system estimated the parameters by minimizing

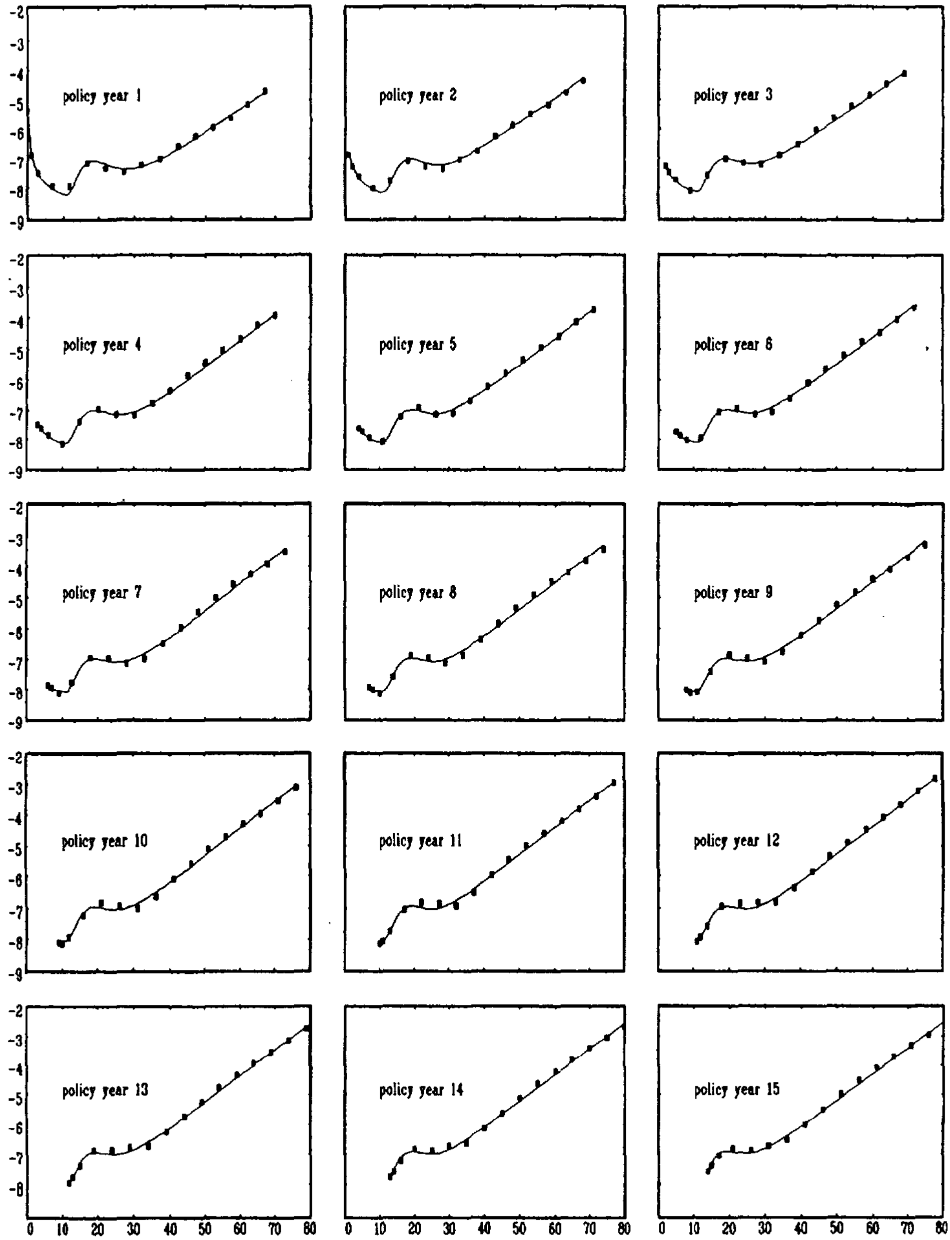
$$\sum_{x \in \mathbf{X}} \sum_{k=0}^{14} (1 - q^{\theta}_{[x]+k}/q^{b}_{[x]+k})^2, \tag{3.1}$$

where $q^{\theta}_{[x]+k}$ is based on the parametric model and $q^{b}_{[x]+k}$ is equal to a graduated select rate from the Basic Tables and $\mathbf{X} = \{0, 1, 3, 7, 12, 17, 22, \ldots, 67\}$. Note that this minimization is subject to the monotonicity constraint given in (1.1). Also note that minimizing (3.1) is actually a difficult and time-consuming exercise, that requires good starting values, good algorithms, a fast computer and some luck.

FIGURE 2

Plots of $\log_e(q_{[y-k]+k})$ from the Basic Tables and from the Formula

(The Values from the Basic Table are given as Solid Squares)



Let us comment on the loss function or fit measure given in (3.1). Note that this measure is based on the relative error $1 - q^{\theta}_{[x]+k}/q^{b}_{[x]+k}$, which means that both small and large mortality rates are given equal consideration. It can be shown that

$$1 - q^{\theta}_{[x]+k}/q^{b}_{[x]+k} \approx \log_e\left\{\log_e(1 - q^{\theta}_{[x]+k})/\log_e(1 - q^{b}_{[x]+k})\right\},$$

which means that the fit measure used by Tenenbein and Vanderhoof [4] is essentially the same as ours. In fact, Carriere [1] found that the parameter estimates based on relative error are almost equal to those based on $\log_e\left\{\log_e(1 - q^{\theta}_{[x]+k})/\log_e(1 - q^{b}_{[x]+k})\right\}$. Next, the Basic Tables give select rates for 70 and over, as a group. We excluded these rates because we were unable to determine the appropriate issue age for this group. Finally, our fit measure does not include the ultimate rates because we were unable to determine what policy year would be appropriate for these rates since they are based on the experience from policy years 16 and over.

A Good-Fitting Formula

Using SYSTAT, we were able to find parameter values a, b, $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_\infty$ that minimized (3.1) for the select rates. Initially, the full eighteen-parameter model was fit to the data with good success. In a search for a parsimonious model, we found that imposing the constraints $\psi_{10} = \psi_{1\infty}$, $\psi_{20} = \psi_{2\infty}$, $\psi_{30} = \psi_{3\infty}$, $m_{10} = m_{1\infty}$, $m_{20} = m_{2\infty}$ $\sigma_{10} = \sigma_{1\infty}$, and $\sigma_{20} = \sigma_{2\infty}$ lead to a reduced model that was almost as good as the full model. The parameter estimates for this reduced twelve-parameter model are given in Table 1. Note that $\psi_{30} = .9798$ and so the Gompertz component of our model explained most of the deaths. Therefore, it is not surprising that restricting the parameters in the Weibull and Inverse-Weibull components had little effect on the fit.

The best way, in our opinion, of verifying that the parameters in Table 1 yield a good-fitting formula is to plot the estimated rates against the table rates. Figure 2 shows fifteen graphs, one for each $k = 0, \ldots, 14$, of $\log_e(q^b_{[y-k]+k})$ at $y \in k + \mathbf{X}$ and of $\log_e(q^\theta_{[y-k]+k})$ at $y = k, \ldots, k+67$. After examining Figure 2, we believe that the rates calculated with our formulas are almost indistinguishable from the tabular rates.

TABLE 1

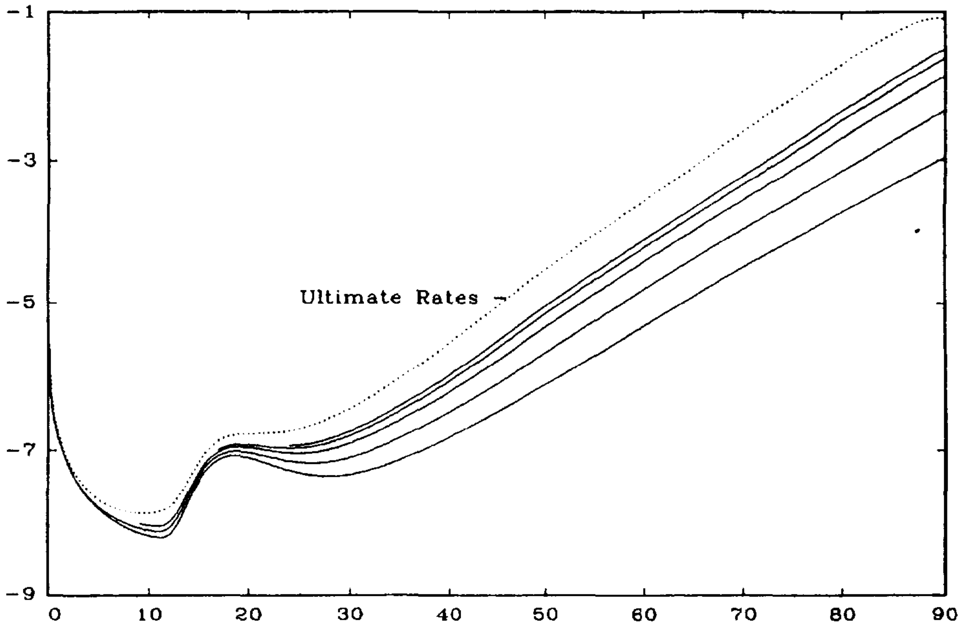
PARAMETER ESTIMATES FOR THE MODEL

k	0	∞
ψ_{1k}	.0133	.0133
ψ_{2k}	.0069	.0069
ψ_{3k}	.9798	.9798
m_{1k}	9.763	9.763
m_{2k}	19.71	19.71
m_{3k}	95.42	73.09
σ_{1k}	25.54	25.54
σ_{2k}	4.671	4.671
σ_{3k}	12.67	10.12
	$a = .2817$	$b = .4766$

Let us illustrate the behavior of our model at some advanced policy years and issue ages. Let q_y denote the ultimate rates, based on our formula, and let $q_{[y-k]+k}$ denote the select rates. Figure 3 plots $\log_e(q_y)$ for $y = 0, \ldots, 90$ and it plots $\log_e(q_{[y-k]+k})$ for the policy years $k+1 = 1, 3, 10, 18, 25$ and the attained ages $y = k, \ldots, 90$. It may be instructive to compare Figure 1 with Figure 3. Note the strict monotonicity in the rates, as you increase the policy year. Also note that the ultimate rates strictly dominate the select rates and finally note the smoothness that our mathematical formula imparts to these rates.

FIGURE 3

A Plot of the Logarithms of the Ultimate Rates and the Select Rates in Policy Years 1, 3, 10, 18, 25 using the Formula at the Issue Ages 0 to 90

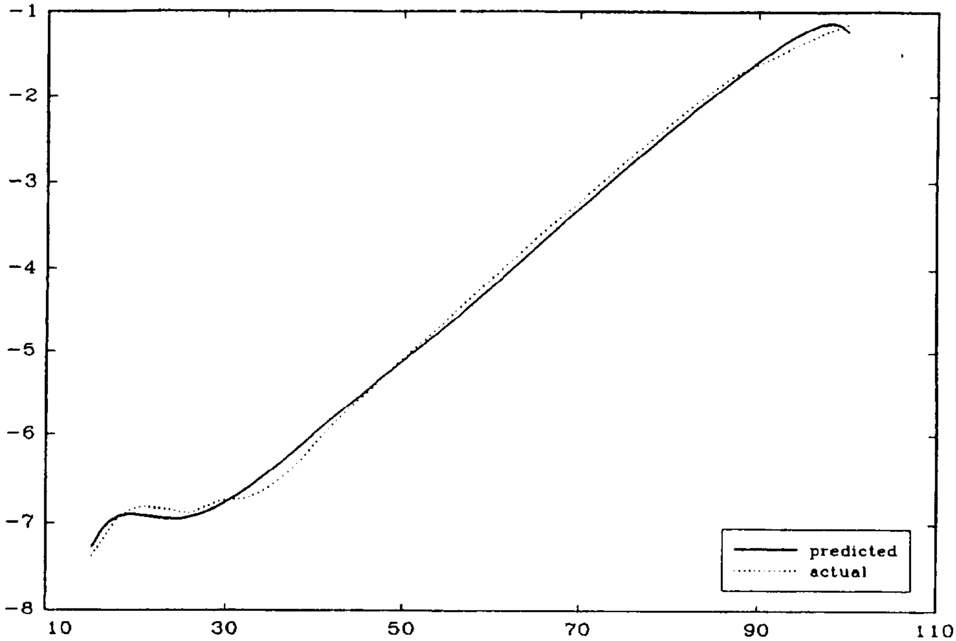


4. VALIDATION

We did not include the ultimate rates in the derivation of our formula because we were unable to determine a policy year that would be appropriate for this data. Let us validate our formula by comparing our rates to the ultimate rates. Let q_y denote the ultimate rates and let $q^{\theta}_{[y-k_u]+k_u}$ denote our select rates, based on the formula, for policy year k_u. The value k_u can be interpreted as the average policy year of the data used in the ultimate rates. To estimate k_u we used a squared error loss function, similar to (3.1), and found that letting $k_u = 19$ yielded good estimates of q_y. To verify this claim, examine Figure 4 where the actual rates are plotted against the predicted values at the attained ages $y = 15, \ldots, 100$. In our opinion, the graph shows that the predicted rates are very close to the actual rates and so we believe that our rates provide reliable estimates.

FIGURE 4

A Comparison of the Ultimate Rates from the Basic Tables With the Predicted Values Based on the Formula



6. CONCLUSION

Based on the success of our mathematical law of select and ultimate mortality, in capturing the pattern of mortality in the 1965-70 Basic Tables, we suggest that future graduations be done with mathematical formulas. There are many advantages of using this approach. First, interpolation becomes a trivial exercise. Second, we do not have to restrict the model to issue ages below 70. Third, we can easily extend the select period beyond 15 years. Fourth, the parameters in the model provide insightful demographic and statistical information about the effects of selection. Therefore, a mathematical model is the most convenient way for practitioners to calculate select rates.

REFERENCES

1. Carriere, J.F. "Parametric Models for Life Tables," TSA XLIV (1992): in press.

2. Committee on Ordinary Insurance and Annuities. "1965-70 Age Last Birthday Basic Tables," *TSA 1974 Reports of Mortality and Morbidity Experience* (1975): 57-62.

3. Panjer, H. and Giuseppe, R. "Parametric Graduation of Canadian Individual Mortality Experience: 1982-1988," Institute of Insurance and Pension Research. Waterloo, Ont., U. of Waterloo: (1990).

4. Tenenbein, A. and Vanderhoof, I.T. "New Mathematical Laws of Select and Ultimate Mortality," *TSA* XXXII (1980): 119-183.