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STOCHASTIC INTEREST RATES AND INSURANCE PORTFOLIOS THE IMPACT OF MODEL AND PARAMETER SELECTION.

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ABSTRACT

The standard deviation of the present value of benefits of portfolios of identical insurance contracts are examined when both the mortality and the force of interest are stochastic. Several models for the force of interest are presented. The impact of the different stochastic models for the force of interest and of different parameters on the mortality risk and the interest risk is investigated.

1. INTRODUCTION.

When valuing a portfolio of insurance contracts, expressions like contingency margins and provision for adverse deviations are often heard. The goal may be defined as finding a suitable set of valuation assumptions such that the actuarial liabilities, including the provisions for adverse deviations, for a portfolio of policies will be sufficient with a high enough probability to pay for all the benefits promised under the policies issued.

Different approaches can be followed to answer this question. A first approach consists of loading each valuation assumption one by one by an appropriate margin determined independently of the other assumptions and independently of the type of contracts in the portfolio. This approach has the advantage of being simple. However, it ignores the fact that each particular assumption may have (and usually does) different impacts on different contracts. Therefore, the probability of adequacy of the loaded actuarial liabilities will vary depending on the type of contracts under consideration. Further, the loaded actuarial liabilities may be inadequate for some contracts and unnecessarily high for other contracts.

A second approach consists of trying different values for each assumption and study the sensitivity of the actuarial liabilities of a particular portfolio to possible fluctuations of one assumption. Appropriate margins for each assumption can then be determined and the loaded actuarial liabilities are obtained from the set of assumptions with margins. This correspond in a way to a simple scenario testing method. This approach takes into account the different impacts that each assumption may have on different contracts. It has the disadvantage of ignoring the interaction effect on the actuarial liabilities of the assumptions, even when the assumptions are assumed to be independent. Of course, this problem could be dealt with by studying many more scenarios where all combinations of two or more assumptions are varied simultaneously. But this is not always possible as testing only a few scenarios is very time consuming.

A third approach would be to study the actuarial liabilities as a random variable and use simulation techniques. This approach considers all possible interaction effects of the assumptions and considers the type of contracts been sold. However, three disadvantages can be identified, firstly, this can be extremely time consuming. Secondly, one faces the problem of choosing an appropriate number of simulations. And thirdly, simulation results cannot be perfectly replicated, they are random variables themselves.

A fourth approach is to study the actuarial liabilities as a random variable using stochastic calculus with appropriate models for each assumption. This approach should be preferred when analytical results can be obtained with models that can be considered realistic enough for studying the problem at hand.

In this (draft) paper, we study the present value of benefits of portfolios of identical n-year temporary insurance policies when both mortality and interest are stochastic. The standard deviation of these present values are presented for five different stochastic processes for the discounting function. The impact of parameters selection is considered for one particular process, namely, the Ornstein-Uhlenbeck process for the force of interest. Finally, the interaction between the mortality risk and the interest risk is investigated.

2. PRESENT VALUE OF BENEFITS AND AVERAGE COST PER POLICY.

Consider a portfolio of c identical n-year temporary insurance policies. The present value of the benefits can be expressed as:

$$Z(c) = \sum_{i=1}^{c} Z_i, \qquad (1)$$

where Z_i, the present value of the benefit for policy i, is given by:

$$Z_i = \begin{cases} e^{-y(K_i+1)} & K_i=0,1,\ldots,n-1 \\ 0 & K_i=n,n+1,\ldots \end{cases}, \qquad (2)$$

c is the number of policies,

$$y(t) = \int_0^t \delta_s \cdot ds, \qquad (3)$$

δ_s is the force of interest at time s and K_i is the curtate-future-lifetime of policyholder i.

The Average Cost Per Policy, ACPP, is then given by: ACPP = Z(c)/c.

We shall be interested in the standard deviation of ACPP under the assumptions that: $\{K_i\}_{i=1}^{c}$ are i.i.d. and $\{K_i\}_{i=1}^{c}$ are independent of $\{\delta_s\}_{s>0}$.

More details can be found in Parker (1993b). The reader is also referred to Waters (1978), Frees (1990) and Norberg (1993).

3. STOCHASTIC DISCOUNTING PROCESSES.

Five models for the discounting function, $e^{-y(t)}$, are considered. They are the Wiener and the Ornstein-Uhlenbeck processes for the force of interest accumulation function, y(t), and the White Noise, the Wiener and the Ornstein-Uhlenbeck processes for the force of interest, δ_t. A comparison of these five models can be found in Parker (1993c).

Table 1 presents the definitions of these five models in terms of W_t, the standard Brownian process.

TABLE 1. FIVE MODELS FOR THE DISCOUNTING FUNCTION.

Modeling the force of interest accumulation function	
1 - Wiener:	$y(t) = \delta \cdot t + \sigma \cdot W_t$
2 - Ornstein-Uhlenbeck:	$y(t) = \delta \cdot t + X(t)$ where $dX(t) = -\alpha \cdot X(t)\,dt + \sigma dW_t$
Modeling the force of interest	
3 - White Noise:	$\left\{\delta_t\right\}_{t>0}$ i.i.d $N(\delta, \sigma^2)$
4 - Wiener:	$\delta_t = \delta + \sigma \cdot W_t$
5 - Ornstein-Uhlenbeck:	$d\delta_t = -\alpha(\delta_t - \delta) \cdot dt + \sigma \cdot dW_t$

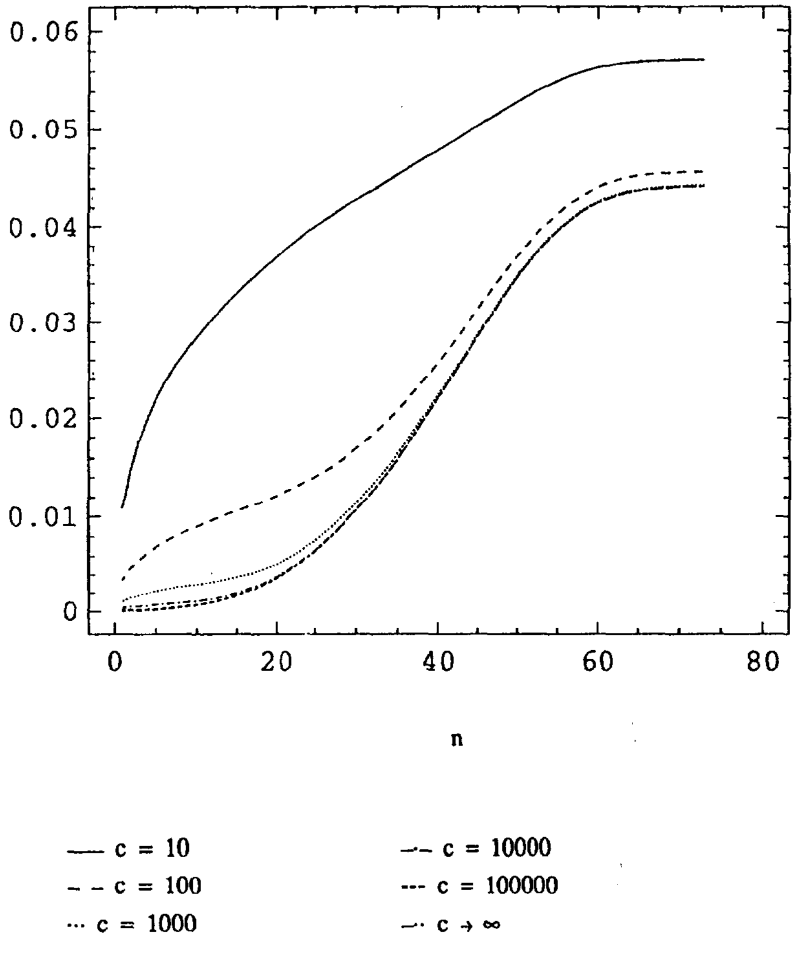
4. STANDARD DEVIATION OF THE ACPP.

From the definitions given in section 2, it is clear that the standard deviation of Z(c) (and ACPP) will depend on 1) the mortality risk due to the K_i's and 2) the interest risk due to the δ_t's (or y(t)'s). A mathematical formulation of this is presented in section 7. For now, we use an intuitive *approach to the problem of splitting the total standard deviation into a* mortality risk and an interest risk.

The mortality risk is a diversifiable one and the central limit theorem applies since the K_i's are assumed independent. This implies that the standard deviation of a limiting portfolio (as the number of policies tends to infinity), is entirely due to the interest risk. If one uses this limiting standard deviation as a measure of the interest risk, then the mortality risk becomes the difference between the total standard deviation of the actual finite size portfolio and the limiting standard deviation. Figure 1 illustrates the standard deviations of portfolios of n-year temporary insurance contracts of different sizes.

Note that the interest risk varies with the term of the contract, n, and that the mortality risk varies with n and with the size of the portfolio.

Figure 1. Standard Deviation of ACPP
n-year temporary insurance
Varying the size, c, of the portfolio



5. IMPACT OF THE MODEL.

In this section, we look at the effect of the model used for the interest rates on the relative importance of the mortality risk and the interest risk.

Figure 2 illustrates the difference between the Ornstein-Uhlenbeck modeling the force of interest and the same process modeling the y(t) function.

Although the two processes should use different parameter values (they could be estimated from past data for example), they use the same values in figure 2 (a similar comment could be made about figure 3). This is acceptable for the point we are trying to make here.

Figure 3 illustrates the differences between the Ornstein-Uhlenbeck modeling the force of interest, the Wiener process modeling the y(t) function and the Wiener process modeling the force of interest.

One should note that the model selection may have a large impact on the relative importance of the mortality and interest risks for certain portfolios.

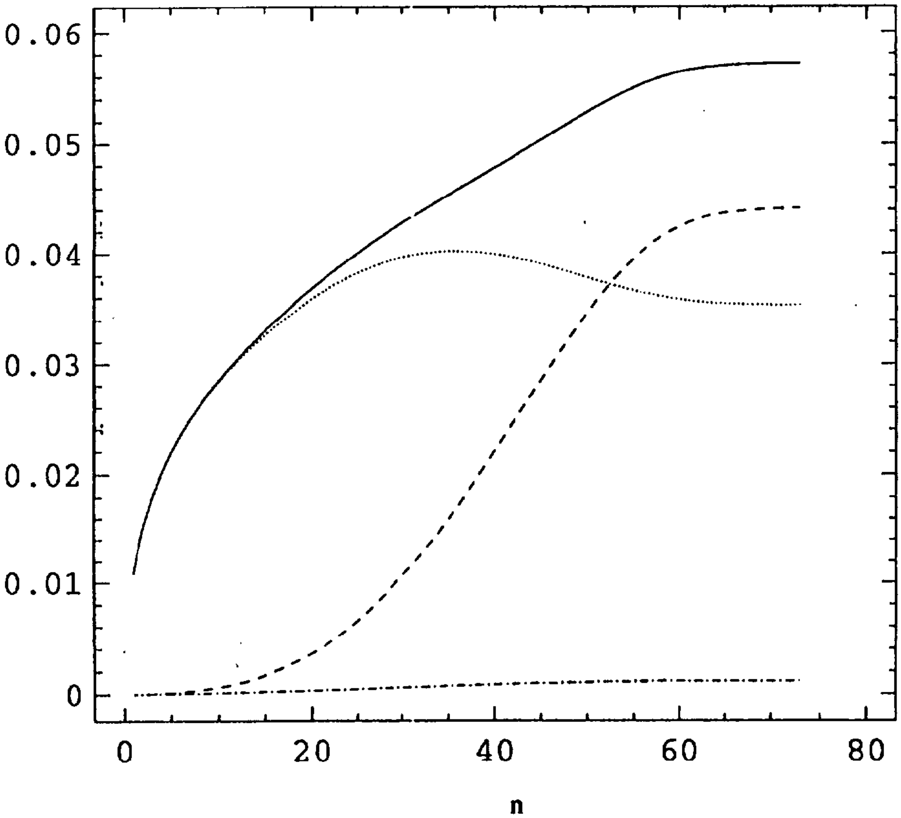
6. IMPACT OF THE PARAMETERS.

In this section, we illustrate the effect of varying certain parameters in the Ornstein-Uhlenbeck model for the force of interest on the relative importance of the mortality and interest risks.

In figures 4, 5 and 6, we vary the level of the interest rates, δ, the friction force, α, and the diffusion coefficient σ respectively. These figures indicate that the parameters of the interest rates model have an impact on the mortality risk as well as on the interest risk.

Interestingly, and probably contrary to what one would expect, the level of interest has a substantial impact on the interest risk. This can be seen from the three curves for c tending to infinity of figure 4.

Figure 2. Standard Deviation of ACPP
n-year temporary insurance
Ornstein-Uhlenbeck for y(t) and Ornstein-Uhlenbeck for δ_t



··· c = 10 y(t): O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$

-·- c → ∞ y(t): O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$

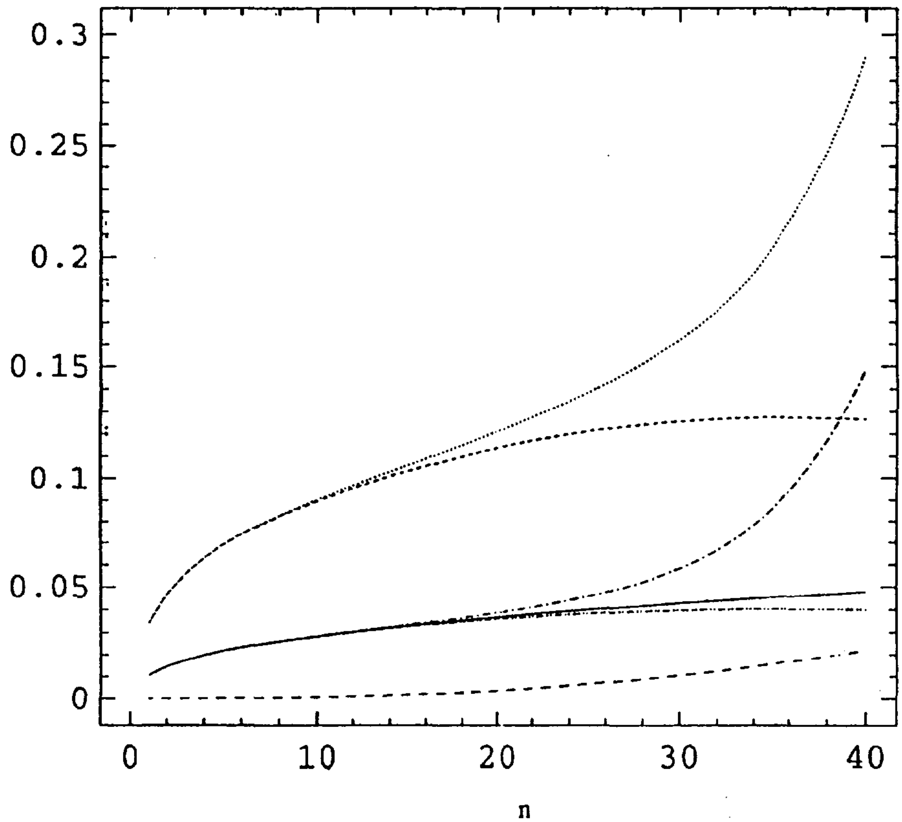
—— c = 10 δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$

- - c → ∞ δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$

Figure 3. Standard Deviation of ACPP

n-year temporary insurance

Ornstein-Uhlenbeck for δ_t, Wiener for y(t) and Wiener for δ_t



--- c = 10 y(t): Wiener, δ=.06, σ=.01

··· c → ∞ y(t): Wiener, δ=.06, σ=.01

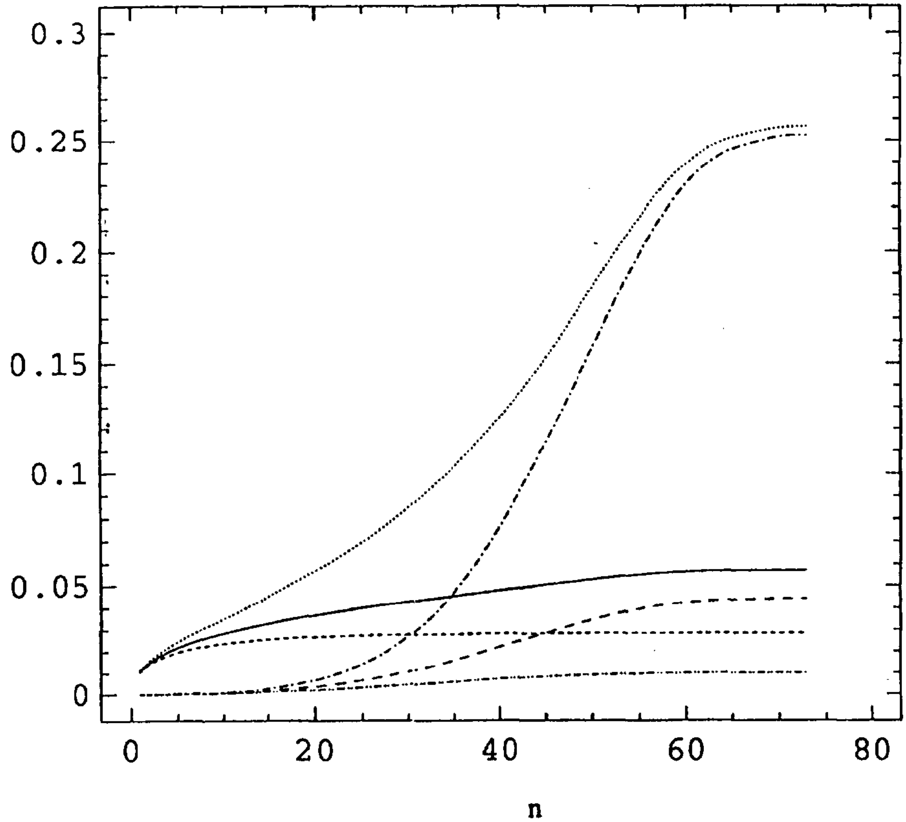
··· c = 10 δ_t: Wiener, δ=.06, σ=.01

–·– c → ∞ δ_t: Wiener, δ=.06, σ=.01

—— c = 10 δ_t: O-U, $\delta=\delta_0$=.06, α=.1, σ=.01

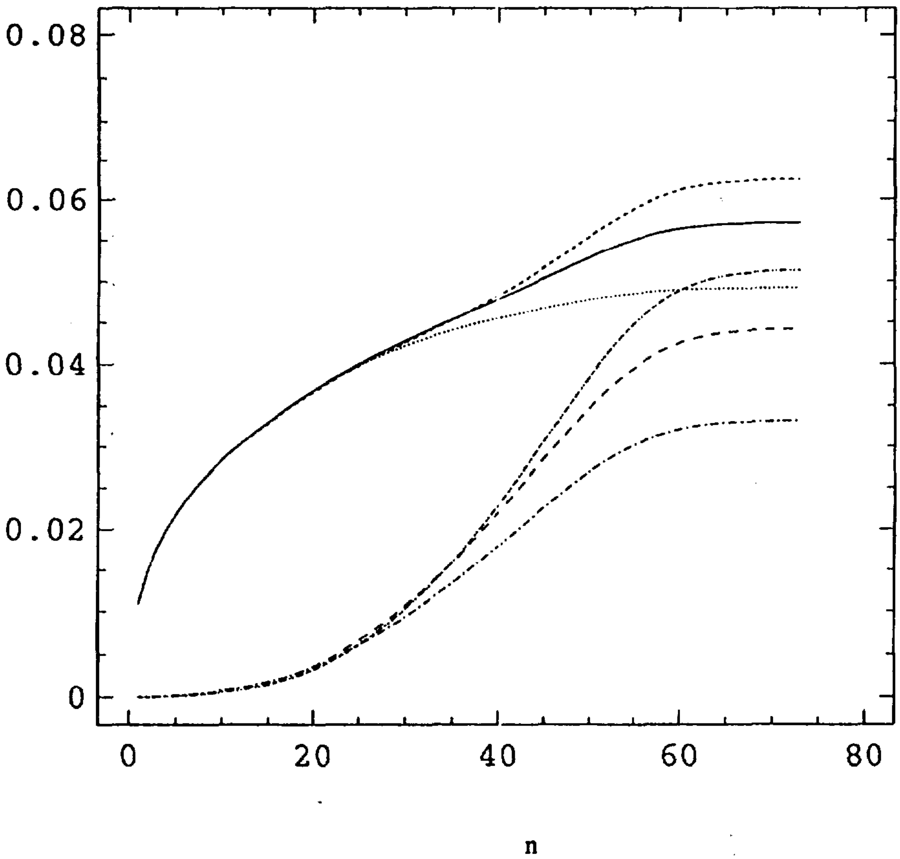
– – c → ∞ δ_t: O-U, $\delta=\delta_0$=.06, α=.1, σ=.01

Figure 4. Standard Deviation of ACPP
n-year temporary insurance
Varying the level of δ_t



··· c = 10	δ_t:	O-U, $\delta = \delta_0 = .02$, $\alpha = .1$, $\sigma = .01$
-·- $c \to \infty$	δ_t:	O-U, $\delta = \delta_0 = .02$, $\alpha = .1$, $\sigma = .01$
—— c = 10	δ_t:	O-U, $\delta = \delta_0 = .06$, $\alpha = .1$, $\sigma = .01$
– – $c \to \infty$	δ_t:	O-U, $\delta = \delta_0 = .06$, $\alpha = .1$, $\sigma = .01$
--- c = 10	δ_t:	O-U, $\delta = \delta_0 = .1$, $\alpha = .1$, $\sigma = .01$
-·· $c \to \infty$	δ_t:	O-U, $\delta = \delta_0 = .1$, $\alpha = .1$, $\sigma = .01$

Figure 5. Standard Deviation of ACPP
n-year temporary insurance
Varying α while keeping $\sigma^2/2\alpha$ constant

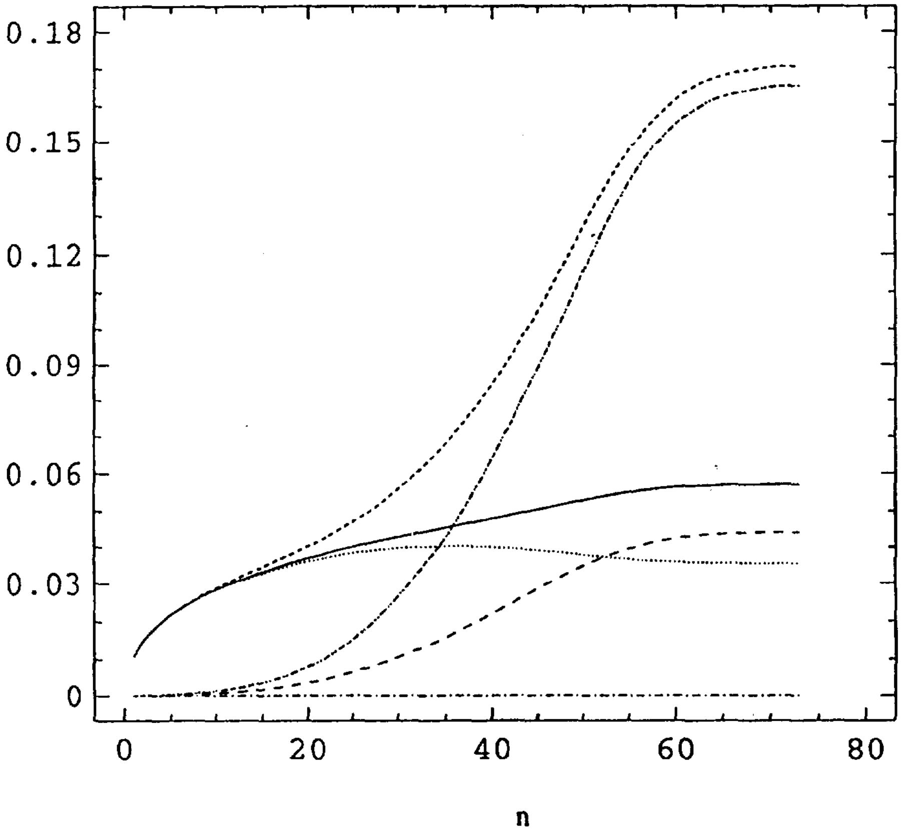


—— c = 10	δ_t:	O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$
– – c → ∞	δ_t:	O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$
--- c = 10	δ_t:	O-U, $\delta=\delta_0=.06$, $\alpha=.05$, $\sigma=.01\times(.5)^{.5}$
-·· c → ∞	δ_t:	O-U, $\delta=\delta_0=.06$, $\alpha=.05$, $\sigma=.01\times(.5)^{.5}$
··· c = 10	δ_t:	O-U, $\delta=\delta_0=.06$, $\alpha=.2$, $\sigma=.01\times(2)^{.5}$
—-— c → ∞	δ_t:	O-U, $\delta=\delta_0=.06$, $\alpha=.2$, $\sigma=.01\times(2)^{.5}$

Figure 6. Standard Deviation of ACPP

n-year temporary insurance

Varying σ



··· c = 10 δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=0$

-·- c → ∞ δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=0$

—— c = 10 δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$

– – c → ∞ δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.01$

--- c = 10 δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.02$

-·· c → ∞ δ_t: O-U, $\delta=\delta_0=.06$, $\alpha=.1$, $\sigma=.02$

7. INTERACTION BETWEEN MORTALITY RISK AND INTEREST RISK.

It appears from the figures of the preceding sections that the mortality and interest risks are not totally independent from one another even if the curtate-future-lifetimes are assumed to be independent of the future interest rates. This interaction is now investigated mathematically.

It can be shown that Z(c) defined in (1) may be written as:

$$Z(c) = \sum_{i=1}^{n} c_i \cdot e^{-y(i)} \tag{4}$$

where c_i is the number of policies payable at time i (see Parker (1992a, section 6.1)).

The variance of the average cost per policy is given by:

$$V[ACPP] = E\left[V\left[\frac{Z(c)}{c} \mid \{c_i\}\right]\right] + V\left[E\left[\frac{Z(c)}{c} \mid \{c_i\}\right]\right] \tag{5}$$

One may define this to be:

$$\text{Total variance} = \text{interest variance} + \text{mortality variance} \tag{6}$$

But this variance, (5), may also be expressed as:

$$V[ACPP] = \sum_i \sum_j \operatorname{cov}\left(e^{-y(i)}, e^{-y(j)}\right) \cdot \left\{ \operatorname{cov}\left(\frac{c_i}{c}, \frac{c_j}{c}\right) + E\left(\frac{c_i}{c}\right) E\left(\frac{c_j}{c}\right) \right\}$$

$$+ \sum_i \sum_j E\left(e^{-y(i)}\right) \cdot E\left(e^{-y(j)}\right) \cdot \operatorname{cov}\left(\frac{c_i}{c}, \frac{c_j}{c}\right) \tag{7}$$

Equation (7) clearly shows that the interest variance term, associated with $\operatorname{cov}\left(e^{-y(i)}, e^{-y(j)}\right)$, also depend on mortality factors due to the presence of c_i and c_j. Similarly, the mortality variance term, associated with $\operatorname{cov}\left(\frac{c_i}{c}, \frac{c_j}{c}\right)$, depends on the model used for the interest rates due to the presence of expected discounting factors in the second term.

8. CONCLUSIONS.

Choosing an interest rate model and its parameters may have a significant impact on the relative importance of the mortality risk and the interest risk of a portfolio of insurance policies.

Somewhat surprisingly, the level of the interest rates does have an impact on the standard deviation of the average cost per policy of a portfolio.

By expressing the total variance of the average cost per policy in the form of equation (7), the *interaction* between mortality and interest risks becomes apparent.

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