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A necessary and sufficient condition on utility functions for decreasing risk aversion: A proof using the general mean value theorem.

Raj Prabaharan

University of Wisconsin-Stevens Point

Abstract: It is known that several utility functions, including quadratic utility functions, have the property that they increase the risk premium with increasing wealth. In this paper, it is assumed that an investor, with utility function u, faces a fixed amount of loss a with probability p, $0 < p < 1$. A necessary and sufficient condition on u, for decreasing risk aversion is derived using the general mean value theorem. Thus, the Pratt-Arrow index for absolute risk aversion is derived (by a simple direct approach), independent of the original work by J. W. Pratt and K. J. Arrow.

1 Introduction

Consider an investor with a wealth of worth x in dollar amount. With the amount of wealth x, one associates an increasing concave function u, called the utility function. The increasing property of u means 'the more the better'. The concavity assumption on u means decreasing marginal utility, i.e., for any $a > 0$, $u(x) - u(x - a)$ decreases with x. For an investor with an increasing, concave utility function, a fixed amount a of loss is valued less with increasing wealth.

For convenience, one considers utility functions with smooth curves by assuming u is twice differentiable. In such a case the above-given conditions on the utility function u can be written as (i) $u' > 0$ and (ii) $u'' < 0$.

Let w be the current wealth of the investor. Assume that the investor faces a random loss of X. Let g be the amount the investor would consider the appropriate (insurance) premium for complete protection against such loss. By appropriate premium, we mean the premium g the investor would be indifferent between paying the premium to get complete protection against the loss and facing the loss himself.

According to the principle of (zero) utility, such premium g is given by

$$u(w - g) = E[u(w - X)]. \tag{1}$$

Since u is concave, from Jensen's inequality, it follows

$$u(w - g) = E[u(w - X)] \leq u(E[w - X])$$

and since u is increasing, one obtains

$$g \geq E[X]. \tag{2}$$

Thus an investor with a concave utility function would be willing to pay, as insurance premium, an amount more than the expected value of loss; such an investor is said to be risk averse. The difference, $g - E[X]$, is called the risk premium.

Now consider a special case of the above problem where the investor with wealth w faces a loss of amount a with probability p and remains with the same wealth w with probability $q = 1 - p$.

From equation 1, the insurance premium g is given by

$$u(w-g) = p.u(w-a) + q.u(w) \tag{3}$$

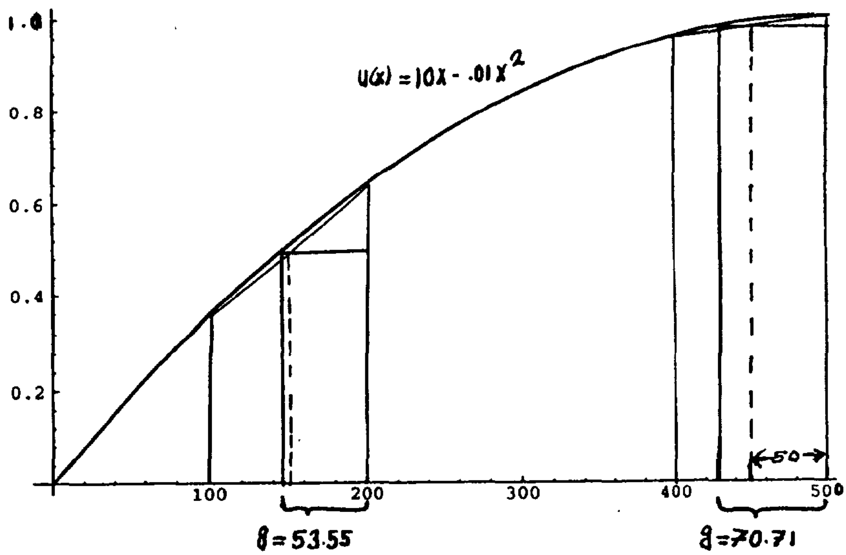
Now consider the insurance premium g as w varies. As one's ability to absorb risks increases with the increasing wealth, one would expect g to be a (strictly) decreasing function of w. But this does not always happen with the utility functions satisfying the conditions (i) and (ii). For example, as illustrated in the following example, in the case of quadratic utility functions, g increases with w. An illustration similar to the one given below is discussed in [2].

Example 1.1 *Let the investor's utility function is given by* $u(x) = 10x - .01x^2, 0 < x < 500$.

Assume that the investor faces a loss of amount $a = 100$ *with probability* $p = \frac{1}{2}$. *The following table gives the values for insurance premium* g, *calculated using equation (3), for different values of wealth* w.

Wealth w	*Premium* g
100	52.77
200	53.55
300	54.95
400	58.11
500	70.71

In the above example, for all the values of w the expected loss is 50. The insurance premium increases with increasing value of w. Insurance premiums when w=200 and when w=500 are illustrated in the following diagram.



In the next section, we give a necessary and sufficient condition on the utility function u, that would guarantee that the premium decreases with increasing wealth.

For the rest of the paper; we do not assume the condition (ii); thus the following results hold irrespective of whether the investor is risk averse or not.

2 Decreasing Risk Aversion

Theorem 2.1 *Let the investor's utility function u be twice differentiable and satisfy condition (i).*

Assume that the investor faces a fixed amount of loss, $a > 0$, with probability $p, 0 < p < 1$.

If $\frac{u''}{u'}$ is an increasing (strictly increasing) then for any $a > 0$, the insurance premium for complete protection against the loss will decrease (strictly decrease) with increasing wealth.

If we further assume u'' is continuous, the converse of the above result also holds.

proof: Assume that $\frac{u''}{u'}$ is an increasing function. Let w denotes the current wealth of the investor. Then the insurance premium for complete protection is given by,

$$u(w - g) = p \cdot u(w - a) + q \cdot u(w). \quad (4)$$

Keeping a fixed and taking derivative with respect to w, we get

$$u'(w - g)(1 - \frac{dg}{dw}) = p \cdot u'(w - a) + q \cdot u'(w). \quad (5)$$

Since $u' > 0$, from the above equation it follows that $\frac{dg}{dw} < 1$.

Suppose $\frac{dg}{dw} > 0$ for some $w = w_1$ and $a = a_1$. For the rest of the proof of our assertion we consider a and w fixed ($a = a_1, w = w_1$). From the above equation it follows that

$$u'(w - g) > p \cdot u'(w - a) + q \cdot u'(w), \quad (6)$$

which can be rearranged as

$$p[u'(w-g) - u'(w-a)] > q[u'(w) - u'(w-g)]. \tag{7}$$

From equation (4) one has

$$p[u(w-g) - u(w-a)] = q[u(w) - u(w-g)].$$

The expressions in both side of the above equation are positive and therefore division of the inequality (6) by these expressions yields

$$\frac{u'(w-g) - u'(w-a)}{u(w-g) - u(w-a)} > \frac{u'(w) - u'(w-g)}{u(w) - u(w-g)} \tag{8}$$

Application of generalized mean value theorem yields

$$\frac{u''(\eta)}{u'(\eta)} > \frac{u''(\zeta)}{u'(\zeta)} \tag{9}$$

for some $\eta \in (w-a, w-g)$ and $\zeta \in (w-g, w)$. Thus we have $\eta < \zeta$ such that $\frac{u''(\eta)}{u'(\eta)} > \frac{u''(\zeta)}{u'(\zeta)}$, which contradicts our assumption. Therefore, it follows that $\frac{dg}{dw} \leq 0$ for all choices of w and a.

Now consider the converse. Assume $\frac{dg}{dw} \leq 0$ for all choices of w and a. Suppose $\frac{u''}{u'}$ is not an increasing function. From our assumption, it follows that $\frac{u''}{u'}$ is continuous. Therefore there exists an interval $[\alpha, \beta], \alpha < \beta$ such that $\frac{u''}{u'}$ is strictly decreasing.

Let $w = \beta$ and $a = \beta - \alpha$. Now repeating the argument given in the above proof, but with the assumption $\frac{dg}{dw} \leq 0$ instead of $\frac{dg}{dw} > 0$ for $w = \beta$ and $a = \beta - \alpha$, one obtains

$$\frac{u''(\eta)}{u'(\eta)} \leq \frac{u''(\zeta)}{u'(\zeta)}$$

for some $\eta < \zeta \in (\alpha, \beta)$. This is a contradiction, since by our construction, $\frac{u''}{u'}$ is strictly decreasing in the interval $[\alpha, \beta]$.

This completes the proof of the theorem.

Corollary 2.2 *Let the investor's utility function satisfy the condition (i) and twice continuously differentiable.*

Assume that the investor faces a fixed amount of loss, $a > 0$, with probability $p, 0 < p < 1$.

For any $a > 0$, the insurance premium for complete protection against the loss will decrease (strictly decrease) with increasing wealth if and only if $\log u'$ *is convex (strictly convex).*

Proof: The result follows from the theorem, by using $(\log u')' = \frac{u''}{u'}$.

If the utility function is differentiable three times one obtains the following corollary.

Corollary 2.3 *Let the investor's utility function satisfy the condition (i) and differentiable three times.*

Assume that the investor faces a fixed amount of loss, $a > 0$, with probability $p, 0 < p < 1$.

For any $a > 0$, the insurance premium for complete protection against the loss will decrease (strictly decrease) with increasing wealth if and only if

$$u'u''' - (u')^2 \geq 0 \ (u'u''' - (u')^2 > 0)$$

Proof: The result follows from the theorem, by considering the derivative of $\frac{u''}{u'}$.

Now consider any quadratic utility function of the form

$$u(x) = ax^2 + bx + c,\ a \neq 0.$$

Since $u'(x) = 2ax + b$ and $u''(x) = 2a$, when $a > 0$

$$u(x) = ax^2 + bx + c,\ x > -b/2a$$

represents the general form of a quadratic utility function of a risk averse investor.

Irrespective of whether $a > 0$ or $a < 0$, $u''(x)/u'(x) = 2a/(2ax + b)$ is a decreasing function x. Therefore by the above theorem, it follows that it is not possible to have a quadratic utility function with decreasing risk aversion.

Also from Corollary 2.2, it follows that any utility function with constant risk aversion is of the form

$$u(x) = -\beta e^{-\alpha x},\ \alpha\beta > 0.$$

References

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