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Examining Changes in Reserves Using Stochastic Interest Models

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Abstract

A fundamental problem in actuarial science is the determination of the reserves necessary to meet future obligations. Reserves are useful quantities because they summarize a vector of discounted cash flows. However, through this summarization, they mask the dynamic nature of interest rates. To study the effects on reserves of the dynamic nature of a stochastic interest environment, we propose looking at a change in discounted reserves. By looking at the appropriate measure of change, we can study potential short-term consequences of changes in the interest environment.

Both the traditional linear ARIMA and newer nonlinear ARCH processes are used to model the force of interest stochastically. We find that, in general, the next period reserve is a function of previous interest rate. However, this is not true when the force of interest can be modelled as a white noise process. Explicit formulas are presented for computing changes in discounted reserves for linear interest rate processes. For nonlinear processes, we describe some approximations and exact simulation algorithms for these computations.

Keywords: Pure discount bond; ARIMA process; ARCH process.

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1 Introduction

In financial statements, assets must equal the capital equity plus liabilities of the firm. An important component of these liabilities for insurance organizations is the reserve, the portion of a firms' assets set aside to meet future uncertain obligations arising from insurance contracts. Although the obligations of each contract are contingent upon uncertain future events, and thus may be modeled stochastically, the reserve set aside is a single number. There are limitations when using a single number to summarize a stochastic quantity. However, reserves play a prominent role in financial statements and thus these quantities are important to managers of insurance organizations.

There are several important problems in actuarial science that rely heavily on the determination of a reserve. To illustrate, if a company or a block of business is to be traded on the open market, a value must be determined for the associated set of obligations. Thus, it is useful to think of a reserve as the "value" associated with a set of stochastic obligations. As another application, reserves have been traditionally used as a measure of financial strength of an organization. In this context, the reserve should be larger than the "value" of obligations, because a conservative approach should be taken for assessing potential future obligations.

Life insurance and annuity reserves are calculated by summarizing discounted cash flows, where the discounting is done with respect to investment earnings, as well as decrements due to mortality, disability, policy lapse, and so on, that may be applicable to a particular policy. For brevity, in this paper we work only with investment earnings and the mortality decrement. Extensions to the multi-decrement case are straightforward.

In the traditional insurance literature, as in Jordan (1967), the deterministic assumption dominates the development of the theory of life contingencies. Namely, mortality happens according to a known mortality table and the interest rate is assumed to have a deterministic value. One step further is to allow the age at death to be a random variable, although the interest rate is assumed to be deterministic. This "semi-stochastic" approach is followed in Bowers et al. (1986). The recent past insurance literature has generalized the traditional theory of life contingencies by introducing stochastic variation in interest rates. This literature includes Boyle (1976), Waters (1978), Panjer and Bellhouse (1980), Bellhouse and Panjer (1981), Giaccotto (1986), Dhaene (1989), Frees (1990), and Beekman and Fuelling (1990, 1992).

In this paper, we compute reserves as (conditional) expectations of sums of future cash flows. Motivation for this approach can be found in, for example, Bowers et al. (1986) for the semi-stochastic approach and Bühlmann (1992) for models using stochastic interest.

Here, we are primarily concerned with quantifying changes in reserves from one financial period to the next. Changes in reserves could be used to quantify the amount of profit released, as in Ramlau-Hansen (1988). In that study, gains and losses emerging from margins built into mortality and other decrements where studied while those arising from investments were ignored. To complement that work, here we focus on changes arising from stochastic interest rates and do not explicitly consider margins built into other decrement rates.

Changes in value of future obligations due to dynamic models of interest have been extensively considered in the financial economics literature, in particular as part of *immunization theory*. Unlike this paper, immunization theory deals with *instantaneous* changes in value. Here, we examine changes in value from one financial period to the next.

The new idea of examining changes in reserves can be illustrated by considering the following simple scenario. Let $\{y_s\}$ represent the random force of interest in the sth period. As argued in Frees (1990), y_s can be interpreted as a one-period spot rate. Consider the case of a T-year pure discount bond. At time 0, the random present value of one unit payable at time T is

$$\nu_T = \prod_{s=1}^{T} e^{-y_s} = exp(-\sum_{s=1}^{T} y_s).$$

Without loss of generality, it is assumed that the time interval is year. Suppose an insurance company has to pay one unit T years later with certainty, but under a stochastic interest rate environment. The reserve at time 0 is denoted by

$$V_0^{(T)} = E(\nu_T) = E[exp(-\sum_{s=1}^{T} y_s)],$$

where the expectation is taken at time zero. After one year, the maturity time of the payment shortens by one and the reserve becomes

$$V_1^{(T-1)} = E[exp(-\sum_{s=2}^{T} y_s)],$$

where the expectation is taken at time one.

This paper examines the change from the initial reserve V_0 to the time one reserve V_1. To examine this change, we discount the time one reserve $V_1^{(T-1)}$ back to time zero and then study its distribution. That is, we investigate the distribution and the statistical properties of the random reserve

$$e^{-y_1} V_1^{(T-1)},$$

from time zero viewpoint. Although V_0 represents the current value of the asset or obligation, V_1 represents the value at the subsequent time period. Thus, for budgeting and other purposes, V_1 and its discounted value $e^{-y_1} V_1$ are important quantities for risk and other financial managers.

An outline of this paper is as follows. Section 2 describes the model that we will be using in the analysis. The linear process for interest rates is investigated in Section 3. Section 4 considers a nonlinear process for interest rates, the Autoregressive Conditionally Heteroskedastic process, that is widely used in economics. Section 5 concludes with some remarks. The proofs of all results are given in the Appendix.

2 The Basic Model

2.1 Insurance Model

We consider here the individual risk model for insurance contracts, using the notation of Bowers et al. (1986). Denote the valuation time to be h so that, at the initial valuation, $h = 0$. Assume that there are n policies in the block of business. For i-th policy, the age at issue is x_i, the duration is k_i when $h = 0$, the curtate random time of decrement is K_i, and the curtate-future-lifetime is J_i (i.e. $J_i = K_i - k_i - h$). Suppose that a death benefit b_{i,K_i+1} is payable at the end of the year of loss and that the annual premiums $a_{i,m}$ are payable at the beginning of each year up to and including the year of loss. Then, at time point $h + \tau + 1$, the random cash flow of i-th policy is

$$F_{i,\tau+1}^{(h)}(J_i) = \begin{cases} -a_{i,k_i+h+\tau+1} & \text{if } J_i > \tau \\ b_{i,k_i+h+\tau+1} & \text{if } J_i = \tau \\ 0 & \text{if } J_i < \tau, \end{cases} \quad (1)$$

where $F_{i,0}^{(h)} = -a_{i,k_i+h}$, and the probability function of J_i is: $\text{Prob}(J_i = \tau) = {}_{\tau|}q_{x_i+k_i+h}, \tau = 0, 1, \ldots$; and $\text{Prob}(J_i > \tau) = {}_{\tau+1}p_{x_i+k_i+h}$. Here, ${}_{\tau|}q_x$ and ${}_{\tau}p_x$ are the traditional deferred decrement probabilities and survival functions calculated from a lifetable.

Because the definition of cash flow is quite general, it can be used for general insurance as well as combinations of whole life insurance, term life insurance, deferred life insurance, annuities, and pure discount bonds.

2.2 Interest Rate Model

Let y_s denote the force of interest in the s-th year ($s = 1, 2, \ldots$). It is natural to assume that y_s has a parametric form,

$$y_s = f(\underline{\theta}; \epsilon_1, \ldots, \epsilon_s),$$

where $\underline{\theta}$ is a vector of parameters, $\{\epsilon_s\}$ are *i.i.d.*, and f is a known function. To illustrate, in Section 3 we will consider the recursive linear process

$$y_t = a + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t - c_1\epsilon_{t-1} - \cdots - c_q\epsilon_{t-q}.$$

The traditional ARMA/ARIMA model belongs to this class of model. In Section 4, we will consider an ARCH model, a nonlinear representation for the force of interest.

Before going to the next section, we make the following basic assumption and notations here.

Assumption

(A1): y_t is a Borel measurable function of $(\epsilon_1, \epsilon_2, \ldots, \epsilon_t)$.

Notations

(1) Let $M(t) = E(e^{t\epsilon})$, the moment generating function of ϵ.

(2) $\sum_{s=i}^{j} = 0$ for $j < i$ and, $\prod_{s=i}^{j} = 1$ for $j < i$.

(3) $E(\cdot|X)$ means taking expectation conditional on the information generated by X.

2.3 The Reserves

To introduce reserves, we first define the loss at valuation time h for i-th policy,

$$L_i^{(h)} = \sum_{\tau=0}^{\infty} F_{i,\tau}^{(h)} exp(-\sum_{s=h+1}^{h+\tau} y_s).$$

So the sum of losses for the whole block of business at time h is

$$S_L^{(h)} = \sum_{i=1}^{n}\sum_{\tau=0}^{\infty} F_{i,\tau}^{(h)} exp(-\sum_{s=h+1}^{h+\tau} y_s).$$

The reserve at time h is defined as $V_h = E_h(S_L^{(h)})$ where $E_h(\cdot) = E(\cdot|\epsilon_1, \ldots, \epsilon_h)$, so that V_h is a function of $(\epsilon_1, \ldots, \epsilon_h)$. The discussion below will concentrate on V_0 and V_1, the reserves at the beginning of valuation and its following period respectively. The notions can be extended to the considerations of further periods.

In the following discussion we assume the random cash flows $(F_{i,\tau}^{(h)})$ are independent of the stochastic interest rates (y_s). The following result for the reserves is a basic one that we will use frequently.

Proposition 1 *Under (A1), we have*

$$V_0 = E_0(S_L^{(0)}) = \sum_{\tau=0}^{\infty}(\sum_{i=1}^{n} c_{\tau,k_i}^{(i)}) \cdot E[exp(-\sum_{s=1}^{\tau+1} y_s)] - \sum_{i=1}^{n} a_{i,k_i}, \tag{2}$$

$$V_1 = E_1(S_L^{(1)}) = \sum_{\tau=0}^{\infty}(\sum_{i=1}^{n} c_{\tau,k_i+1}^{(i)}) \cdot E[exp(-\sum_{s=2}^{\tau+2} y_s)|\epsilon_1] - \sum_{i=1}^{n} a_{i,k_i+1}, \tag{3}$$

where

$$c_{\tau,k_i}^{(i)} = b_{i,k_i+\tau+1} \cdot {}_{\tau|}q_{x_i+k_i} - a_{i,k_i+\tau+1} \cdot {}_{\tau+1}p_{x_i+k_i}, \tag{4}$$

provided the expectations exist. □

In fact, $E(F_{i,\tau+1}^{(0)}) = c_{\tau,k_i}^{(i)}$, the expected cash flow of i-th policy at time $h = 0$, and $E(F_{i,\tau+1}^{(1)}) = c_{\tau,k_i+1}^{(i)}$, the expected cash flow of i-th policy at time $h = 1$. Although V_0 is a constant, V_1 is usually stochastic since it is a function of ϵ_1, the disturbance of force of interest generated in period one. When considering the discounted reserve, we simply multiply V_1 by e^{-y_1}. In actuarial science, it is traditional to present recursive calculations of reserves. In Appendix 1, we provide the recursive calculations relating V_1 to V_0.

3 Linear Interest Rate Process

Define a linear interest process to mean that the force of interest $\{y_t\}$ can be represented as

$$y_t = a + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t - c_1\epsilon_{t-1} - \cdots - c_q\epsilon_{t-q}, \tag{5}$$

where $(a, \phi_1, \ldots, \phi_p, c_1, \ldots, c_q)$ are parameters, $\{\epsilon_t\}$ are *i.i.d.*, $E(\epsilon_t) = 0$, $(y_0, y_{-1}, \ldots, y_{1-p})$ and $(\epsilon_0, \epsilon_{-1}, \ldots, \epsilon_{1-q})$ are known. The linear interest rate process can also be written in the form

$$y_t = \gamma_t + \sum_{i=0}^{t-1} \beta_i \epsilon_{t-i}, t \geq 1, \tag{6}$$

where

$$\gamma_t = \sum_{i=1}^{p} y_{i-p} \sum_{j=max(0,i-t)}^{i-1} \phi_{p-j}\alpha_{j-i+t} - \sum_{i=1}^{q} \epsilon_{i-q} \sum_{j=max(0,i-t)}^{i-1} c_{q-j}\alpha_{j-i+t} + a\sum_{i=0}^{t-1}\alpha_i, \tag{7}$$

$$\alpha_0 = 1, \beta_0 = 1; \alpha_i = \sum_{j=1}^{min(i,p)} \phi_j \alpha_{i-j}, i \geq 1; \beta_i = \alpha_i - \sum_{j=1}^{min(i,q)} c_j \alpha_{i-j}, i \geq 1, \tag{8}$$

see, for example, Box and Jenkins (1976) or Dhaene (1989). In other words, y_t is affine (linear plus constant) in $(\epsilon_1, \ldots, \epsilon_t)$.

If the force of interest follows a linear process, then the two discount factors in Proposition 1 can be expressed explicitly by using the next theorem.

Theorem 1 *For a one unit T-year default-free pure discount bond, if $\{y_t\}$ follows a linear interest process, then*

$$V_0^{(T)} = E[exp(-\sum_{s=1}^{T} y_s)] = \prod_{l=0}^{T-1} M(-\sum_{i=0}^{l}\beta_i) \cdot exp(-\sum_{s=1}^{T}\gamma_s),$$

and

$$V_1^{(T-1)} = E[exp(-\sum_{s=2}^{T} y_s)|\epsilon_1] = \prod_{l=0}^{T-2} M(-\sum_{i=0}^{l}\beta_i) \cdot exp(-\sum_{s=2}^{T}\gamma_s) \cdot exp(-\epsilon_1 \sum_{i=1}^{T-1}\beta_i).$$

where $\{\beta_i\}$ and $\{\gamma_s\}$ are defined in (7) and (8). □

Therefore, supposing a linear process for the force of interest, we can combine Proposition 1 and Theorem 1 to write down the reserves V_0 and V_1. To illustrate, it is of interest to write down the special case of independent interest.

Corollary 1 *(Independent Interest Case) If $\phi_1 = \cdots = \phi_p = c_1 = \cdots = c_q = 0$ so that $\{y_t\}$ are iid, then*

$$V_0 = \sum_{\tau=0}^{\infty}(\sum_{i=1}^{n} c_{\tau,k_i}^{(i)}) \cdot [e^{-a}M(-1)]^{\tau+1} - \sum_{i=1}^{n} a_{i,k_i},$$

and

$$V_1 = \sum_{\tau=0}^{\infty}(\sum_{i=1}^{n} c_{\tau,k_i+1}^{(i)}) \cdot [e^{-a}M(-1)]^{\tau+1} - \sum_{i=1}^{n} a_{i,k_i+1}. \square$$

From equation (3), it is evident that, in general, V_1 is a function of ϵ_1, and is thus a random variable. However, in the *i.i.d.* case, V_1 becomes a constant, which means that it is independent of ϵ_1. The intuition is that in this *i.i.d.* case, $\{y_t\}$ are purely random, and thus ϵ_1 does not provide any sequential information about $(y_2, y_3, \ldots)$ and hence the next-period reserve V_1. Therefore, by employing the expectation approach for pricing and assuming that the force of interest is generated by a white noise series, the measure V_1 (also for $V_h, h \geq 1$) is a deterministic value, which may not capture the real events. Examining an autocorrelated interest environment may model practical situations more appropriately than the independent interest benchmark case.

3.1 Default-free Pure Discount Bond Example

The simplest special case for the cash flows is a pure discount bond. Suppose that an insurance company has to pay one unit T years later with certainty. Then, using Theorem 1 and a linear process for the interest rates as in (5), we have

$$e^{-y_1} \cdot V_1^{(T-1)} = \prod_{l=0}^{T-2} M(-\sum_{i=0}^{l} \beta_i) \cdot exp(-\sum_{s=1}^{T} \gamma_s) \cdot exp(-\epsilon_1 \sum_{i=0}^{T-1} \beta_i), T \geq 2.$$

Making normality assumption for ϵ's yields

$$e^{-y_1} \cdot V_1^{(T-1)} \stackrel{d}{=} e^U$$

where

$$U \sim N(-\sum_{s=1}^{T} \gamma_s + \frac{1}{2}\sigma^2 \sum_{l=0}^{T-2} (\sum_{i=0}^{l} \beta_i)^2, \sigma^2 (\sum_{i=0}^{T-1} \beta_i)^2), T \geq 2,$$

$$\sigma^2 = Var(\epsilon_t).$$

Thus, the discounted reserve $e^{-y_1} \cdot V_1^{(T-1)}$ has a lognormal distribution in this special case. It is easy to check that if σ^2 increases, so does the mean and variance of $e^{-y_1} \cdot V_1^{(T-1)}$.

For a single payment with fixed payment date as above, the exact distribution of $e^{-y_1} V_1$ can be obtained. However, for a general insurance contract such as life insurance or deferred annuity, analytically the exact distribution is extremely difficult to find. Appendix 2 presents a moment-matching method to approximate the distribution. A simulation example for the distribution under a general insurance contract is provided in next subsection.

3.2 A Simulation Example for Linear Process

To illustrate the distributions of V_1 and $e^{-y_1} V_1$, we consider a block of whole life business that was also used in Example 4.1 in Frees (1990). For simplicity, policies are categorized into three groups of size 10 so the total size is 30. Assume that, for each category, ages at issue are $x = 30, 30, 40$

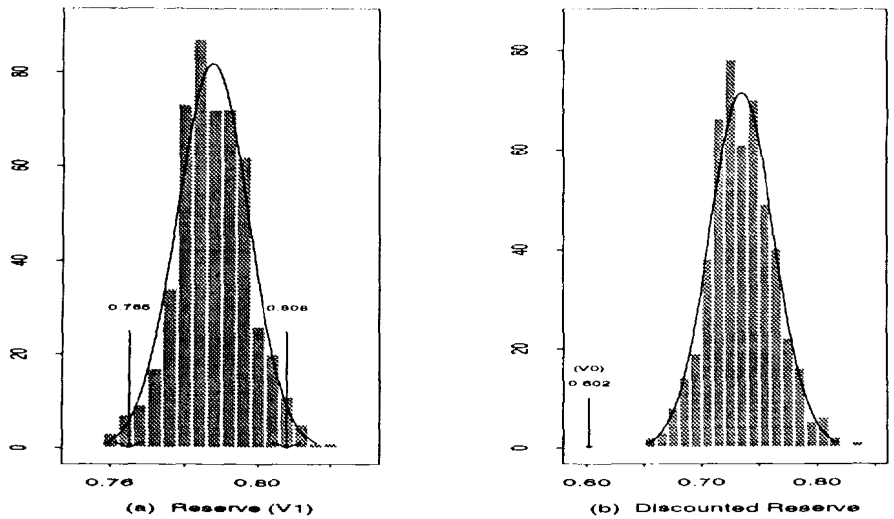


Figure 1: Frequency Histograms for Random Reserve V_1 and Discounted Reserve $e^{-y_1}V_1$. Each histogram was generated from 500 simulations using an AR(2) model for the force of interest. For comparison, $V_1 = 0.765$ and 0.808 corresponding to constant force of interest 0.08 and 0.079 respectively, are plotted on (a). The initial reserve, $V_0 = 0.602$, is plotted on (b).

and durations are $k = 5, 10, 5$, respectively. All death benefits are \$1. The mortality decrements are the 1979-81 U.S. Life Table that appear in Bowers et al. (1986). Here, assume that we have the following stationary AR(2) model for the interest rate:

$$y_t = 0.08 + 0.6(y_{t-1} - 0.08) - 0.3(y_{t-2} - 0.08) + \epsilon_t, t \geq 1,$$

where $\epsilon_t \sim iid\ N(0, \sigma = 0.025), y_0 = 0.06, y_{-1} = 0.07$, as in Giaccotto (1986) and Dhaene (1989). To compute the level premiums for each of three categories, first use constant force of interest $0.08(= E(y_t))$ to get the net level annual premiums, and then add 20% as the relative security loading to obtain the final level premiums.

Combining Proposition 1 and Theorem 1 yields analytic expressions for V_1 and its discounted version, $e^{-y_1}V_1$. To approximate these expressions, we performed 500 simulations. The resulting frequency histograms are shown in Figure 1. Normally distributed curves are superimposed for comparison purposes. The sample mean and standard deviation of the simulated distribution of reserve V_1 is 0.7875 and 0.009748, respectively. For discounted reserve $e^{-y_1}V_1$, the corresponding measures are 0.7349 and 0.02799. It is interesting that the dispersion for discounted reserve $e^{-y_1}V_1$ is almost three times as for reserve V_1 (0.02799/0.009748=2.9) in this example. The difference originates from the additional random portion — the discount factor e^{-y_1}.

Because the mean of the force of interest $E(y_t) = 0.08$, suppose that we had used $\delta = 0.08$ as constant force of interest to calculate the next-period reserve. Then, $V_1 = 0.765$, which underestimates the "true" reserve, as shown in Figure 1(a). For comparison, choosing another $\delta = 0.079$ as constant force of interest gives $V_1 = 0.808$, which overestimates the "true" distribution. This example indicates that ignoring the stochastic interest rate environment would easily miss the target distribution.

Since V_0 and discounted reserve $e^{-y_1}V_1$ are both valued at time 0, we can compare $V_0 = 0.602$ and the distribution of $e^{-y_1}V_1$ to see the changes in reserves. To illustrate, we may wish to know how much to add to the current reserve so that the next-period funds can meet future obligations measured by V_1 with 95% probability. That is to say, we want to find A such that $P[(V_0 + A)e^{y_1} > V_1] = 0.95$. Hence $A = (95\text{-th percentile of } e^{-y_1}V_1) - V_0 \approx 0.782 - 0.602 = 0.18$. The sample values of (skewness, kurtosis) for V_1 and $e^{-y_1}V_1$ in this simulation example are (0.0643, 3.19) and (0.155, 3.21) respectively. The kurtosis defined here is equal to 3 for a normal variate. Thus, a normal distribution could be used to approximate the reserve V_1 even though slightly positive skewness and slight thicker tails feature are exhibited from the sample quantities. The positivity of skewness of $e^{-y_1}V_1$ is larger than that of V_1 but a normal approximation to the distribution of discounted reserve $e^{-y_1}V_1$ seems adequate.

4 Nonlinear Interest Rate Process

4.1 Introduction to ARCH Process

Under the traditional linear time series setting, the conditional variance of one-step-ahead prediction is time invariant. Recently, applied researchers have recognized the importance of explicitly modeling time-varying second- and higher-order moments. One of the most prominent tools that has emerged for describing such changing variances is the Autoregressive Conditional Heteroskedasticity (ARCH) model of Engle (1982) and its various extensions. Bollerslev et al. (1992) contains an overview of some of the developments in the formulation of ARCH models and a survey of the numerous empirical applications using financial data. In particular, this survey includes a discussion of the modeling of interest rates.

The simplest non-trivial ARCH model is the first order linear model given by:

$$\begin{aligned} \epsilon_t|\psi_{t-1} &\sim N(0, h_t), \\ h_t &= \delta_0 + \delta_1 \epsilon_{t-1}^2, \end{aligned} \tag{9}$$

where ψ_{t-1} is the information set (σ-field) available at time $t-1$ and $\delta_0 > 0, \delta_1 \geq 0$, unknown parameters. The nonlinear ARCH process is serially uncorrelated with nonconstant variances conditional on the past, but constant unconditional variances. This model captures the tendency for volatility clustering, that is, for large (small) changes to be followed by other large (small) changes, but of unpredictable sign. The non-linearity stems from the variance of disturbance term ϵ_t. If δ_1 were equal to zero, then the model would be the conventional Gaussian white noise. However, for $\delta_1 \neq 0$, the effect of (9) is to make the variance of the disturbance terms at time t dependent on the realized value of the disturbance term in the previous period.

We have just seen a simple example of ARCH process for the innovations. To certain extent, reserves V_0 and V_1, as obtained from (2) and (3), involve the moment generating function of ϵ_t's. Because of the nonexistence of higher moments under ARCH model (9) and hence its moment generating function (see Engle (1982) and Bollerslev (1986)), we consider a model similar to (9) but in absolute value form:

$$\begin{aligned} &\epsilon_t|\psi_{t-1} \sim N(0, h_t), \\ &h_t = \delta_0 + \delta_1|\epsilon_{t-1}|, \end{aligned} \tag{10}$$

where $\delta_0 > 0, \delta_1 \geq 0$. This model was mentioned in the seminal paper of Engle (1982), but is seldom used in the ARCH literature. Perhaps it is due to the mathematical tractability difficulty in absolute values. Recent papers that discuss the absolute version of ARCH model include Engle and Bollerslev (1986), Schwert (1990), and Higgins and Bera (1992). The absolute value form of the model in (10) not only shares the features described above for model (9), but also possesses the existence of the moment generating function, a desirable characteristic. We note that δ_1 in model (10) depends on the measurement unit.

In the discussion below we use the following model for the force of interest:

$$\left.\begin{aligned} &\text{(a) } y_t = a + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \epsilon_t - c_1\epsilon_{t-1} - \cdots - c_q\epsilon_{t-q}, \\ &\text{(b) } \epsilon_t|\psi_{t-1} \sim N(0, h_t), \\ &\text{(c) } h_t = \delta_0 + \delta_1|\epsilon_{t-1}|, t \geq 1. \end{aligned}\right\} \tag{11}$$

That is, we allow for conditional (prediction) variance to depend on the absolute value of immediately previous innovation.

4.2 Approximate Results for Reserves Under ARCH Processes

We will examine the effect of δ_1 on the reserves V_0 and V_1 under model (11), supposing δ_1 to be small. Before stating the result for reserves, a lemma for the finiteness of the moment generating function of innovations is given first. Engle (1982) said that "The absolute value form ... can be shown to have finite variance for any parameter values". In fact, we can prove a stronger, new result that the moment generating function is finite, hence so are all moments of ϵ_t's.

Lemma 1 *If ϵ_t's follows model (11) (b) and (c), then the moment generating function of $(\epsilon_1, \ldots, \epsilon_T)$ is finite for all $\delta_0 > 0, \delta_1 \geq 0$.*

With the above lemma, we are ready to state the basic main result for a pure discount bond under the ARCH process.

Theorem 2 *For a one unit T-year default-free pure discount bond, under model (11) for the force of interest, we have that, as $\delta_1 \to 0$,*

$$(i)\ V_0^{(T)} = E[exp(-\sum_{s=1}^{T} y_s)] = \chi_0^{(T)} + O(\delta_1^3)$$

$$(ii)\ V_1^{(T-1)} = E[exp(-\sum_{s=2}^{T} y_s)|\epsilon_1] = \chi_1^{(T)} + O(\delta_1^3).$$

Here,

$$\chi_0^{(T)} = exp(-\sum_{s=1}^{T} \gamma_s + \frac{\delta_0}{2}\sum_{l=0}^{T-1} \zeta_l^2) \cdot [1 + \frac{\delta_1}{2} C_1^{(T)}(\epsilon_0) + \frac{\delta_1^2}{2} C_2^{(T)}(\epsilon_0)],$$

$$\chi_1^{(T)} = exp(-\sum_{s=2}^{T} \gamma_s - \epsilon_1 \sum_{i=1}^{T-1} \beta_i + \frac{\delta_0}{2}\sum_{l=0}^{T-2} \zeta_l^2) \cdot [1 + \frac{\delta_1}{2} C_1^{(T-1)}(\epsilon_1) + \frac{\delta_1^2}{2} C_2^{(T-1)}(\epsilon_1)],$$

$$C_1^{(T)}(x) = |x|\zeta_{T-1}^2 + \sqrt{\delta_0}\sum_{l=0}^{T-2} \zeta_l^2 q(\sqrt{\delta_0}\zeta_{l+1}),$$

$$\begin{aligned} C_2^{(T)}(x) &= \frac{|x|}{\sqrt{\delta_0}}\zeta_{T-2}^2[q(\sqrt{\delta_0}\zeta_{T-1}) - \frac{1}{\sqrt{2\pi}}e^{-\delta_0\zeta_{T-1}^2/2}] + \frac{1}{4}\delta_0 \sum_{l=0}^{T-2} \zeta_l^4[1 + \delta_0\zeta_{l+1}^2 - q^2(\sqrt{\delta_0}\zeta_{l+1})] \\ &+ \sum_{l=0}^{T-3} \zeta_l^2 q(\sqrt{\delta_0}\zeta_{l+2})[q(\sqrt{\delta_0}\zeta_{l+1}) - \frac{1}{\sqrt{2\pi}}e^{-\delta_0\zeta_{l+1}^2/2}] + \frac{1}{4}[C_1^{(T)}(x)]^2, \end{aligned}$$

$$q(x) = \frac{2}{\sqrt{2\pi}}e^{-x^2/2} + x[2\Phi(x) - 1],$$

$\Phi(x) =$ *distribution function of a standard normal variable,*

and

$$\zeta_l = \sum_{i=0}^{l} \beta_i . \square$$

We can see that the terms outside the square brackets of $\chi_0^{(T)}$ and $\chi_1^{(T)}$ correspond to the results of the linear process case ($\delta_1 = 0$) under normality. It can be shown that both coefficients of the linear and quadratic order terms (with respect to δ_1)—C_1 and C_2—are nonnegative. Thus for a pure discount bond, when δ_1 small, the reserves V_0 and V_1 are both larger in the ARCH process

(11) case than in the linear process case. Moreover, we expect that V_0 and V_1 would increase if δ_1 increases.

Once we have Theorem 2 for a pure discount bond we can have corresponding results for V_0 and V_1 in the general block of business case. That is, by retrieving (2) and (3) in Proposition 1 and substituting $\chi_0^{(\tau+1)}$ and $\chi_1^{(\tau+2)}$ for those two discount factors, respectively, the desired results are obtained.

The above analysis studies the case where δ_1 is assumed to be small. For larger values of δ_1, we resort to simulation techniques, as described in the following subsection.

4.3 A Simulation Example for an ARCH Process

Unlike the ARMA model, the ARCH model takes into account the conditional heteroscedasticity. The model that we use for an ARCH process resembles that for linear process. The insurance model is the same as in Section 3.2. However, the model for interest rate variations becomes:

$$y_t = 0.08 + 0.6(y_{t-1} - 0.08) - 0.3(y_{t-2} - 0.08) + \epsilon_t,$$
$$\epsilon_t|\psi_{t-1} \sim N(0, h_t), h_t = (0.025)^2 + \delta_1|\epsilon_{t-1}|, t \geq 1,$$

with $\delta_1 = 0, 0.001, 0.003, \ldots, 0.08$, where $y_0 = 0.06, y_{-1} = 0.07, \epsilon_0 = -0.01$. We shall use this ARCH model as a nonlinear time series example to do the simulations for illustrating the next period reserve. The largest value of δ_1 chosen to investigate is 0.08. The reason is as follows. In Appendix 3, we show that $sd(\epsilon_t) \leq \sqrt{\delta_0 + \delta_1(\delta_1 + \sqrt{\delta_1^2 + 2\pi\delta_0})/\pi}$. With $\delta_0 = (0.025)^2, \delta_1 = 0.08$, we have $sd(\epsilon_t) \leq 0.072$, which is about 2.9 times the standard deviation of ϵ_t in the linear process case $(0.072/0.025 = 2.88)$.

4.3.1 The Valuation Algorithm

Part of the simulation ideas that we use comes from Tilley (1993), where he presents a simulation algorithm for valuing American-style options.

For nonlinear time series models, like ARCH processes, it is difficult to compute analytically, $E[exp(-\sum_{s=1}^{T} y_s)]$ or $E[exp(-\sum_{s=2}^{T} y_s)|\epsilon_1]$, and hence the distribution of the reserve, since the error term ϵ's are no longer independent. However, using simulations can gain some insight into the distribution of V_1 and $e^{-y_1}V_1$. For simplicity, consider first a one unit T-year pure discount bond,

$$V_1 = E[exp(-\sum_{s=2}^{T} y_s)|\epsilon_1].$$

Since it involves an expectation that could not be expressed simply, we need to approximate it by simulation. Figure 2 helps us to understand how the simulation works.

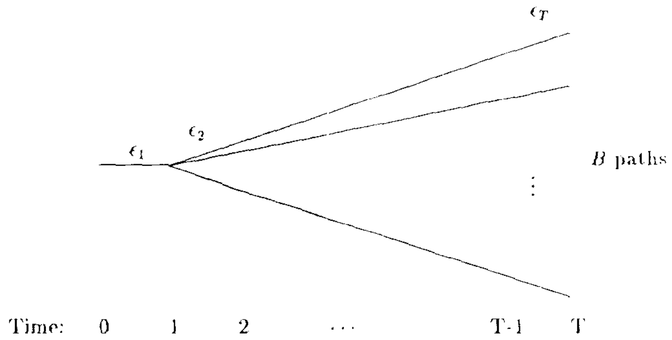


Figure 2: Diagram exhibiting the simulation paths for the ARCH process.

Given a value of ϵ_1 (its 98-th percentile, for example), we simulate $(\epsilon_2, \ldots, \epsilon_T)$ on each path from time 1 to T. Suppose we have $B(= 1200$, say) such repetitions. Then the expectation $E[exp(-\sum_{s=2}^{T} y_s)|\epsilon_1]$ can be approximated by

$$B^{-1}\sum_{k=1}^{B} exp(-\sum_{s=2}^{T} y_s^{(k)})$$

where $y_s^{(k)}$ is the simulated s-th period force of interest on path k ($k = 1, \ldots, B$) for a certain percentile of ϵ_1.

Now extend a pure discount bond to the case of whole block of business with n insureds. Let ω be the limiting age. From (3), we have

$$V_1 = \sum_{\tau=0}^{M-2} f_\tau E[exp(-\sum_{s=2}^{\tau+2} y_s)|\epsilon_1] - \sum_{i=1}^{n} a_{i,k_i+1}$$

where $M = \omega - \min_{1\leq i\leq n}(x_i + k_i)$ and $f_\tau = \sum_{i=1}^{n} c_{\tau,k_i+1}^{(i)}$, since $f_\tau = 0$ for $\tau > M - 2$. In this case we use the same ideas as above with replacing $T = \tau + 2, 0 \leq \tau \leq M - 2$, then we obtain a value of V_1 which corresponds to a given percentile of ϵ_1.

4.3.2 Sensitivity of the Reserve to Variance Autoregression Parameter

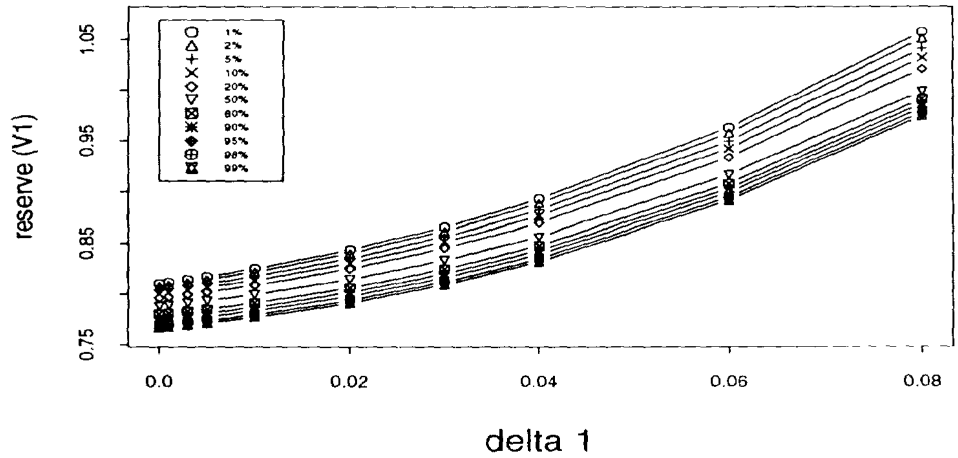
In our numerical example, we use different values of δ_1 in the simulation. Thus, we can study the sensitivity of V_1 and $e^{-y_1}V_1$ to a change in the value of the variance autoregression parameter δ_1. For example, we might like to see how large this parameter can be before it matters. Therefore, we include the case $\delta_1 = 0$ for comparison purposes. For each percentile of ϵ_1, we implement the simulation method described above to obtain the values of $V_1(\mu_p, \delta_1)$ where μ_p denotes the $100p$-th percentile of the distribution of ϵ_1. By examining many different percentiles of ϵ_1, especially the

Table 1: Simulated values of reserve V_1 for different combinations of percentiles of ϵ_1 and values of δ_1. The value in parentheses beneath V_1 corresponds to its simulation standard error. Number of simulation (B) is 1200.

	Percentile of ϵ_1										
	1%	2%	5%	10%	20%	50%	80%	90%	95%	98%	99%
$\delta_1 = 0$	0.8107	0.8080	0.8041	0.8006	0.7963	0.7883	0.7804	0.7762	0.7728	0.7690	0.7665
	(0.0075)	(0.0075)	(0.0075)	(0.0074)	(0.0074)	(0.0073)	(0.0073)	(0.0072)	(0.0072)	(0.0072)	(0.0072)
0.001	0.8120	0.8093	0.8053	0.8017	0.7974	0.7893	0.7813	0.7772	0.7737	0.7700	0.7674
	(0.0077)	(0.0076)	(0.0076)	(0.0076)	(0.0075)	(0.0075)	(0.0074)	(0.0074)	(0.0073)	(0.0073)	(0.0073)
0.003	0.8147	0.8119	0.8078	0.8042	0.7998	0.7914	0.7834	0.7782	0.7757	0.7719	0.7693
	(0.0080)	(0.0079)	(0.0079)	(0.0078)	(0.0078)	(0.0077)	(0.0077)	(0.0076)	(0.0076)	(0.0076)	(0.0076)
0.005	0.8176	0.8147	0.8105	0.8068	0.8022	0.7936	0.7855	0.7813	0.7778	0.7740	0.7713
	(0.0082)	(0.0082)	(0.0082)	(0.0081)	(0.0081)	(0.0080)	(0.0080)	(0.0079)	(0.0079)	(0.0079)	(0.0078)
0.010	0.8253	0.8223	0.8178	0.8138	0.8090	0.7998	0.7915	0.7872	0.7837	0.7797	0.7771
	(0.0090)	(0.0090)	(0.0089)	(0.0089)	(0.0088)	(0.0087)	(0.0087)	(0.0086)	(0.0086)	(0.0086)	(0.0086)
0.020	0.8436	0.8402	0.8351	0.8306	0.8252	0.8148	0.8063	0.8018	0.7981	0.7940	0.7912
	(0.0108)	(0.0108)	(0.0107)	(0.0106)	(0.0105)	(0.0103)	(0.0103)	(0.0103)	(0.0103)	(0.0102)	(0.0102)
0.030	0.8660	0.8622	0.8565	0.8514	0.8453	0.8336	0.8248	0.8202	0.8164	0.8122	0.8093
	(0.0129)	(0.0128)	(0.0127)	(0.0126)	(0.0124)	(0.0122)	(0.0122)	(0.0122)	(0.0121)	(0.0121)	(0.0121)
0.040	0.8930	0.8887	0.8824	0.8767	0.8698	0.8565	0.8477	0.8430	0.8391	0.8347	0.8317
	(0.0152)	(0.0151)	(0.0149)	(0.0148)	(0.0146)	(0.0143)	(0.0143)	(0.0143)	(0.0143)	(0.0142)	(0.0142)
0.060	0.9628	0.9574	0.9495	0.9423	0.9336	0.9165	0.9079	0.9031	0.8990	0.8945	0.8914
	(0.0208)	(0.0206)	(0.0204)	(0.0201)	(0.0199)	(0.0193)	(0.0194)	(0.0194)	(0.0194)	(0.0194)	(0.0194)
0.080	1.0576	1.0508	1.0408	1.0318	1.0207	0.9983	0.9907	0.9859	0.9818	0.9771	0.9739
	(0.0281)	(0.0279)	(0.0275)	(0.0271)	(0.0267)	(0.0257)	(0.0260)	(0.0260)	(0.0261)	(0.0261)	(0.0261)

tails, we can present the whole picture of sensitivity of variance autoregression parameter on the next-period reserve.

The simulated values of V_1, and associated simulation standard errors, are summarized in Table 1. Plotting those values of V_1 together in Figure 3(a) presents a more clear picture. We see that for each fixed value of ϵ_1, as δ_1 increases the reserve V_1 also increases. This aspect is consistent with the discussion following Theorem 2 for a pure discount bond. On the other hand, fixing δ_1, larger value of ϵ_1 reduces the reserve V_1, as shown in Table 1 and Figure 3(a). However, it is not necessary that this is always true. That is, recall from Theorem 1 that there need not be a monotonic relationship between the value of ϵ_1 and reserve V_1. From Figure 3(a), it seems that V_1 increases "quadraticly" for $\delta_1 \in [0, 0.08]$ so the approximation up to second order term appears adequate. When $\delta_1 < 0.02$, the linear order term already gives good approximation to the reserve V_1. The plot also indicates that the dispersion of V_1 tends to increase as δ_1 increases. A corresponding graph for discounted reserve $e^{-y_1}V_1$ is shown in Figure 3(b) in which similar features appear. In this example, the discounted reserve is less influenced by δ_1 than the reserve, as indicated by the growing rate of curves in the figure. As noted in linear process case, the dispersion of discounted reserve $e^{-y_1}V_1$ is larger than that of reserve V_1, for various values of δ_1. In addition, the discounted reserve seems to



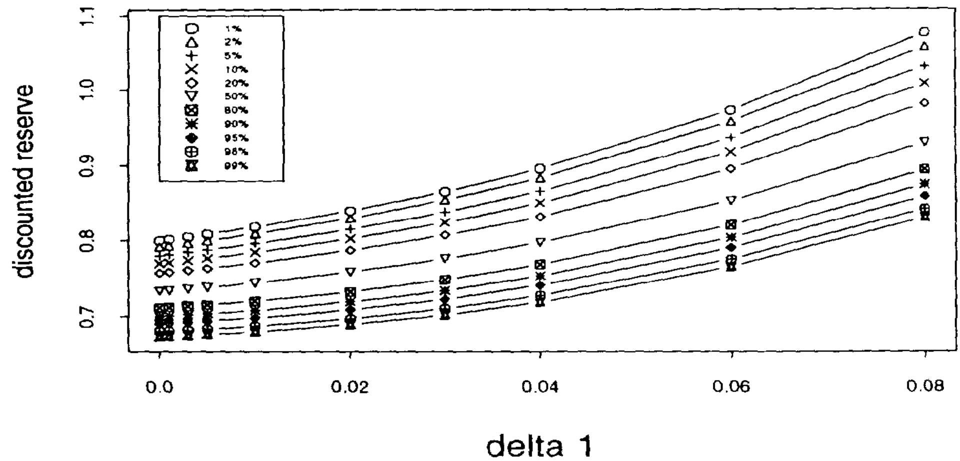


Figure 3: Using simulation method. (a)Upper—Plot of reserve V_1 against δ_1 for given percentile of ϵ_1. (b)Lower—Plot of discounted reserve $e^{-\nu_1}V_1$ against δ_1 for given percentile of ϵ_1. The 1% denotes the first percentile of ϵ_1, and so forth.

Table 2: Values of V_1 obtained from approximation result of Theorem 2.

	Percentile of ϵ_1										
	1%	2%	5%	10%	20%	50%	80%	90%	95%	98%	99%
$\delta_1 = 0$	0.8088	0.8061	0.8022	0.7987	0.7945	0.7865	0.7786	0.7744	0.7710	0.7672	0.7647
0.001	0.8099	0.8072	0.8033	0.7998	0.7955	0.7874	0.7794	0.7752	0.7718	0.7680	0.7654
0.003	0.8124	0.8096	0.8055	0.8019	0.7975	0.7893	0.7812	0.7770	0.7735	0.7697	0.7670
0.005	0.8150	0.8121	0.8080	0.8043	0.7998	0.7913	0.7832	0.7789	0.7754	0.7715	0.7689
0.010	0.8223	0.8193	0.8149	0.8110	0.8062	0.7972	0.7889	0.7845	0.7809	0.7769	0.7742
0.020	0.8406	0.8372	0.8323	0.8278	0.8225	0.8123	0.8036	0.7990	0.7953	0.7912	0.7883
0.030	0.8637	0.8599	0.8543	0.8493	0.8432	0.8318	0.8228	0.8181	0.8143	0.8100	0.8071
0.040	0.8918	0.8874	0.8810	0.8754	0.8685	0.8555	0.8465	0.8417	0.8378	0.8335	0.8305
0.060	0.9628	0.9572	0.9489	0.9416	0.9328	0.9162	0.9071	0.9024	0.8985	0.8942	0.8913
0.080	1.0541	1.0468	1.0363	1.0269	1.0155	0.9942	0.9856	0.9811	0.9775	0.9734	0.9706

Table 3: Values of ratio = (approximation value − simulated value)/simulation standard error.

	Percentile of ϵ_1										
	1%	2%	5%	10%	20%	50%	80%	90%	95%	98%	99%
$\delta_1 = 0$	-0.247	-0.248	-0.248	-0.247	-0.247	-0.248	-0.248	-0.247	-0.248	-0.248	-0.247
0.001	-0.267	-0.266	-0.264	-0.264	-0.261	-0.258	-0.261	-0.263	-0.264	-0.267	-0.267
0.003	-0.297	-0.294	-0.290	-0.288	-0.284	-0.276	-0.284	-0.288	-0.292	-0.294	-0.298
0.005	-0.316	-0.315	-0.309	-0.305	-0.299	-0.287	-0.298	-0.305	-0.309	-0.314	-0.318
0.010	-0.334	-0.330	-0.323	-0.317	-0.309	-0.291	-0.309	-0.318	-0.324	-0.331	-0.335
0.020	-0.279	-0.276	-0.271	-0.265	-0.256	-0.235	-0.257	-0.265	-0.271	-0.277	-0.280
0.030	-0.178	-0.178	-0.175	-0.173	-0.167	-0.147	-0.167	-0.173	-0.176	-0.178	-0.179
0.040	-0.083	-0.085	-0.089	-0.089	-0.088	-0.068	-0.088	-0.090	-0.089	-0.086	-0.084
0.060	-0.002	-0.013	-0.026	-0.034	-0.041	-0.019	-0.041	-0.035	-0.026	-0.013	-0.004
0.080	-0.127	-0.143	-0.165	-0.181	-0.194	-0.161	-0.194	-0.181	-0.165	-0.144	-0.128

grow linearly for δ_1 up to about 0.03.

Next, we compare the values of reserve V_1 obtained from two different methods: (1) by simulation and (2) by the approximation result in Theorem 2. Table 2 shows the values of V_1 obtained from approximation approach. The corresponding plot for Table 2 would look like as Figure 3(a) so that it is not presented here. To compare these two methods let us consider the following ratio:

$$\frac{\text{approximation value} - \text{simulated value}}{\text{simulation standard error}}.$$

The ratio values are reported in Table 3 in which we find that all of them are less than 0.34 in absolute value. Theorem 2 therefore provides us good approximations to the reserve V_1 under the ARCH model (11) for the force of interest. It is noted that all of the ratio values are negative from which we induce that in this example the approximation method would slightly underestimate the "true" reserve V_1.

5 Concluding Remarks

This paper has examined the discrete-time short-term consequences on reserves due to changes in the interest rate environment. Generally, when viewed at initial time, the next period reserve is random variable, which is a function of the disturbance generated in the first period. However, in the special case of white noise process for the force of interest, the next period reserve is a deterministic value. *Linear and nonlinear ARCH process models for the force of interest are considered in the* paper. Under linear interest rate processes, explicit expressions are given for the changes in reserves. In particular, for a pure discount bond, the next period reserve and its discounted value have a lognormal distribution. As an extension from linear processes to nonlinear processes, approximation formulas and simulation algorithms are presented. Compared with simulation results, we have found that the approximation formulas perform well in the sense that the discrepancies between the approximations and simulations are small, relative to the simulation standard error. Even though we used only one type of insurance contract for our comparisons, the results are applicable as long as the cash flows are independent of interest rates. This is because, under this assumption, those two factors can be separated, as shown in Proposition 1.

We considered in this paper only the initial and next period reserves. The values will evolve as the time goes by. To handle this one-period ahead problem sequentially, stochastic control theory may be used to access not only the dynamic nature of the time development, but also the mobility of the cash flows (claims, premiums, and expenses) in an insurance organization. See Martin-Löf (1983), and Vandebroek and Dhaene (1990) for applying the control theory on insurance context, where they assume fixed interest rates.

In this paper, the cash flows are assumed to be independent of interest rate variations. For many insurance contracts, this assumption appears to be unnecessarily restrictive. We hope to investigate this issue in the future.

Appendix 1: Expected Value of Discounted Reserve

For single policy we have the following result which can be served as a stepping stone to compute the expected value of discounted reserve for a whole block of business.

Proposition 2 *Suppose that (A1) holds. For single policy, we have*

$$V_{h-1} + a_{k+h-1} = E(p_{x+k+h-1} \cdot e^{-y_h} \cdot V_h + q_{x+k+h-1} \cdot e^{-y_h} \cdot b_{k+h} | \epsilon_1, \ldots, \epsilon_{h-1}), h \geq 1.$$

In particular, for $h = 1$,

$$V_0 + a_k = p_{x+k} E(e^{-y_1} \cdot V_1) + q_{x+k} \cdot b_{k+1} E(e^{-y_1}). \Box \tag{12}$$

(Compare Bowers et al. (1986, 7.8.2).) In words, the resources required at the beginning of policy year h equal the present value of year-end requirements in expected value sense. Furthermore, we can employ (12) to calculate the expected discounted reserve $E(e^{-y_1} \cdot V_1)$ for each policy and then add them up to obtain the whole block of business expected discounted reserve.

A recursive expression for $E[exp(-\sum_{s=1}^{h} y_s) \cdot V_h]$, the expected value of discounted period-h reserve, is obtained from Proposition 2:

$$\begin{aligned} & E[exp(-\sum_{s=1}^{h-1} y_s) \cdot V_{h-1}] + a_{k+h-1} E[exp(-\sum_{s=1}^{h-1} y_s)] \\ = \quad & p_{x+k+h-1} E[exp(-\sum_{s=1}^{h} y_s) \cdot V_h] + q_{x+k+h-1} \cdot b_{k+h} E[exp(-\sum_{s=1}^{h} y_s)], h \geq 1. \end{aligned}$$

Note that $V_0^{(h)} = E[exp(-\sum_{s=1}^{h} y_s)]$, its value can therefore be computed by using Theorems 1 and 2 under different models for $\{y_t\}$.

<u>**Proof**</u>

For single policy, let the loss at time h be

$$L_h = \sum_{\tau=0}^{\infty} F_\tau^{(h)} exp(-\sum_{s=h+1}^{h+\tau} y_s)$$

where $F_\tau^{(h)}$ is defined similarly as (1). By definition,

$$\begin{aligned} V_h \quad = \quad & E_h(L_h) = \sum_{\tau=0}^{\infty} E_J(F_\tau^{(h)}) \cdot E[exp(-\sum_{s=h+1}^{h+\tau} y_s) | \epsilon_1, \ldots, \epsilon_h] \\ = \quad & -a_{k+h} + \sum_{\tau=0}^{\infty} E_J(F_{\tau+1}^{(h)}) \cdot E[exp(-\sum_{s=h+1}^{h+\tau+1} y_s) | \epsilon_1, \ldots, \epsilon_h] \\ = \quad & -a_{k+h} + \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau|}q_{x+k+h} - a_{k+h+\tau+1} \cdot {}_{\tau+1}p_{x+k+h}) \\ & \qquad \cdot E[exp(-\sum_{s=h+1}^{h+\tau+1} y_s) | \epsilon_1, \ldots, \epsilon_h] \end{aligned} \tag{13}$$

So

$$e^{-y_h} \cdot V_h = -e^{-y_h} \cdot a_{k+h} + \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau}|q_{x+k+h} - a_{k+h+\tau+1} \cdot {}_{\tau+1}p_{x+k+h}) \cdot E[exp(-\sum_{s=h}^{h+\tau+1} y_s)|\epsilon_1, \ldots, \epsilon_h],$$

which implies

$$E(p_{x+k+h-1} \cdot e^{-y_h} \cdot V_h|\epsilon_1, \ldots, \epsilon_{h-1}) = -a_{k+h} \cdot p_{x+k+h-1} \cdot E(e^{-y_h}|\epsilon_1, \ldots, \epsilon_{h-1}) + \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau+1}|q_{x+k+h-1} - a_{k+h+\tau+1} \cdot {}_{\tau+2}p_{x+k+h-1}) \cdot E[exp(-\sum_{s=h}^{h+\tau+1} y_s)|\epsilon_1, \ldots, \epsilon_{h-1}]. \quad (14)$$

From (13), we obtain

$$\begin{aligned} V_{h-1} + a_{k+h-1} &= \sum_{\tau=0}^{\infty} (b_{k+h+\tau} \cdot {}_{\tau}|q_{x+k+h-1} - a_{k+h+\tau} \cdot {}_{\tau+1}p_{x+k+h-1}) \cdot E(e^{-\sum_{s=h}^{h+\tau} y_s}|\epsilon_1, \ldots, \epsilon_{h-1}) \\ &= (b_{k+h} \cdot q_{x+k+h-1} - a_{k+h} \cdot p_{x+k+h-1}) \cdot E(e^{-y_h}|\epsilon_1, \ldots, \epsilon_{h-1}) \\ &\quad + \sum_{\tau=0}^{\infty} (b_{k+h+\tau+1} \cdot {}_{\tau+1}|q_{x+k+h-1} - a_{k+h+\tau+1} \cdot {}_{\tau+2}p_{x+k+h-1}) \cdot E(e^{-\sum_{s=h}^{h+\tau+1} y_s}|\epsilon_1, \ldots, \epsilon_{h-1}) \\ &= b_{k+h} \cdot q_{x+k+h-1} \cdot E(e^{-y_h}|\epsilon_1, \ldots, \epsilon_{h-1}) + E(p_{x+k+h-1} \cdot e^{-y_h} \cdot V_h|\epsilon_1, \ldots, \epsilon_{h-1}) \end{aligned}$$

by noting (14). The proof is completed. □

Appendix 2: An Approximation Method to the Distribution Under Linear Process

If $\{y_t\}$ follow (5) and $\epsilon_t \sim iid\ N(0, \sigma^2)$, then $e^{-y_1}V_1$ has the form

$$X \equiv \sum_{j=0}^{\infty} \theta_j e^{\lambda_j \epsilon}$$

where $\epsilon \sim N(0, \sigma^2)$, with

$$\theta_j = \begin{cases} -e^{-\gamma_1} \sum_{i=1}^{n} a_{i,k_i+1} & , j = 0 \\ \sum_{i=1}^{n} c_{j-1,k_i+1}^{(i)} \cdot exp[\frac{1}{2}\sigma^2 \sum_{l=0}^{j-1} (\sum_{u=0}^{l} \beta_u)^2 - \sum_{s=1}^{j+1} \gamma_s] & , j \geq 1 \end{cases}$$

$$\lambda_j = -\sum_{s=0}^{j} \beta_s, j \geq 0.$$

Although X looks to have a simple form in terms of ϵ, the exact distribution of X is very difficult to compute. Nevertheless, its first four moments can be relatively easily derived by using Proposition 3 given below.

Proposition 3 *Let* $X \equiv \sum_{j=0}^{\infty} \theta_j e^{\lambda_j \epsilon}$ *where* $\epsilon \sim N(0, \sigma^2)$. *Then*

$$E(X) = \sum_{j=0}^{\infty} \theta_j e^{\frac{1}{2}\sigma^2 \lambda_j^2},$$

$$E(X^2) = \sum_{j=0}^{\infty} \theta_j^2 e^{2\sigma^2 \lambda_j^2} + 2\sum_{j<l}^{\infty} \theta_j \theta_l e^{\frac{1}{2}\sigma^2(\lambda_j+\lambda_l)^2},$$

$$E(X^3) = \sum_{j=0}^{\infty} \theta_j^3 e^{\frac{9}{2}\sigma^2 \lambda_j^2} + 3\sum_{j\neq l}^{\infty} \theta_j^2 \theta_l e^{\frac{1}{2}\sigma^2(2\lambda_j+\lambda_l)^2} + 6\sum_{j<l<m}^{\infty} \theta_j \theta_l \theta_m e^{\frac{1}{2}\sigma^2(\lambda_j+\lambda_l+\lambda_m)^2},$$

$$\begin{aligned} E(X^4) \;=\; & \sum_{j=0}^{\infty} \theta_j^4 e^{8\sigma^2 \lambda_j^2} + 4\sum_{j\neq l} \theta_j^3 \theta_l e^{\frac{1}{2}\sigma^2(3\lambda_j+\lambda_l)^2} + 6\sum_{j<l} \theta_j^2 \theta_l^2 e^{2\sigma^2(\lambda_j+\lambda_l)^2} \\ & + 12\Big(\sum_{j<l<m} + \sum_{l<j<m} + \sum_{l<m<j}\Big)\theta_j^2 \theta_l \theta_m e^{\frac{1}{2}\sigma^2(2\lambda_j+\lambda_l+\lambda_m)^2} \\ & + 24\sum_{j<l<m<n} \theta_j \theta_l \theta_m \theta_n e^{\frac{1}{2}\sigma^2(\lambda_j+\lambda_l+\lambda_m+\lambda_n)^2}, \end{aligned}$$

provided they exist. □

Proof

The proof is simply based on the expansions

$$(\sum a_i)^2 = \sum a_i^2 + 2\sum_{i<j} a_i a_j,$$

$$(\sum a_i)^3 = \sum a_i^3 + 3\sum_{i\neq j} a_i^2 a_j + 6\sum_{i<j<k} a_i a_j a_k,$$

$$\begin{aligned} (\sum a_i)^4 \;=\; & \sum a_i^4 + 4\sum_{i\neq j} a_i^3 a_j + 6\sum_{i<j} a_i^2 a_j^2 + 12\Big(\sum_{i<j<k} + \sum_{j<i<k} + \sum_{j<k<i}\Big) a_i^2 a_j a_k \\ & + 24\sum_{i<j<k<l} a_i a_j a_k a_l, \end{aligned}$$

and the normality assumption on ϵ. □

By employing Proposition 3 to calculate the first four moments of $e^{-y_1} V_1$, we can approximate its distribution by a curve of Pearson family or normal power method. We can then use the approximated distribution to compute its percentiles.

Appendix 3: An Upper Bound for the Variance of Innovation Under ARCH Model

First, $E(\epsilon_t) = EE(\epsilon_t|\psi_{t-1}) = 0$, by (11)(b). Thus,

$$Var(\epsilon_t) = E(\epsilon_t^2) = EE(\epsilon_t^2|\psi_{t-1}) = \delta_0 + \delta_1 E|\epsilon_{t-1}|, \text{ by (11)(c).}$$

Since for $X \sim N(0,\sigma^2)$, $E|X| = \frac{2}{\sqrt{2\pi}}\sigma$, we have

$$\begin{aligned}
E|\epsilon_{t-1}| &= EE(|\epsilon_{t-1}||\psi_{t-2}) = E(\frac{2}{\sqrt{2\pi}}h_{t-1}^{1/2}) = \frac{2}{\sqrt{2\pi}}E\sqrt{\delta_0 + \delta_1|\epsilon_{t-2}|} \\
&\le \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \delta_1 E|\epsilon_{t-2}|}, \text{ by the Jensen's inequality} \\
&\le \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \delta_1 E|\epsilon_{t-3}|}}, \text{ by the same argument} \\
&\le \cdots \le \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \cdots + \delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \delta_1 E|\epsilon_{t-k}|}}}, \text{ in general.}
\end{aligned}$$

Assuming that the series starts indefinitely far in the past with finite first absolute moment, the limit of the upper bound on the right hand side as k goes to infinity tends to L where

$$L = \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \delta_1\frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \cdots}}.$$

Hence $L = \frac{2}{\sqrt{2\pi}}\sqrt{\delta_0 + \delta_1 L}$, which implies $L = (\delta_1 + \sqrt{\delta_1^2 + 2\pi\delta_0})/\pi$. Thus

$$Var(\epsilon_t) \le \delta_0 + \delta_1(\delta_1 + \sqrt{\delta_1^2 + 2\pi\delta_0})/\pi.$$

Appendix 4: Proofs

Proof of Proposition 1

By definition,

$$S_L^{(0)} = \sum_{i=1}^{n}\sum_{\tau=0}^{\infty} F_{i,\tau}^{(0)} exp(-\sum_{s=1}^{\tau} y_s) = \sum_{\tau=0}^{\infty}(\sum_{i=1}^{n} F_{i,\tau+1}^{(0)}) exp(-\sum_{s=1}^{\tau+1} y_s) - \sum_{i=1}^{n} a_{i,k},$$

and

$$S_L^{(1)} = \sum_{i=1}^{n}\sum_{\tau=0}^{\infty} F_{i,\tau}^{(1)} exp(-\sum_{s=2}^{\tau+1} y_s) = \sum_{\tau=0}^{\infty}(\sum_{i=1}^{n} F_{i,\tau+1}^{(1)}) exp(-\sum_{s=2}^{\tau+2} y_s) - \sum_{i=1}^{n} a_{i,k_i+1}.$$

Then we have the desired results for V_0 and V_1 by utilizing the independence between cash flows and interest rates, the definition of cash flow $F_{i,\tau+1}^{(h)}$ in (1) and the probability function of J_t. □

Proof of Theorem 1

The y_t expressing as (6) implies

$$\sum_{s=1}^{T} y_s = \sum_{s=1}^{T} \gamma_s + \sum_{s=1}^{T}\sum_{i=0}^{s-1} \beta_i \epsilon_{s-i} = \sum_{s=1}^{T} \gamma_s + \sum_{s=1}^{T} \epsilon_s (\sum_{i=0}^{T-s} \beta_i), \tag{15}$$

and

$$\sum_{s=2}^{T} y_s = \sum_{s=2}^{T} \gamma_s + \sum_{s=2}^{T}\sum_{i=0}^{s-1} \beta_i \epsilon_{s-i} = \sum_{s=2}^{T} \gamma_s + \epsilon_1 \sum_{i=1}^{T-1} \beta_i + \sum_{s=2}^{T} \epsilon_s (\sum_{i=0}^{T-s} \beta_i). \tag{16}$$

Therefore,

$$E[exp(-\sum_{s=1}^{T} y_s)] = exp(-\sum_{s=1}^{T} \gamma_s) \cdot \prod_{l=0}^{T-1} M(-\sum_{i=0}^{l} \beta_i)$$

and

$$E[exp(-\sum_{s=2}^{T} y_s)|\epsilon_1] = exp(-\sum_{s=2}^{T} \gamma_s) \cdot exp(-\epsilon_1 \sum_{i=1}^{T-1} \beta_i) \cdot \prod_{l=0}^{T-2} M(-\sum_{i=0}^{l} \beta_i)$$

by using the *iid* property of ϵ's and re-indexing. □

Proof of Lemma 1

Let $\underline{t} = (t_1, \ldots, t_T)'$. Then the moment generating function of $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_T)'$ is

$$\begin{aligned} M_{\underline{\epsilon}}(\underline{t}) &= E[exp(\sum_{i=1}^{T} t_i\epsilon_i)] = E[exp(\sum_{i=1}^{T-1} t_i\epsilon_i)E(e^{t_T\epsilon_T}|\psi_{T-1})] \\ &= e^{\delta_0 t_T^2/2} E[exp(\sum_{i=1}^{T-1} t_i\epsilon_i) \cdot exp(\frac{\delta_1}{2} t_T^2 |\epsilon_{T-1}|)], \text{ by (11) (b) and (c)} \\ &\leq e^{\delta_0 t_T^2/2} \{E[exp(2\sum_{i=1}^{T-1} t_i\epsilon_i)]\}^{1/2} \{E[exp(\delta_1 t_T^2 |\epsilon_{T-1}|)]\}^{1/2}, \text{ by the Hölder inequality.} \end{aligned}$$

Since for all $b > 0$ and $X \sim N(0, \sigma^2)$,

$$\begin{aligned} E(e^{b|X|}) &= E[e^{-bX} I(X < 0)] + E[e^{bX} I(X \geq 0)], \text{ where } I \text{ is an indicator function} \\ &\leq E(e^{-bX}) + E(e^{bX}) = 2e^{b^2\sigma^2/2}. \end{aligned}$$

Thus,

$$\begin{aligned} E[exp(\delta_1 t_T^2 |\epsilon_{T-1}|)] &= EE[exp(\delta_1 t_T^2 |\epsilon_{T-1}|)|\psi_{T-2}] \leq E[2exp\{\frac{1}{2}\delta_1^2 t_T^4 (\delta_0 + \delta_1|\epsilon_{T-2}|)\}] \\ &= 2e^{\delta_0\delta_1^2 t_T^4/2} E[exp(\frac{1}{2}\delta_1^3 t_T^4 |\epsilon_{T-2}|)] \leq C_1 < \infty, \end{aligned}$$

by a recursive argument, where

$$C_1 = 2^{T-1} exp[\frac{2\delta_0}{\delta_1^2} \sum_{j=2}^{T} \frac{(\delta_1 t_T)^{(2^j)}}{2^{(2^j-1)}} + \frac{2|\epsilon_0|}{\delta_1} \cdot \frac{(\delta_1 t_T)^{(2^T)}}{2^{(2^T-1)}}].$$

Hence,

$$
\begin{aligned}
M_{\underline{\epsilon}}(\underline{t}) &= E[exp(\sum_{i=1}^{T} t_i\epsilon_i)] \leq C_1^{1/2} e^{\delta_0 t_T^2/2}\{E[exp(2\sum_{i=1}^{T-1} t_i\epsilon_i)]\}^{1/2} \\
&= C_2\{E[exp(2\sum_{i=1}^{T-1} t_i\epsilon_i)]\}^{1/2}, \text{ say} \\
&\leq C_3 < \infty, \text{ for } t_i \in \Re, i = 1,\ldots,T,
\end{aligned}
$$

by a recursive argument, where

$$C_3 = C_2^{2(1-2^{-(T-1)})} \cdot exp[2^{T-2}t_1^2(\delta_0 + \delta_1\epsilon_0^2)].\square$$

To prove Theorem 2, we will require the following lemma.

Lemma 2 *If the ARCH model (11) (b) and (c) governs the innovation process $\{\epsilon_t\}$, then for a fixed positive integer T and as $\delta_1 \to 0$ we have that*
(i) the joint density of $\underline{\epsilon} = (\epsilon_1,\ldots,\epsilon_T)'$ is

$$
\begin{aligned}
& f(\epsilon_1,\ldots,\epsilon_T) \\
&= (2\pi\delta_0)^{-T/2} exp(-\frac{1}{2\delta_0}\sum_{i=1}^{T}\epsilon_i^2)\cdot[1 + \frac{\delta_1}{2}\Psi_{1,0}(\underline{\epsilon}) + \frac{\delta_1^2}{2}\Psi_{2,0}(\underline{\epsilon})] + O(\delta_1^3),
\end{aligned}
$$

(ii) the joint density of $(\epsilon_2,\ldots,\epsilon_T)$ given ϵ_1 is

$$
\begin{aligned}
& f(\epsilon_2,\ldots,\epsilon_T|\epsilon_1) \\
&= (2\pi\delta_0)^{-(T-1)/2} exp(-\frac{1}{2\delta_0}\sum_{i=2}^{T}\epsilon_i^2)\cdot[1 + \frac{\delta_1}{2}\Psi_{1,1}(\underline{\epsilon}) + \frac{\delta_1^2}{2}\Psi_{2,1}(\underline{\epsilon})] + O(\delta_1^3),
\end{aligned}
$$

where

$$
\begin{aligned}
\Psi_{1,k}(\underline{\epsilon}) &= \frac{1}{\delta_0^2}\sum_{i=k}^{T-1}|\epsilon_i|\epsilon_{i+1}^2 - \frac{1}{\delta_0}\sum_{i=k}^{T-1}|\epsilon_i|, \\
\Psi_{2,k}(\underline{\epsilon}) &= \frac{1}{4}[\Psi_{1,k}(\underline{\epsilon})]^2 - \frac{1}{\delta_0^3}\sum_{i=k}^{T-1}\epsilon_i^2\epsilon_{i+1}^2 + \frac{1}{2\delta_0^2}\sum_{i=k}^{T-1}\epsilon_i^2, \quad k = 0, 1.\ddagger
\end{aligned}
$$

Proof of Lemma 2

Taylor's series expansion is used to derive the results. From the conditional densities of (11) (b) and (c) we have

$$f(\epsilon_1,\ldots,\epsilon_T) = (2\pi)^{-T/2}[\prod_{i=0}^{T-1}(\delta_0 + \delta_1|\epsilon_i|)]^{-1/2}\cdot exp(-\frac{1}{2}\sum_{i=1}^{T}\frac{\epsilon_i^2}{\delta_0 + \delta_1|\epsilon_{i-1}|}). \tag{17}$$

When $\delta_1 = 0$, then $f(\epsilon_1,\ldots,\epsilon_T) = \phi(\underline{\epsilon})$ where $\phi(\underline{\epsilon}) = (2\pi\delta_0)^{-T/2}exp[-\sum_{i=1}^{T}\epsilon_i^2/(2\delta_0)]$. From (17),

$$\ln f = -\frac{T}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=0}^{T-1}\ln(\delta_0 + \delta_1|\epsilon_i|) - \frac{1}{2}\sum_{i=1}^{T}\frac{\epsilon_i^2}{\delta_0 + \delta_1|\epsilon_{i-1}|}.$$

Differentiating with respect to δ_1 gives

$$\frac{1}{f}\frac{df}{d\delta_1} = -\frac{1}{2}\sum_{i=0}^{T-1}\frac{|\epsilon_i|}{\delta_0+\delta_1|\epsilon_i|} + \frac{1}{2}\sum_{i=1}^{T}\frac{\epsilon_i^2|\epsilon_{i-1}|}{(\delta_0+\delta_1|\epsilon_{i-1}|)^2}. \tag{18}$$

Differentiating again on (18) gives

$$-\frac{1}{f^2}(\frac{df}{d\delta_1})^2 + \frac{1}{f}(\frac{d^2f}{d\delta_1^2}) = \frac{1}{2}\sum_{i=0}^{T-1}\frac{\epsilon_i^2}{(\delta_0+\delta_1|\epsilon_i|)^2} - \sum_{i=1}^{T}\frac{\epsilon_i^2\epsilon_{i-1}^2}{(\delta_0+\delta_1|\epsilon_{i-1}|)^3}. \tag{19}$$

Thus,

$$\frac{df}{d\delta_1} = \frac{f}{2}[\sum_{i=1}^{T}\frac{\epsilon_i^2|\epsilon_{i-1}|}{(\delta_0+\delta_1|\epsilon_{i-1}|)^2} - \sum_{i=0}^{T-1}\frac{|\epsilon_i|}{\delta_0+\delta_1|\epsilon_i|}], \tag{20}$$

so,

$$\frac{df}{d\delta_1}\bigg|_{\delta_1=0} = \frac{\phi(\underline{\epsilon})}{2}(\frac{1}{\delta_0^2}\sum_{i=0}^{T-1}|\epsilon_i|\epsilon_{i+1}^2 - \frac{1}{\delta_0}\sum_{i=0}^{T-1}|\epsilon_i|) = \frac{\phi(\underline{\epsilon})}{2}\Psi_{1,0}(\underline{\epsilon}).$$

Also,

$$\frac{d^2f}{d\delta_1^2} = \frac{1}{f}(\frac{df}{d\delta_1})^2 + f[\frac{1}{2}\sum_{i=0}^{T-1}\frac{\epsilon_i^2}{(\delta_0+\delta_1|\epsilon_i|)^2} - \sum_{i=1}^{T}\frac{\epsilon_i^2\epsilon_{i-1}^2}{(\delta_0+\delta_1|\epsilon_{i-1}|)^3}], \tag{21}$$

and

$$\frac{d^2f}{d\delta_1^2}\bigg|_{\delta_1=0} = \frac{\phi(\underline{\epsilon})}{4}[\Psi_{1,0}(\underline{\epsilon})]^2 + \phi(\underline{\epsilon})(\frac{1}{2\delta_0^2}\sum_{i=0}^{T-1}\epsilon_i^2 - \frac{1}{\delta_0^3}\sum_{i=0}^{T-1}\epsilon_i^2\epsilon_{i+1}^2) = \phi(\underline{\epsilon})\Psi_{2,0}(\underline{\epsilon}).$$

From Taylor's formula, as $\delta_1 \to 0$,

$$f(\epsilon_1,\ldots,\epsilon_T) = \phi(\underline{\epsilon}) + \frac{\phi(\underline{\epsilon})}{2}\Psi_{1,0}(\underline{\epsilon})\cdot\delta_1 + \frac{\phi(\underline{\epsilon})\Psi_{2,0}(\underline{\epsilon})}{2!}\cdot\delta_1^2 + O(\delta_1^3),$$

so result (i) is proved.

For $f(\epsilon_2,\ldots,\epsilon_T|\epsilon_1)$ in (ii), the derivation is essentially the same as $f(\epsilon_1,\ldots,\epsilon_T)$ with replacing $\prod_{i=0}^{T-1}$ and $\sum_{i=1}^{T}$ by $\prod_{i=1}^{T-1}$ and $\sum_{i=2}^{T}$ respectively. The proof of Lemma 2 is completed. □

Proof of Theorem 2

The proof of Theorem 2 is organized as follows. The first step is to write down the Taylor's expansion for $f(\underline{\epsilon})$ with an explicit form for the remainder. The second step is to derive the linear and quadratic order expressions (with respect to δ_1) for $V_0^{(T)}$. The boundedness of the remainder term is established in the last step.

For the first step, we differentiate (19) to give

$$\frac{2}{f^3}(\frac{df}{d\delta_1})^3 - \frac{3}{f^2}\frac{df}{d\delta_1}\frac{d^2f}{d\delta_1^2} + \frac{1}{f}\frac{d^3f}{d\delta_1^3} = -\sum_{i=0}^{T-1}\frac{|\epsilon_i|^3}{(\delta_0+\delta_1|\epsilon_i|)^3} + 3\sum_{i=1}^{T}\frac{\epsilon_i^2|\epsilon_{i-1}|^3}{(\delta_0+\delta_1|\epsilon_{i-1}|)^4}.$$

So, by substituting $df/d\delta_1$ and $d^2f/d\delta_1^2$ from (20) and (21), we have

$$\frac{d^3f}{d\delta_1^3} = \frac{3}{f}\frac{df}{d\delta_1}\frac{d^2f}{d\delta_1^2} - \frac{2}{f^2}(\frac{df}{d\delta_1})^3 + fT_3 = f_{\delta_1}(\underline{\epsilon}) \cdot (\frac{1}{8}T_1^3 + \frac{3}{2}T_1T_2 + T_3)$$

where

$$T_1 = \sum_{i=0}^{T-1} \frac{|\epsilon_i|\epsilon_{i+1}^2}{(\delta_0 + \delta_1|\epsilon_i|)^2} - \sum_{i=0}^{T-1} \frac{|\epsilon_i|}{\delta_0 + \delta_1|\epsilon_i|},$$

$$T_2 = \frac{1}{2}\sum_{i=0}^{T-1} \frac{\epsilon_i^2}{(\delta_0 + \delta_1|\epsilon_i|)^2} - \sum_{i=0}^{T-1} \frac{\epsilon_i^2\epsilon_{i+1}^2}{(\delta_0 + \delta_1|\epsilon_i|)^3},$$

$$T_3 = 3\sum_{i=0}^{T-1} \frac{|\epsilon_i|^3\epsilon_{i+1}^2}{(\delta_0 + \delta_1|\epsilon_i|)^4} - \sum_{i=0}^{T-1} \frac{|\epsilon_i|^3}{(\delta_0 + \delta_1|\epsilon_i|)^3}.$$

By Taylor's formula with Lagrange form of the remainder,

$$f(\underline{\epsilon}) = \phi(\underline{\epsilon})[1 + \frac{\delta_1}{2}\Psi_{1,0}(\underline{\epsilon}) + \frac{\delta_1^2}{2}\Psi_{2,0}(\underline{\epsilon})] + R_3(\delta_1, \underline{\epsilon}),$$

where

$$R_3(\delta_1, \underline{\epsilon}) = \frac{\delta_1^3}{3!} \cdot \left.\frac{d^3f}{d\delta_1^3}\right|_{\delta_1=\xi}, 0 < \xi < \delta_1.$$

The first step is done.

For the second step, by using (15)

$$V_0^{(T)} = E[exp(-\sum_{s=1}^{T} y_s)] = exp(-\sum_{s=1}^{T}\gamma_s)E[exp(\sum_{s=1}^{T}\pi_s\epsilon_s)],$$

where $\pi_s = -\sum_{i=0}^{T-s}\beta_i, 1 \le s \le T$. Let $e_i = \epsilon_i/\sqrt{\delta_0}, 0 \le i \le T$; $\pi_s^* = \sqrt{\delta_0}\,\pi_s, 1 \le s \le T$; $\phi(\underline{\epsilon}) = (2\pi\delta_0)^{-T/2}\, exp[-\sum_{i=1}^{T}\epsilon_i^2/(2\delta_0)]$, $\phi_0(\underline{e}) = (2\pi)^{-T/2}exp(-\sum_{i=1}^{T} e_i^2/2)$, and Φ is the distribution function of a standard normal variate. Hence

$$\begin{aligned} & E[exp(\sum_{s=1}^{T}\pi_s\epsilon_s)] \\ = & \int exp(\sum_{s=1}^{T}\pi_s\epsilon_s)[1 + \frac{\delta_1}{2}\Psi_{1,0}(\underline{\epsilon}) + \frac{\delta_1^2}{2}\Psi_{2,0}(\underline{\epsilon})]\phi(\underline{\epsilon})\prod_{i=1}^{T} d\epsilon_i + \int exp(\sum_{s=1}^{T}\pi_s\epsilon_s)R_3(\delta_1,\underline{\epsilon})\prod_{i=1}^{T} d\epsilon_i \\ = & \int exp(\sum_{s=1}^{T}\pi_s^* e_s)\phi_0(\underline{e})\prod_{i=1}^{T} de_i + \frac{\delta_1}{2}\int exp(\sum_{s=1}^{T}\pi_s^* e_s)\Upsilon_{1,0}(\underline{e})\,\phi_0(\underline{e})\prod_{i=1}^{T} de_i \\ & + \frac{\delta_1^2}{2}\int exp(\sum_{s=1}^{T}\pi_s^* e_s)\Upsilon_{2,0}(\underline{e})\,\phi_0(\underline{e})\prod_{i=1}^{T} de_i + \frac{\delta_1^3}{6}\int exp(\sum_{s=1}^{T}\pi_s\epsilon_s)\cdot\left.\frac{d^3f}{d\delta_1^3}\right|_{\delta_1=\xi}\prod_{i=1}^{T} d\epsilon_i \end{aligned} \tag{22}$$

where

$$\Upsilon_{1,0}(\underline{e}) = \frac{1}{\sqrt{\delta_0}}(\sum_{i=0}^{T-1}|e_i|e_{i+1}^2 - \sum_{i=0}^{T-1}|e_i|),$$

$$\Upsilon_{2,0}(\underline{e}) = \frac{1}{\delta_0}[\frac{1}{4}(\sum_{i=0}^{T-1}|e_i|e_{i+1}^2)^2 - \frac{1}{2}\sum_{i=0}^{T-1}|e_i|\cdot\sum_{i=0}^{T-1}|e_i|e_{i+1}^2 + \frac{1}{4}(\sum_{i=0}^{T-1}|e_i|)^2 - \sum_{i=0}^{T-1}e_i^2e_{i+1}^2 + \frac{1}{2}\sum_{i=0}^{T-1}e_i^2]$$

By viewing $e_i \sim iid\, N(0,1)$, the first integral of (22) is $\prod_{s=1}^{T} M_\Phi(\pi_s^*) = exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2) = exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1}\zeta_l^2)$, by noting $\zeta_l = \sum_{i=0}^{l}\beta_i$, i.e. $\pi_s = -\zeta_{T-s}$, and after re-indexing. The second integral (linear order term with respect to δ_1) is

$$\frac{1}{\sqrt{\delta_0}}\{\sum_{i=0}^{T-1} E[exp(\sum_{s=1}^{T}\pi_s^* e_s)\cdot|e_i|e_{i+1}^2] - \sum_{i=0}^{T-1} E[exp(\sum_{s=1}^{T}\pi_s^* e_s)\cdot|e_i|]\}.$$

For $i = 0$,

$$E[exp(\sum_{s=1}^{T}\pi_s^* e_s)\cdot|e_i|] = |e_0|exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2) = \frac{|\epsilon_0|}{\sqrt{\delta_0}}exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2).$$

For $i = 1, \ldots, T$,

$$\begin{aligned} E[exp(\sum_{s=1}^{T}\pi_s^* e_s)\cdot|e_i|] &= E(|e_i|\cdot e^{\pi_i^* e_i})\cdot E[exp(\sum_{\substack{s=1\\ s\neq i}}^{T}\pi_s^* e_s)] \\ &= e^{(\pi_i^*)^2/2}q(\pi_i^*)\cdot exp(\frac{\delta_0}{2}\sum_{\substack{s=1\\ s\neq i}}^{T}\pi_s^2) = exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2)\cdot q(\sqrt{\delta_0}\pi_i), \end{aligned}$$

since for $X \sim N(0,1), E(|X|e^{aX}) = e^{a^2/2}q(a)$ where $q(a) = \frac{2}{\sqrt{2\pi}}e^{-a^2/2} + a[2\Phi(a) - 1]$. Define π_0 such that $q(\sqrt{\delta_0}\pi_0) = |\epsilon_0|/\sqrt{\delta_0}$. Then

$$E[exp(\sum_{s=1}^{T}\pi_s^* e_s)\cdot|e_i|] = exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2)\cdot q(\sqrt{\delta_0}\pi_i), i = 0, 1, \ldots, T.$$

Similar technique is used to find that

$$E[exp(\sum_{s=1}^{T}\pi_s^* e_s)\cdot|e_i|e_{i+1}^2] = exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2)\cdot q(\sqrt{\delta_0}\pi_i)\cdot(1+\delta_0\pi_{i+1}^2), i = 0, 1, \ldots, T-1,$$

by using, for $X \sim N(0,1), E(X^2 e^{aX}) = e^{a^2/2}(1+a^2)$. Hence the second integral becomes

$$\begin{aligned} & \frac{1}{\sqrt{\delta_0}}exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2)\sum_{i=0}^{T-1}[q(\sqrt{\delta_0}\pi_i)\cdot(1+\delta_0\pi_{i+1}^2) - q(\sqrt{\delta_0}\pi_i)] \\ = \; & \sqrt{\delta_0}exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2)\sum_{i=0}^{T-1}q(\sqrt{\delta_0}\pi_i)\cdot\pi_{i+1}^2 \\ = \; & \sqrt{\delta_0}exp(\frac{\delta_0}{2}\sum_{s=1}^{T}\pi_s^2)[\frac{|\epsilon_0|}{\sqrt{\delta_0}}\pi_1^2 + \sum_{i=1}^{T-1}q(\sqrt{\delta_0}\pi_i)\cdot\pi_{i+1}^2] \\ = \; & exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1}\zeta_l^2)\cdot[|\epsilon_0|\zeta_{T-1}^2 + \sqrt{\delta_0}\sum_{l=0}^{T-2}\zeta_l^2 q(\sqrt{\delta_0}\zeta_{l+1})] \\ = \; & exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1}\zeta_l^2)\cdot C_1^{(T)}(\epsilon_0), \end{aligned}$$

by noting $\pi_s = -\zeta_{T-s}, q(a) = q(-a)$, and re-indexing. The linear order term for $V_0^{(T)}$ is proved. For the quadratic order term (the third integral of (22)) the derivations are more tedious but the ideas are the same as for the linear order term. That proof is omitted. The third integral can be proved to be equal to $exp(\frac{\delta_0}{2}\sum_{l=0}^{T-1}\zeta_l^2)\cdot C_2^{(T)}(\epsilon_0)$. The second step is done.

For the last (remainder) term in (22), by using calculus, it is easy to show that there exist constants a_1, a_2 such that

$$\frac{|\epsilon_i|}{(\delta_0+\xi|\epsilon_i|)^2}\le a_1, \text{ and, } \frac{|\epsilon_i|}{(\delta_0+\xi|\epsilon_i|)}\le a_2, \text{ for } \epsilon_i\in\Re, 0\le a_1,a_2<\infty.$$

Therefore,

$$\begin{aligned}\left|\frac{1}{8}T_1^3+\frac{3}{2}T_1T_2+T_3\right| &\le \frac{1}{8}(a_1\sum_{i=1}^{T}\epsilon_i^2+Ta_2)^3+\frac{3}{2}(a_1\sum_{i=1}^{T}\epsilon_i^2+Ta_2)(\frac{1}{2}Ta_2^2+a_1a_2\sum_{i=1}^{T}\epsilon_i^2)\\ &\quad+(3a_1a_2^2\sum_{i=1}^{T}\epsilon_i^2+Ta_2^3).\end{aligned}\tag{23}$$

We claim that

$$R:=\int exp(\sum_{s=1}^{T}\pi_s\epsilon_s)\cdot\frac{d^3f}{d\delta_1^3}\Big|_{\delta_1=\xi}\prod_{i=1}^{T}d\epsilon_i=O(1).$$

Because

$$|R|\le\int exp(\sum_{s=1}^{T}\pi_s\epsilon_s)f_\xi(\underline{\epsilon})\cdot\left|\frac{1}{8}T_1^3+\frac{3}{2}T_1T_2+T_3\right|\prod_{i=1}^{T}d\epsilon_i,$$

and note (23), it suffices to prove that for $i=1,\ldots,T$ and all $r>0$,

$$\int|\epsilon_i|^r\cdot exp(\sum_{s=1}^{T}\pi_s\epsilon_s)f_\xi(\underline{\epsilon})\prod_{i=1}^{T}d\epsilon_i=E[|\epsilon_i|^r\cdot exp(\sum_{s=1}^{T}\pi_s\epsilon_s)]<\infty.$$

It is obviously true since

$$\begin{aligned}E[|\epsilon_i|^r\cdot exp(\sum_{s=1}^{T}\pi_s\epsilon_s)] &\le \{E[|\epsilon_i|^{2r}\}^{1/2}\{E[exp(2\sum_{s=1}^{T}\pi_s\epsilon_s)]\}^{1/2}, \text{ by the Hölder inequality}\\ &< \infty, \text{ by Lemma 1.}\end{aligned}$$

Result (i) follows.

The proof for $V_1^{(T-1)}$ follows the same steps as for $V_0^{(T)}$, by noting that from (16)

$$V_1^{(T-1)}=E[exp(-\sum_{s=2}^{T}y_s)|\epsilon_1]=exp(-\sum_{s=2}^{T}\gamma_s-\epsilon_1\sum_{i=1}^{T-1}\beta_i)\cdot E[exp(\sum_{s=2}^{T}\pi_s\epsilon_s)|\epsilon_1]$$

and using Lemma 2(ii). □

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