"Makeham-Type" Mortality Models

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Abstract

This paper presents two types of mortality models based on the Makeham Law of Mortality, called the **Inverse-Makeham Model** and the **Modified-Makeham Select Model.**

The Makeham Law of Mortality, given as

$$\mu(x) = Bc^{x} + A$$

has Inverse-Makeham form as

$$\mu_x = \frac{\left[\frac{1}{\sigma}\exp\left\{-\frac{x-m}{\sigma}\right\}\right]}{\exp\left\{e^{-(x-m)/\sigma}\right\}-1} + e^{-D/\sigma}$$

where σ is the dispersion of the informative parametrization of the Makeham model about its mode m, and the Modified-Makeham Select form, given as

$$\mu'_{[x-t]+t} = s^{t}\left[\left(B_s c_s^{x} + A_s\right) - \left(B_u c_u^{x} + A_u\right)\right] + \left[B_u c_u^{x} + A_u\right] \text{ or,}$$

$$\mu'_{[x-t]+t} = s^{t}\left[\mu_{[x]} - \mu_x\right] + \left[\mu_x\right]$$

with u meaning ultimate mortality and s meaning select mortality.

We investigate the effectiveness of these models in exhibiting the pattern of mortality by simulating the nonlinear models and estimating its parameters via nonlinear regression, using the NLIN procedure of the SAS software and comparing the theoretical values given by the models to the actual values in the existing mortality tables. The correctness of these models to exhibit the behavior of mortality gives alternative mortality models that can be used as benchmarks for constructing new ones utilizing other unchartered parametric distributions.

Keywords: Makeham Law of Mortality, ultimate mortality rate, select mortality rate

1. Introduction

Mortality tables are a familiar tool for most actuaries. Mortality forecasts are based on these tables, aside from the underwriting information and the existing analytic mortality models. But mortality models are of equal importance. Bowers [1] discussed that mortality models are of analytic form for three reasons: philosophical, practical and ease of estimation. The philosophical justification provides that, since many physical phenomena are explained efficiently by simple formulas, using biological arguments, human survival can be governed by an equally simple law, one that's easier to estimate and interpret for practical purposes. For practical reasons, using the statistical concept of model parsimony makes it much easier to communicate a function with few parameters than to communicate a life table with perhaps 100 parameters or mortality probabilities. Lastly, a simple analytic survival function will have fewer parameters to be estimated from the mortality data. In addition, analytic mortality models fit to age ranges where accurate data is abundant can be used to predict mortality at advanced ages, where the data is sparse and unreliable.

With the changes in mortality patterns, specifically the decelerating rate of increase in mortality at high ages in many developed countries, new mortality models are needed to describe the new conditions.

Like the other countries in the world, Philippine studies show an increase in the life expectancy of the Filipinos. Based on the 2000 Report on the Philippine Human Development Index released recently by the National Statistical Coordination Board, the average life expectancy for the Philippines was estimated 67.4, 68 and 68.9 years for 1994, 1997 and 2000, respectively, with an average growth rate of 1.1 percent per year [8].

While insurance industries in the Philippines do not yet have mortality data that include these changes, the models in this paper may be used to forecast mortality for advanced ages near the tail end of the available data.

Forecasting is not a problem within the range of the data set, since interpolation variance is small when the fit is good. However, when we extrapolate outside the range, the prediction variance can be volatile. In practice, we do not usually extrapolate so far from the boundary, since the prediction variance becomes very large especially in nonlinear cases, as in the models presented in this paper. The purpose of this research is to benchmark plausible models that would fit well to the available data at hand. The models and the estimations can always be further enhanced and continually improved when data is updated.

In this paper, two modifications of the Makeham Law of Mortality will be derived and their effectiveness in showing the mortality patterns of the 1993 Philippine Intercompany Mortality (PICM) Table will be demonstrated. The models are called the Inverse Makeham and the Modified-Makeham Select.

2. Some Preliminaries

Consider the survival function, $s(x)$. The force of mortality, $\mu(x)$, is given as

$$\mu(x) = -\frac{s'(x)}{s(x)} \qquad (2.1)$$

Alternatively, the survival function may be defined as

$$s(x) = \exp\left\{-\int_0^x \mu_s ds\right\}. \qquad (2.2)$$

The Makeham Law of Mortality is given as

$$\mu(x) = Bc^x + A \qquad (2.3)$$

where $B > 0,\ A \geq -B,\ c > 1,\ x \geq 0.$ This is basically the Gompertz Law, but with the addition of the accident hazard, A.

3. Our Mortality Models

3.1 The Inverse-Makeham Model

Jacques F. Carriere [3] developed a model using the informative parametrization of the Gompertz model instead of using its standard parametrization. It is important to note that the transformation from the standard to the informative parametrization has practical scientific basis. The standard Makeham Law given in equation (2.3) given by (2.3) is centered by the location parameter *m* and scaled by the variance *s,* where these parameters can be explained in practical terms. Willekens [11] showed that the Gompertz function is a type of extreme value distribution and argued that the application of extreme value theory may uncover new theories of aging. We can say the same thing about the Makeham function, since it is a generalized Gompertz function. The Makeham Law, being a simple extension of the Gompertz Law with the addition of the accident term, the constant *A,* has the same assumption as Gompertz.

Adapting the parametrization method developed by Carriere, the Makeham Model given in equation (2.3) the Makeham model (2.3) is transformed to its informative parametrization, as

$$\mu(x) = \frac{1}{\sigma}\exp\left\{\frac{x-m}{\sigma}\right\} + \exp\left\{-\frac{D}{\sigma}\right\}. \qquad (3.1)$$

where $B = \frac{1}{\sigma}\exp\left(-\frac{m}{\sigma}\right)$, $c = \exp\left\{\frac{1}{\sigma}\right\}$ and $A = \exp\left\{-\frac{D}{\sigma}\right\}$, with $m > \sigma$.

The corresponding survival function, $s(x)$, of the Makeham model is thus

$$\begin{aligned} s(x) &= \exp\left\{-\int_0^x \left[\frac{1}{\sigma}\exp\left\{\frac{s-m}{\sigma}\right\} + e^{-D/\sigma}\right] ds\right\} \\ &= \exp\left\{-\left[\int_0^x \frac{1}{\sigma}\exp\left\{\frac{s-m}{\sigma}\right\} ds + \int_0^x e^{-D/\sigma}\, ds\right]\right\} \\ &= \exp\left\{e^{-m/\sigma} - e^{x-m/\sigma} - e^{-D/\sigma}x\right\} \qquad (3.2) \end{aligned}$$

The probability density function is

$$f(x) = \left[\frac{1}{\sigma}\exp\left\{\frac{x-m}{\sigma}\right\} + e^{-D/\sigma}\right]\left[\exp\left\{e^{-m/\sigma} - e^{x-m/\sigma} - e^{-D/\sigma}x\right\}\right] \qquad (3.3)$$

The nonlinear model we have just constructed has three parameters: D, m and σ. Here, A is the accident parameter, $\mathrm{m} > 0$ is a location parameter and $\sigma > 0$ is the dispersion parameter since

$$\lim_{\sigma \to 0} s(m-\varepsilon) - s(m+\varepsilon) = 1 \qquad (3.4)$$

The Inverse-Makeham Model associated with equation (3.2) has survival function given by

$$s(x) = \frac{\left[1-\exp\left\{-e^{-(x-m)/\sigma}\right\}\right]\left[\exp\left\{-e^{-D/\sigma}x\right\}\right]}{1-\exp\left\{-e^{m/\sigma}\right\}}. \qquad (3.5)$$

Here, $s(0) = 1$ and $s(\infty) = 0$.

Now, the force of mortality is then

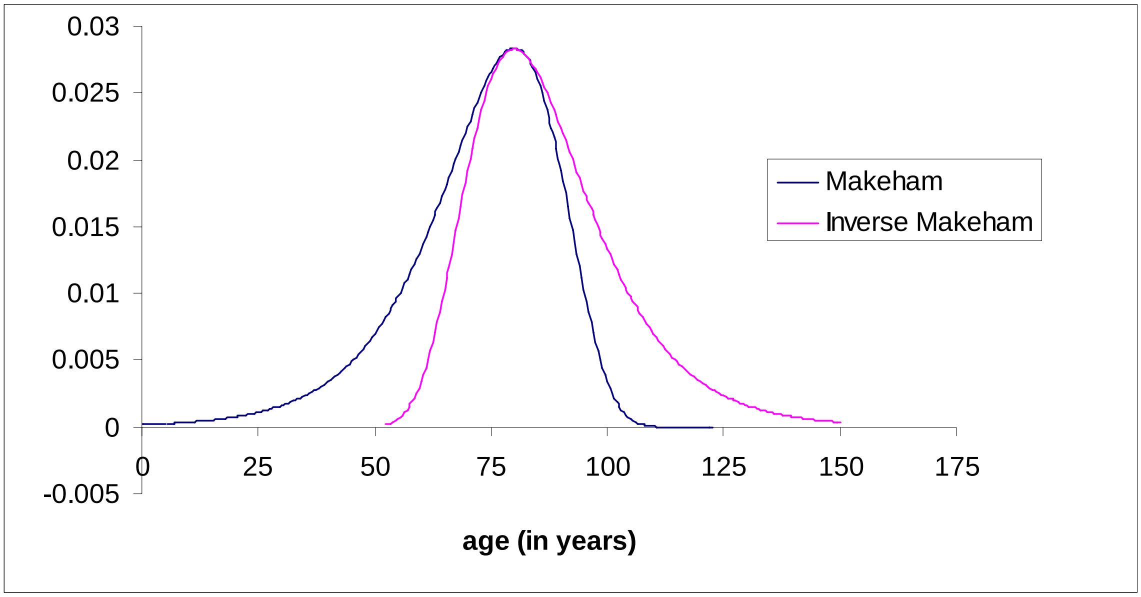
$$\mu_x = \frac{\left[\frac{1}{\sigma}\exp\left\{-\frac{x-m}{\sigma}\right\}\right]}{\exp\left\{e^{-(x-m)/\sigma}\right\}-1} + e^{-D/\sigma} \qquad (3.6)$$

and we call this the Inverse-Makeham Model. And the probability density function is

$$f(x) = \frac{\frac{1}{\sigma}\left(\exp\left\{-e^{-(x-m)/\sigma} - \frac{x-m}{\sigma} - e^{-D/\sigma}x\right\}\right) + \left(\exp\left\{-e^{-D/\sigma}x - \frac{D}{\sigma}\right\}\right)\left(1-\exp\left\{-e^{-(x-m)/\sigma}\right\}\right)}{1-\exp\left\{-e^{m/\sigma}\right\}}. \qquad (3.7)$$

Figure 1 gives the density function, $f(x)$, plots for the Makeham and the Inverse-Makeham models with parameter values $m = 80$, $\sigma = 13$ and $D = 234$. The Makeham is skewed to the left, while the Inverse-Makeham is skewed to the right. It is also noticeable that the Inverse-Makeham density is approximately equal to the Makeham density reflected around m.

Figure 1. Graphs of the Makeham and the Inverse-Makeham Density Functions



3.2 The Modified-Makeham Select Model

Tenenbein and Vanderhoof [10] proposed a formula that describes how mortality rates for an initially select group move from the select Gompertz curve to the ultimate Gompertz curve. They based their algorithm on the study by David Brillinger, who pointed out that the probability of death is the sum of the probabilities of failure of the individual system.

Tenenbein and Vanderhoof's algorithm is as follows: Beginning with a group of select lives, some will die at the end of the first year according to a specified law of mortality. For the survivors, a certain proportion s will remain select and the rest $(1-s)$ will become ultimate. The mortality rate for the combined group in the second year will then be

$$q_{[x]+1} = s^t q_{[x+1]} + (1 - s^t) q_{x+1}. \qquad (3.8)$$

For the succeeding years, this splitting pattern continues so that

$$q_{[x]+t} = s^t q_{[x+t]} + (1 - s^t) q_{x+t}. \qquad (3.9)$$

The corresponding force of mortality will be

$$\mu_{[x]+t} = s^t \mu_{[x+t]} + (1-s^t)\mu_{x+t}. \qquad (3.10)$$

Assuming the Makeham Law, equation (3.10) becomes

$$\mu_{[x]+t} = s^t\left[B_s c_s^{x+t} + A_s\right] + (1-s^t)\left[B_u c_u^{x+t} + A_u\right] \qquad (3.11)$$

$$= s^t\left[\left(B_s c_s^{x+t} + A_s\right) - \left(B_u c_u^{x+t} + A_u\right)\right] + \left[B_u c_u^{x+t} + A_u\right] \qquad (3.12)$$

With a change in variables, (3.12) may be rewritten as

$$\mu'_{[x-t]+t} = \mathrm{s}^{\mathrm{t}}\left[\left(\mathrm{B}_{\mathrm{s}} c_s^x + A_s\right) - \left(\mathrm{B}_{\mathrm{u}} c_u^x + A_u\right)\right] + \left[\mathrm{B}_{\mathrm{u}} c_u^x + A_u\right]$$

$$= \mathrm{s}^{\mathrm{t}}\left[\mu_{[x]} - \mu_x\right] + \left[\mu_x\right] \qquad (3.13)$$

and we call equation (3.13) the Modified-Makeham Select Mortality Model.

In the discussion of Vanderhoof's paper, Smith [10] said that "the healthy lives would be zeroth-duration or first-policy-year standard select risks, and the lives in average health would be standard ultimate risks. If the ultimate lives remain ultimate while the select lives tend to become ultimate in increasing proportion with advance in duration, then we can express the mortality of a cohort at duration *t* as the weighted mean of zeroth-duration select mortality rates and ultimate mortality rates."

We say that equation (3.13) is a weighted arithmetic mean of the select and ultimate μ's. It is worth noting that as time increases to infinity, equation (3.13) reverts to the ultimate function, i.e.,

$$\lim_{t\to\infty}\mu'_{[x-t]+t} = \lim_{t\to\infty} \mathrm{s}^{\mathrm{t}}\left[\left(\mathrm{B}_{\mathrm{s}} c_s^x + A_s\right) - \left(\mathrm{B}_{\mathrm{u}} c_u^x + A_u\right)\right] + \left[\mathrm{B}_{\mathrm{u}} c_u^x + A_u\right]$$

$$= \left[\left(\mathrm{B}_{\mathrm{s}} c_s^x + A_s\right) - \left(\mathrm{B}_{\mathrm{u}} c_u^x + A_u\right)\right]\lim_{t\to\infty}\left(s^t\right) + \left[\mathrm{B}_{\mathrm{u}} c_u^x + A_u\right]$$

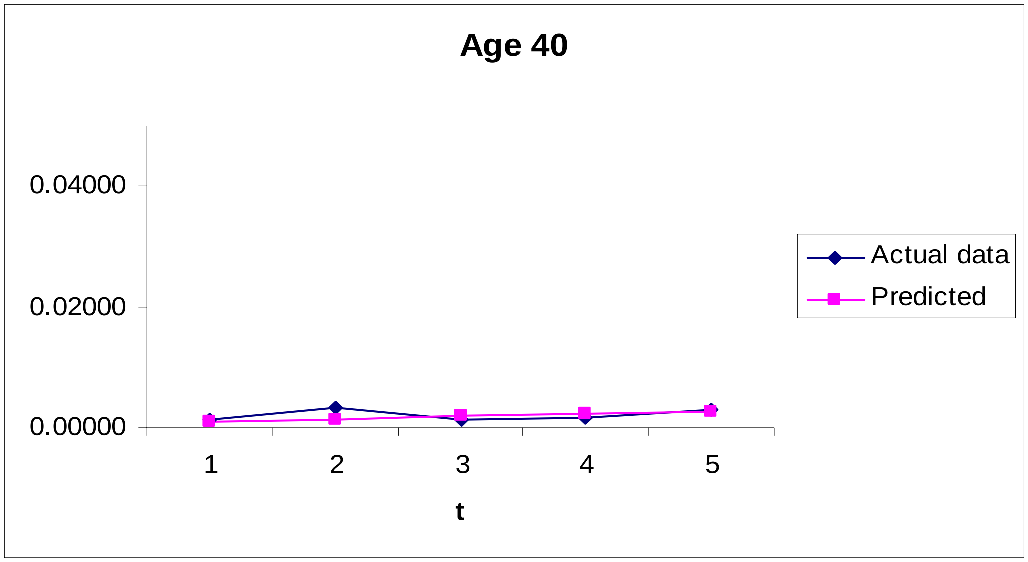
Since $0 < s < 1$, then the first term approaches 0. Therefore,

$$\lim_{t\to\infty}\mu'_{[x-t]+t} = B_u c_u^x + A_u \qquad (3.14)$$

which is the ultimate mortality rate.

Figure 2 shows the graph of the predicted values of $\mu'_{[\mathrm{x-t}]+t}$ against the actual data from the 1993 PICM.

Figure 2. Graphs of the Predicted Mortality Rates against Actual Data for Age 40



4. Methodology

To evaluate the efficiency of the models to describe actual mortality patterns, they were fitted in the 1993 Philippine Intercompany Mortality Table (PICM). This table shows the latest published mortality patterns in the Philippines based on policy exposures and deaths compiled from 90 percent of the insurance companies in the country. The data were gathered through actuarial mortality studies as a joint project of the Insurance Institute for Asia and the Pacific (IIAP), Philippine Life Insurance Association (PLIA) and the Actuarial Society of the Philippines (ASP). The data consists of mortality by age at selection with five-year durations, which we consider as the select period. The 1993 PICM is shown in Appendix I.

The data were analyzed using the NLIN procedure of the SAS software. The methods used were Newton, Gauss-Newton and Marquardt. The DUD is not used here, because it can only be really useful when the three methods mentioned here all fail to converge to some useful estimates.

The Inverse-Makeham Model was fitted in the Ultimate Mortality Rates for all the data available in the 1993 PICM, that is, ages 6 to 92 years, while the Modified-Makeham Select Model was fitted for the age range 25 to 61 years, the interval with complete set of data. This is believed to be a valid range because in practice, many Philippine insurance premiums are computed from ages 20 to 25 until 60 to 65. Age 21 is the age when the majority of the Filipinos start to be gainfully employed, and age 65 is the compulsory retirement age.

The Analysis of Variance (ANOVA) table that contains the Goodness-of-Fit Test is included in the SAS output. This validates the null hypothesis that our models do not fit the data against the alternative hypothesis that the models do fit the data. To obtain a conclusion from this test, it makes use of the F-distribution. The decision rule is to reject the null hypothesis if the p-value is less than 0.05.

5. Results and Discussion

Tables 5.1 to 5.3 are the ANOVA tables for the different methods applied to the 1993 PICM in validating the Inverse-Makeham Model.

TABLE 5.1
SAS Output—ANOVA Table for the Newton Method

Source	DF	Sum of Squares	Mean Square	Approx F-value	p-value
Regression	3	0.1184	0.0395	227.35	<.0001
Residual	73	0.0127	0.000174		
Uncorrected Total	76	0.1311			
Corrected Total	75	0.0870			

TABLE 5.2
SAS Output—ANOVA Table for the Gauss-Newton Method

Source	DF	Sum of Squares	Mean Square	Approx F-value	p-value
Regression	3	0.1184	0.0395	218.23	<.0001
Residual	74	0.0127	0.000171		
Uncorrected Total	77	0.1311			
Corrected Total	76	0.0875			

TABLE 5.3
SAS Output—ANOVA Table for the Marquardt Method

Source	DF	Sum of Squares	Mean Square	Approx F-value	p-value
Regression	3	0.1184	0.0395	218.23	<.0001
Residual	74	0.0127	0.000171		
Uncorrected Total	77	0.1311			
Corrected Total	76	0.0875			

Table 5.4 gives the estimates for the parameters in the Inverse-Makeham model. It is worth noting that, for the three methods used, the estimates for the parameters obtained by the Newton method vary slightly from the estimates for the Gauss-Newton and the Marquardt methods, which give values equal to each other.

TABLE 5.4
Parameter Estimates for the Inverse-Makeham Model

Parameter	Newton	Gauss-Newton	Marquardt
D	53.4970	53.6548	53.6548
m	78.8117	78.7930	78.7930
σ	9.5828	9.5816	9.5816

When parameters D, m and σ were estimated for equation (3.1), the parametrized Makeham Model, using the three NLIN procedures, the results give equal values for the Gauss-Newton and the Marquardt methods. While the parameter estimates differ in the Newton method, the large difference is in the D value only, with the m and σ having values close to those obtained from the other two methods. Table 5.5 summarizes this.

TABLE 5.5
Parameter Estimates for the Parametrized Makeham Model

Parameter	Newton	Gauss-Newton	Marquardt
D	415.5	3195.6	3195.6
m	81.3537	81.3536	81.3536
σ	16.6103	16.6104	16.6104

The parameter estimates were likewise obtained for the original Makeham Model, equation (2.3). Table 5.6 gives these values. However, since the Gauss-Newton and the Marquardt methods failed to satisfy the specific initial condition, $A \geq -B$, of equation (2.3), then we used the Newton estimates to compute the mortality rates found in Appendix 3.

TABLE 5.6
Parameter Estimates for the Original Makeham Model

Parameter	Newton	Gauss-Newton	Marquardt
A	0.00701	-0.00408	-0.00408
B	4.524E-6	0.000809	0.000809
c	1.1180	1.0555	1.0555

In validating the Modified-Makeham Select Model, Tables 5.7 to 5.9 present the ANOVA tables for the different methods applied to the 1993 PICM.

TABLE 5.7
SAS Output—ANOVA Table for the Newton Method

Source	DF	Sum of Squares	Mean Square	Approx F-value	p-value
Regression	6	0.00582	0.000969	110.59	<.0001
Residual	179	0.000866	4.84E-6		
Uncorrected Total	185	0.00668			
Corrected Total	184	0.00354			

TABLE 5.8
SAS Output—ANOVA Table for the Gauss-Newton Method

Source	DF	Sum of Squares	Mean Square	Approx F-value	p-value
Regression	6	0.00582	0.000969	110.59	<.0001
Residual	179	0.000866	4.84E-6		
Uncorrected Total	185	0.00668			
Corrected Total	184	0.00354			

TABLE 5.9
SAS Output—ANOVA Table for the Marquardt Method

Source	DF	Sum of Squares	Mean Square	Approx F-value	p-value
Regression	6	0.00582	0.000969	110.59	<.0001
Residual	179	0.000866	4.84E-6		
Uncorrected Total	185	0.00668			
Corrected Total	184	0.00354			

With regard to the Modified-Makeham Select Model, Table 5.10 gives the estimates for the parameters. Again it is worth noting that, for the three methods used, the estimates for each of the parameters are approximately equal.

TABLE 5.10
Parameter Estimates for Each Method

Parameter	Newton	Gauss-Newton	Marquardt
As	-42.1820	-43.2080	-43.2080
Au	0.00191	0.00191	0.00191
Bs	42.1823	43.2084	43.2084
Bu	0.00002	0.00002	0.00002
Cs	1.0000	1.0000	1.0000
Cu	1.1295	1.1295	1.1295
S	0.8487	0.8487	0.8487

Notice that, using the same initial conditions, all the methods of nonlinear estimation converge to the same optimal estimates. This is a very good indicator that of the models developed in this paper, the Inverse-Makeham Model fits well in the Ultimate Mortality Rates of the 1993 PICM, while the Modified-Makeham Select Model fits well in the core of the 1993 PICM. As in solving linear systems of equations, nonlinear optimization, or in our particular case, nonlinear estimation applied to a particular problem, may give none, one or many (even infinite) solutions. Although this may not be theoretically surprising, our optimal estimates, which may generally be only local in the nonlinear surface being optimized, can be useful as benchmarks for making the further enhancements and continuous improvements in mortality table constructions. This is especially true when data can be updated with better data that is more complete, precise and accurate, and not messy with missing values that can spoil or bias the estimation process. In economic terms, our estimated model can be useful in improving the insurance industry's continuous updates and improvements of its operations for better risk management and profitability.

6. Conclusion

Since data for advanced ages is not available in the Philippines, we can use the models developed in this paper to forecast these mortality rates. However, the actuary must be cautioned that regular revision of the parameter values is necessary while the mortality table is being updated or once data is available in order to minimize prediction variance. It is now safer to consider older clients for insurance coverage.

The missing values from the original data were likewise interpolated. This will be a useful tool to the actuary. While the available data were also predicted and some of the values do not conform with actual, the actuary can be helped by comparing these two values and deciding which one is more acceptable. The actuary can also use the predicted values to modify the mortality table, especially if she has doubts about the accuracy of the data collected.

It was also noted that the Inverse-Makeham Model did not give values for ages 0 to 15. Therefore the model's best use will be for the advanced age mortality rates.

Appendix 2 gives the predicted values for the 1993 PICM using the models developed in this paper. The select mortality rates are estimated using the Modified-Makeham Select Model, while the ultimate mortality rates are estimated using the parametrized Makeham and Inverse-Makeham Models.

Since different models were used for the select and the ultimate years, the values are expectedly not consistent for the select and the ultimate years. The purpose of Appendix 2 is illustrative for the models developed.

From Appendix 2, we can say that the Modified-Makeham Select Model is not appropriate for ages above 91 because it produces very large force of mortality rates after the second select year.

Focusing on the mortality rates for advanced ages generated by the Makeham Models presented in this paper, Appendix 3 gives the mortality rates for ages 90 to 99 approximated through the original Makeham, the parametrized Makeham and the Inverse-Makeham Models. It is clearly seen that the Inverse-Makeham Model gives the lowest mortality rates at all ages in this range. Since this model fits well in the 1993

PICM, the Filipino actuary may consider these rates for premium computation because they will result in lower premium rates and, thus, a better position for competition.

7. Recommendation

The models presented in this study, having described the underlying behavior of the 1993 PICM, may now be simulated for other mortality tables that exhibit similar characteristics as the data in the 1993 PICM. This will give the actuary some alternative mortality models for predicting the probability of mortality, aside from the known parametric models.

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APPENDIX 1
1993 Philippine Intercompany Force of Mortality Table $\left(\mu_{[x]+t}\right)$ Values

	Select Year					Ultimate Year
Age	**1**	**2**	**3**	**4**	**5**	
0	0.0006618089	-	-	-	-	-
1	0.0015704725	-	-	-	-	-
2	0.0009773575	-	-	-	-	-
3	0.0007454378	-	0.0008206767	-	-	-
4	0.0004309428	0.0004967734	-	0.0004346845	-	-
5	-	-	0.0004346845	-	0.0006923997	-
6	-	-	-	-	-	0.0002354077
7	-	0.0012453351	0.0005861618	-	-	0.0007347499
8	-	0.0020429053	-	-	0.0006472494	0.0006187114
9	0.0005737245	0.0006961322	-	-	-	0.0003713189
10	0.0006027716	0.0007057190	0.0007132643	-	0.0006798110	0.0002147631
11	0.0021111870	0.0014433511	-	-	-	0.0002512316
12	0.0014482282	0.0008680567	-	0.0007334089	-	0.0002355277
13	0.0007155660	-	0.0009526036	-	-	0.0001515215
14	-	0.0008853518	0.0018735340	-	-	0.0004256106
15	0.0011562882	-	-	0.0009920619	0.0009930529	0.0004150661
16	0.0015255531	0.0019323658	0.0009930529	0.0010235436	0.0010460269	0.0007193687
17	-	-	-	0.0011068023	0.0010899137	0.0004778942
18	0.0007668740	-	-	-	0.0012422312	0.0002983145
19	0.0013395868	-	0.0023094748	0.0008752729	0.0018885722	0.0008723604
20	0.0005188046	0.0018231509	-	0.0011370061	-	0.0011365556
21	0.0006983338	-	0.0023571058	-	0.0011976068	0.0004839371
22	0.0014612771	0.0010465775	0.0010875512	0.0011111071	0.0032123340	0.0008950404
23	0.0009525635	-	0.0013995790	0.0050864843	0.0025873242	0.0013990382
24	0.0011941327	0.0016409356	0.0007097218	0.0015631110	0.0011668605	0.0005111206
25	0.0008163932	0.0015482679	-	0.0015560200	0.0023337210	0.0010073872
26	0.0006432468	0.0014117661	0.0015847851	0.0016086832	0.0003319551	0.0013829859
27	0.0003172603	0.0001082859	0.0013373438	0.0010043542	0.0004747227	0.0005754255
28	0.0011277156	0.0013972357	0.0010025023	0.0020759232	0.0020296383	0.0013917881
29	0.0005658100	0.0008334172	0.0019149223	0.0013484387	0.0041520077	0.0013901158
30	0.0009916415	0.0008267016	0.0010291994	0.0022786642	0.0009729231	0.0009653058
31	0.0013306750	0.0021204466	0.0010467076	0.0018533263	0.0013069737	0.0015947509

Age	Select Year 1	2	3	4	5	Ultimate Year
32	0.0011527341	0.0010964909	0.0018481868	0.0019509519	0.0022172964	0.0014857031
33	0.0006836636	0.0015509020	0.0016766849	0.0002654652	0.0016774662	0.0015223782
34	0.0007042779	0.0009353473	0.0016122490	0.0008571472	0.0016276038	0.0015766122
35	0.0007160763	0.0030736488	0.0020164216	0.0021848350	0.0023174933	0.0014542970
36	0.0010720544	0.0005708529	0.0015915458	0.0009157492	0.0035705167	0.0012977317
37	0.0012389672	0.0018334197	0.0028607781	0.0011986781	0.0006709150	0.0020571244
38	0.0009407323	0.0013599844	0.0015926075	0.0011061115	0.0023768224	0.0012779963
39	0.0009626932	0.0019420145	0.0017887589	0.0025649266	0.0030338174	0.0020702415
40	0.0013240662	0.0031609204	0.0016658367	0.0017179148	0.0029617918	0.0028280351
41	0.0012437832	0.0018498499	0.0013910671	0.0037027266	0.0034331565	0.0020448192
42	0.0011038590	0.0033277910	0.0009892792	0.0029900959	0.0045693938	0.0026935243
43	0.0015235600	0.0015405961	0.0041562854	0.0018587364	0.0033348545	0.0022782432
44	0.0011936721	0.0036417230	0.0053547612	0.0029202799	0.0035328833	0.0029217341
45	0.0018407732	0.0027444927	0.0061614528	0.0029824029	0.0033876817	0.0024083578
46	0.0035695232	0.0027774836	0.0040136539	0.0034328856	0.0074159705	0.0026550515
47	0.0030586930	0.0030375387	0.0034744891	0.0050077579	0.0061972031	0.0030171771
48	0.0013930799	0.0031991518	0.0039321709	0.0041285809	0.0035581427	0.0068228125
49	0.0059391418	0.0022675790	0.0058479661	0.0019442689	0.0069686648	0.0049385647
50	0.0028843357	0.0021082207	0.0040147282	0.0043212130	0.0078363441	0.0044503280
51	0.0013059123	0.0034071578	0.0072072399	0.0023952062	0.0080178573	0.0060079014
52	0.0050414267	0.0039003264	0.0078462515	0.0044563347	0.0060150542	0.0041074741
53	0.0020927783	0.0020181651	0.0042680351	0.0030911928	0.0158147770	0.0072616823
54	0.0008288434	0.0087146525	0.0055457694	0.0046457649	0.0106572978	0.0077869397
55	0.0018984309	0.0097979940	0.0062129505	0.0081577643	0.0141445035	0.0073976755
56	0.0054200721	0.0054945173	0.0072046412	0.0128826258	0.0131689107	0.0067024715
57	0.0044576606	0.0041152360	0.0093531446	0.0034364278	0.0091324845	0.0069422216
58	0.0049342233	0.0029069812	0.0063091710	0.0163194915	0.0160484909	0.0092244750
59	0.0016934831	0.0124418397	0.0183011694	0.0047961733	0.0120681683	0.0116673091
60	0.0052817037	0.0199342128	0.0114778897	0.0210217978	0.0198026241	0.0104416548
61	0.0041407913	0.0038835010	0.0107721013	0.0173314558	0.0200340583	0.0099150521
62	0.0022598916	-	0.0179216273	0.0040322586	0.0192684194	0.0123000970
63	0.0046838521	-	-	0.0257841218	0.0243419766	0.0127390280
64	0.0075047302	-	0.0071684924	0.0243914577	0.0208582282	0.0142932029
65	0.0117648452	-	0.0210534058	0.0233110741	0.0238106469	0.0172495314
66	-	0.0248460014	0.0361485156	0.0516243702	0.0173164480	0.0162770656
67	-	0.0322608604	0.0124225210	0.0175957614	0.0561945597	0.0161554392
68	-	-	0.0243914577	0.0106952913	0.0112995603	0.0270849890
69	-	-	0.0487901667	0.0307716584	0.0134230170	0.0165513104
70	-	0.0476280512	-	-	0.0555698469	0.0335345501
71	-	0.1177830344	-	0.0444517617	0.0870113734	0.0337869962
72	-	0.2876820725	-	-	0.1000834638	0.0261952184
73	-	-	-	-	0.0425596179	0.0258299699
74	-	-	-	-	-	0.0405671832
75	-	-	-	0.3364722426	-	0.0443468872
76	-	-	0.4054651031	-	-	0.0427696713
77	-	-	-	-	-	0.0532813640

Age	Select Year 1	2	3	4	5	Ultimate Year
78	-	-	-	-	-	0.0555904508
79	-	-	-	-	-	0.0502129729
80	-	-	-	-	-	0.0788696834
81	-	-	-	-	-	0.0846343825
82	-	-	-	-	-	0.0999109140
83	-	-	-	-	-	0.0848994451
84	-	-	-	-	-	0.1019045057
85	-	-	-	-	-	0.0938187565
86	-	-	-	-	-	0.1222176352
87	-	-	-	-	-	0.0800427110
88	-	-	-	-	-	0.0327898182
89	-	-	-	-	-	0.0312525427
90	-	-	-	-	-	0.1267517067
91	-	-	-	-	-	0.1133286885
92	-	-	-	-	-	0.0909717763
93	-	-	-	-	-	-
94	-	-	-	-	-	-
95	-	-	-	-	-	-
96	-	-	-	-	-	-
97	-	-	-	-	-	-
98	-	-	-	-	-	-
99	-	-	-	-	-	-

APPENDIX 2
Predicted Force of Mortality Values for the 1993 PICM Using the Models Developed in this Paper*

	Select Year					Ultimate Year	
Age	1	2	3	4	5	Par M	Inv M
0	0.0005466190	0.0007559245	0.0009335622	0.0010843232	0.0012122741	0.000449318	
1	0.0005470109	0.0007566490	0.0009345689	0.0010855695	0.0012137237	0.000477199	
2	0.0005474535	0.0007574672	0.0009357059	0.0010869771	0.0012153610	0.000506811	
3	0.0005479534	0.0007583915	0.0009369903	0.0010885670	0.0012172103	0.000538259	
4	0.0005485181	0.0007594354	0.0009384409	0.0010903629	0.0012192991	0.00057166	
5	0.0005491559	0.0007606145	0.0009400794	0.0010923913	0.0012216584	0.000607132	
6	0.0005498763	0.0007619463	0.0009419301	0.0010946823	0.0012243232	0.000644806	
7	0.0005506900	0.0007634505	0.0009440204	0.0010972701	0.0012273331	0.000684818	
8	0.0005516090	0.0007651496	0.0009463815	0.0011001930	0.0012307328	0.000727313	
9	0.0005526471	0.0007670687	0.0009490483	0.0011034944	0.0012345728	0.000772444	
10	0.0005538196	0.0007692363	0.0009520604	0.0011072233	0.0012389100	0.000820376	
11	0.0005551439	0.0007716846	0.0009554627	0.0011114351	0.0012438089	0.000871283	
12	0.0005566398	0.0007744500	0.0009593055	0.0011161924	0.0012493423	0.000925348	
13	0.0005583294	0.0007775735	0.0009636460	0.0011215657	0.0012555921	0.000982768	
14	0.0005602377	0.0007811015	0.0009685485	0.0011276348	0.0012626514	0.001043751	
15	0.0005623932	0.0007850863	0.0009740859	0.0011344899	0.0012706248	0.001108518	

	Select Year					Ultimate Year	
Age	1	2	3	4	5	Par M	Inv M
16	0.0005648278	0.0007895872	0.0009803405	0.0011422328	0.0012796308	0.001177304	0.003698697
17	0.0005675777	0.0007946710	0.0009874050	0.0011509783	0.0012898031	0.001250359	0.003698697
18	0.0005706838	0.0008004131	0.0009953843	0.0011608564	0.0013012926	0.001327947	0.003698697
19	0.0005741920	0.0008068988	0.0010043970	0.0011720137	0.0013142701	0.001410349	0.003698697
20	0.0005781546	0.0008142244	0.0010145768	0.0011846159	0.0013289281	0.001497864	0.003698697
21	0.0005826303	0.0008224987	0.0010260749	0.0011988501	0.0013454844	0.001590811	0.003698697
22	0.0005876856	0.0008318444	0.0010390620	0.0012149276	0.0013641847	0.001689524	0.003698697
23	0.0005933956	0.0008424005	0.0010537310	0.0012330871	0.0013853067	0.001794363	0.003698697
24	0.0005998451	0.0008543236	0.0010702995	0.0012535983	0.0014091639	0.001905708	0.003698697
25	0.0006071297	0.0008677907	0.0010890137	0.0012767657	0.0014361107	0.002023961	0.003698697
26	0.0006153577	0.0008830018	0.0011101514	0.0013029332	0.0014665471	0.002149553	0.003698697
27	0.0006246513	0.0009001828	0.0011340264	0.0013324895	0.0015009251	0.002282938	0.003698697
28	0.0006351483	0.0009195887	0.0011609932	0.0013658732	0.0015397549	0.0024246	0.003698697
29	0.0006470047	0.0009415076	0.0011914522	0.0014035802	0.0015836133	0.002575052	0.003698697
30	0.0006603965	0.0009662651	0.0012258557	0.0014461703	0.0016331512	0.00273484	0.003698697
31	0.0006755226	0.0009942286	0.0012647144	0.0014942757	0.0016891044	0.002904543	0.003698697
32	0.0006926075	0.0010258134	0.0013086053	0.0015486108	0.0017523035	0.003084777	0.003698697
33	0.0007119048	0.0010614885	0.0013581801	0.0016099823	0.0018236868	0.003276195	0.003698697
34	0.0007337012	0.0011017835	0.0014141749	0.0016793014	0.0019043144	0.00347949	0.003698697
35	0.0007583202	0.0011472966	0.0014774209	0.0017575974	0.0019953831	0.003695401	0.003698697
36	0.0007861274	0.0011987038	0.0015488573	0.0018460326	0.0020982453	0.003924709	0.003698697
37	0.0008175356	0.0012567681	0.0016295447	0.0019459203	0.0022144282	0.004168247	0.003698697
38	0.0008530112	0.0013223518	0.0017206812	0.0020587433	0.0023456567	0.004426897	0.003698697
39	0.0008930809	0.0013964286	0.0018236198	0.0021861770	0.0024938793	0.004701596	0.003698697
40	0.0009383395	0.0014800983	0.0019398890	0.0023301133	0.0026612967	0.004993341	0.003698697
41	0.0009894592	0.0015746033	0.0020712150	0.0024926894	0.0028503947	0.00530319	0.003698697
42	0.0010471989	0.0016813466	0.0022195478	0.0026763191	0.0030639809	0.005632266	0.003698697
43	0.0011124159	0.0018019132	0.0023870896	0.0028837288	0.0033052265	0.005981761	0.003698697
44	0.0011860784	0.0019380932	0.0025763281	0.0031179981	0.0035777134	0.006352944	0.003698697
45	0.0012692803	0.0020919085	0.0027900730	0.0033826052	0.0038854873	0.006747159	0.003698697
46	0.0013632568	0.0022656428	0.0030314978	0.0036814790	0.0042331180	0.007165836	0.003698697
47	0.0014694032	0.0024618758	0.0033041872	0.0040190569	0.0046257669	0.007610494	0.003698697
48	0.0015892957	0.0026835209	0.0036121899	0.0044003512	0.0050692637	0.008082743	0.003698697
49	0.0017247142	0.0029338691	0.0039600788	0.0048310231	0.0055701935	0.008584296	0.003698698
50	0.0018776694	0.0032166373	0.0043530195	0.0053174670	0.0061359936	0.009116972	0.003698701
51	0.0020504322	0.0035360241	0.0047968459	0.0058669053	0.0067750648	0.009682702	0.003698721
52	0.0022455679	0.0038967714	0.0052981478	0.0064874960	0.0074968958	0.010283537	0.003698829
53	0.0024659737	0.0043042355	0.0058643684	0.0071884531	0.0083122038	0.010921655	0.003699296
54	0.0027149220	0.0047644663	0.0065039145	0.0079801842	0.0092330943	0.01159937	0.003701027
55	0.0029961091	0.0052842968	0.0072262818	0.0088744444	0.0102732401	0.012319138	0.003706538
56	0.0033137099	0.0058714455	0.0080421957	0.0098845114	0.0114480847	0.01308357	0.003721853
57	0.0036724400	0.0065346299	0.0089637705	0.0110253820	0.0127750718	0.013895437	0.003759491
58	0.0040776257	0.0072836967	0.0100046891	0.0123139954	0.0142739036	0.014757682	0.003842262
59	0.0045352830	0.0081297676	0.0111804068	0.0137694842	0.0159668342	0.015673431	0.004006886
60	0.0050522068	0.0090854048	0.0125083799	0.0154134588	0.0178789993	0.016646005	0.004305883
61	0.0056360723	0.0101647969	0.0140083255	0.0172703281	0.0200387898	0.017678929	0.00480618
62	0.0062955484	0.0113839703	0.0157025140	0.0193676620	0.0224782732	0.018775949	0.005583661

Age	Select Year 1	2	3	4	5	Ultimate Year Par M	Inv M
63	0.0070404266	0.0127610267	0.0176161000	0.0217366007	0.0252336696	0.019941041	0.006714218
64	0.0078817666	0.0143164119	0.0197774953	0.0244123169	0.0283458899	0.02117843	0.008263163
65	0.0088320601	0.0160732194	0.0222187914	0.0274345383	0.0318611427	0.022492602	0.010275544
66	0.0099054165	0.0180575335	0.0249762353	0.0308481374	0.0358316207	0.023888322	0.012769569
67	0.0111177727	0.0202988164	0.0280907681	0.0347037976	0.0403162757	0.025370649	0.0157344
68	0.0124871290	0.0228303453	0.0316086330	0.0390587658	0.0453816935	0.026944958	0.019132237
69	0.0140338169	0.0256897072	0.0355820614	0.0439777023	0.0511030828	0.028616957	0.022903674
70	0.0157808008	0.0289193565	0.0400700487	0.0495336412	0.0575653921	0.030392708	0.026974796
71	0.0177540193	0.0325672454	0.0451392304	0.0558090741	0.0648645705	0.032278648	0.031264552
72	0.0199827695	0.0366875359	0.0508648712	0.0628971756	0.0731089924	0.034281615	0.035691293
73	0.0225001428	0.0413414040	0.0573319824	0.0709031863	0.0824210670	0.03640887	0.040177879
74	0.0253435160	0.0465979481	0.0646365845	0.0799459753	0.0929390553	0.038668128	0.044655155
75	0.0285551060	0.0525352145	0.0728871326	0.0901598055	0.1048191230	0.041067577	0.049063919
76	0.0321825970	0.0592413570	0.0822061267	0.1016963267	0.1182376595	0.043615918	0.053355666
77	0.0362798480	0.0668159450	0.0927319305	0.1147268274	0.1333938964	0.04632239	0.057492417
78	0.0409076930	0.0753714421	0.1046208259	0.1294447780	0.1505128661	0.049196804	0.061445965
79	0.0461348440	0.0850348761	0.1180493333	0.1460687031	0.1698487423	0.052249583	0.06519677
80	0.0520389110	0.0959497247	0.1332168323	0.1648454266	0.1916886145	0.055491794	0.068732712
81	0.0587075547	0.1082780463	0.1503485225	0.1860537357	0.2163567502	0.058935192	0.072047835
82	0.0662397877	0.1222028855	0.1696987666	0.2100085209	0.2442194094	0.062592261	0.075141148
83	0.0747474449	0.1379309914	0.1915548673	0.2370654508	0.2756902830	0.06647626	0.078015561
84	0.0843568437	0.1556958870	0.2162413330	0.2676262530	0.3112366347	0.070601271	0.080676941
85	0.0952106597	0.1757613366	0.2441246960	0.3021446792	0.3513862389	0.074982248	0.083133328
86	0.1074700448	0.1984252619	0.2756189546	0.3411332416	0.3967352170	0.079635075	0.085394272
87	0.1213170203	0.2240241655	0.3111917196	0.3851708228	0.4479568876	0.084576621	0.087470302
88	0.1369571792	0.2529381271	0.3513711577	0.4349112707	0.5058117646	0.089824802	0.089372502
89	0.1546227386	0.2855964468	0.3967538330	0.4910931067	0.5711588482	0.095398645	0.091112174
90	0.1745759879	0.3224840189	0.4480135648	0.5545504904	0.6449683791	0.101318358	0.09270059
91	0.1971131831	0.3641485316	0.5059114318	0.6262256053	0.7283362443	0.107605404	0.094148792
92	0.2225689450	0.4112085986	0.5713070727	0.7071826476	0.8225002480	0.114282575	0.095467461
93	0.2513212281	0.4643629444	0.6451714490	0.7986236268	0.9288584902	0.121374082	0.096666815
94	0.2837969318	0.5244007779	0.7286012620	0.9019062129	1.0489901247	0.128905633	0.097756548
95	0.3204782392	0.5922135108	0.8228352359	1.0185638939	1.1846788059	0.136904534	0.09874579
96	0.3619097759	0.6688079927	0.9292725093	1.1503287445	1.3379391714	0.145399787	0.09964309
97	0.4087066966	0.7553214600	1.0494934097-	1.2991571434	1.5110467541	0.15442219	0.100456416
98	0.4615638185	0.8530384213	1.1852829166	1.4672588198	1.7065717689	0.164004455	0.101193159
99	0.5212659377	0.9634097290	1.3386571647	1.6571296634	1.9274172730	0.174181322	0.101860152

* The Select Force of Mortality Rates are estimated using the Modified-Makeham Select Model, while the Ultimate Force of Mortality Rates are estimated using the parametrized Makeham and the Inverse-Makeham Models. (Parameter values are obtained from the Gauss-Newton and Marquardt procedures that yield equal values.)

APPENDIX 3
Predicted Force of Mortality Rates for Ages 90 to 99
Using Three Types of Makeham Models

Age	Original Makeham	Parametrized Makeham	Inverse Makeham
90	0.110592149	0.101318358	0.09270059
91	0.122814842	0.107605404	0.094148792
92	0.136479814	0.114282575	0.095467461
93	0.151757252	0.121374082	0.096666815
94	0.168837427	0.128905633	0.097756548
95	0.187933064	0.136904534	0.09874579
96	0.209281985	0.145399787	0.09964309
97	0.23315008	0.15442219	0.100456416
98	0.259834609	0.164004455	0.101193159
99	0.289667913	0.174181322	0.101860152