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Surplus Dependent Risk Models

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Abstract

The main objective of this study is to analyse and control a surplus dependent risk process, where larger surpluses allow for a wider underwriting policy.

We focus here mostly on the diffusion approximation of the risk process. The results are derived in two papers "Brownian motion in Risk Theory", submitted for publication, and "Surplus dependent risk processes defined by stochastic differential equations", in preparation. We only include here a long summary of these papers.

Keywords: Risk Theory, Surplus process

Summary of the Presentation

Under our general assumption that larger surpluses allow for a wider underwriting, the risk process $R(t)$ is defined by the following stochastic differential equations,

$$dR(t) = \sqrt{2aR(t)}dW(t) + cR(t)dt.$$

This model can be obtained by weak convergence arguments using changes of both, diffusion and drift coefficients, in Brownian motion with drift,

$$Y(t) = \sqrt{2a}W(t) + ct,$$

or directly from the corresponding piecewise deterministic Markov process $S(t)$ with infinitesimal generator,

$$Gf(x) = \lambda x[(1+\theta)p_1 + \int_0^\infty [f(x-y) - f(x)]P(dy)],$$

P being the probability distribution of individual claims.

The process $R(t)$ was defined as a state variable by Cox et al.(1985) with the intuitive motivation that neither the mean nor the variance dominates one another. The process $R(t)$ is a diffusion process with some known analitical properties; for more details see Ikeda & Watanabe(1981).

Because the scale function is the same for both processes $Y(t)$ and $R(t)$, the probability of ruin is also the same for both processes, with any initial surplus.

More properties can be proved.

The probability of ruin is the same for the process $S(t)$ and $U(t)$, the latter being the classical compound Poisson risk process. This result is obtained by the martingale approach of Schmidli(1992).

Using heavy traffic approximations, see Grandell(1991) or Schmidli(1992), we also get:

$$\psi(U_n) \to \psi(Y)$$
$$\psi(S_n) \to \psi(R)$$

ψ being the probability of ruin.

We also introduce the more realistic version of the process $R(t)$,

$$dR_1(t) = \sqrt{2ag(R_1(t), D)}dW(t) + h(R_1(t), D)dt,$$

where

$$g(x, D) = \begin{cases} x & \text{if } x \le D \\ D & \text{if } x > D \end{cases}$$

and

$$h(x, D) = \begin{cases} cx & \text{if } x \le D \\ cD + \beta(x - D) & \text{if } x > D,\ 0 < \beta < c \end{cases}$$

for which we consider two control problem examples.

In a first model we maximize the expected discounted payment of dividends with a fixed dividend barrier and interest rate β. Choosing $a = 0.1$, $c = 0.3$, $R_1(0) = 1$, $D = 10$, $\beta = 0.1$ and a time horizon of 50, we see, through simulations, that the optimal dividend barrier is 70 when dividend payments are 55. In this case the total expected discounted payment is 12.4. This is an example of a singular control.

Our second example is based on a continuous consumption rate and an optimal choice between:

(i) an insurance business investment, or

(ii) an investment in risk-free assets.

The insurance business surplus is modelled by the process $R(t)$ and the risk-free asset value by $V(t)$, such that

$$dV(t) = rV(t)dt.$$

Let

u_1 - fixed fraction of wealth invested in the insurance business

u_2 - consumption-fixed fraction of the current wealth

P - penalty in the case of ruin

$U(w) = 1 - e^{-\lambda w}$ - utility function

x - initial surplus

Our goal is to maximize with respect to u_1 and u_2, the total discounted expected utility of the consumption, discounted at rate β. Here the penalty is not discounted. It implies that we need to maximize

$$\int_0^{\infty}[1 - exp(\frac{-\lambda C e^{Csx}}{A\lambda(e^{Cs} - 1) + C})]e^{-\beta s}ds - Pe^{xC/A}$$

where $A = u_2^2 u_1^2 a$, $C = [(1 - u_1)r + u_1 c - u_2]u_2$.

This is an elementary problem; for more general problems in finance see Duffie(1988), Karatzas & Shreve(1988) and Fleming & Soner(1993).

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