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A Numerical Method for Computing the Probability Distribution of Total Risk of a Portfolio

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ABSTRACT

Consider a portolio of n statistically independent insurance risks. The total loss S associated with the portolio is given by $S = X_1 + X_2 \cdots X_n$ where X_1 represents the loss associated with the i-th risk, $i = 1,2,\cdots,n$. The Laplace transform of S is the product of the Laplace transforms of X_i. In the present paper, we propose and investigate a numerical method of computing the probability distribution of S.

1. THE DISTRIBUTION OF THE TOTAL LOSS

Consider a portfolio of n independent insurance risks. Let X_i be the loss associated with the ith risk, $i = 1,2,\cdots,n$. The total loss S, its expected value and variance, are given by

$$\begin{aligned} S &= X_1 + X_2 + \cdots + X_n, \\ E(S) &= \sum_{i=1}^{n} E(X_i), \\ Var(S) &= \sum_{i=1}^{n} Var(X_i) \end{aligned}$$

Since the X_i's are independent, the moment generating function of S is

$$\begin{aligned} M_S(t) &= E(e^{tS}) \\ &= E\left(e^{t\sum_i X_i} \right) \\ &= E\left(\prod_i e^{tX_i} \right) \\ &= \prod_i E(e^{tX_i}) \\ &= \prod_i M_{X_i}(t) \end{aligned}$$

2. THE CONTINUOUS CASE

In the continuous case, we can think of the moment generating function as a special case of the Laplace transform,

$$L_S(z) = E(e^{zS}),$$

with $z = -t$, because S is non-negative. In this case, we can invert $L_S(z) = M_S(t)$ to obtain the probability density function.

If the X_i's are not identically distributed, we have to use numerical methods to approximate the probability density function of S in most cases. Even if the X_i's are identically distributed we may have to use numerical methods.

The inversion of the Laplace transform leads to a Fredholm equation of the first kind:

Find f such that

$$\int_0^{\infty} e^{-zt} f(t)dt = g(z),$$

where g is known.

3. THE DISCRETE CASE

In the discrete case,

$$M_S(t) = \prod_{i=1}^{n} M_{X_i}(t),$$

where

$$M_{X_i}(t) = \sum_{j=0}^{k_i} e^{x_{i_j} t} p_i(x_{i_j}).$$

Without loss of generality, we may assume that

$$k_i = N \text{ and } x_{i_0} = 0 \text{ for } i = 1,2,\cdots,n$$

and

$$x_{i_j} - x_{i_{j-1}} = 1 \text{ for } j = 1,2,\cdots,N; i = 1,2,\cdots,n.$$

Then

$$M_S(t) = \sum_{i=1}^{n}\left(\sum_{j=0}^{N} e^{jt} p_i(j)\right).$$

Since

$$M_S(t) = \sum_{k=0}^{nN} e^{kt} p_s(k),$$

we can equate coefficients of e^{kt}, $k = 0,1,\cdots,nN$ to obtain the probability density function of S.

The following example taken from Insurance Risk Models by Panjer and Willmot (pp. 131-132) illustrates the above result.

Probability Distribution of the Losses X_1, X_2, X_3

i	$x=0$	$x=1$	$x=2$	$x=3$
1	.3	.2	.4	.1
2	.6	.1	.3	0
3	.4	.2	0	.4

The moment generating function of the sum S is:

$$M_S(t) = \left(.3 + .2e^t + .4e^{2t} + .1e^{3t}\right)$$
$$\left(.6 + .1e^t + .3e^{2t} + 0\right).$$
$$\left(.4 + .2e^t + 0 + .4e^{3t}\right)$$

$$M_S(t) = .072 + .096e^t + .170e^{2t} +$$
$$.206e^{3t} + .144e^{4t} + .178e^{5t} +$$
$$.070e^{6t} + .052e^{7t} + .012e^{8t} +$$
$$0e^{9t}$$

The Probability Distribution of S is

x	0	1	2	3	4	5	6	7	8	9
$P_S(x)$	.072	.096	.170	.206	.144	.178	.070	.052	.012	0

This is exactly the same table given in Insurance Risk Models.

4. NUMERICAL METHODS

In the continuous case, where the Laplace transform cannot be inverted analytically, there are several methods available for numerical approximation.

Davies and Martin (Journal of Computational Physics, vol 33, 1979; pp. 1-32) compared some of these methods. The method of Stehfest (Communications of ACM, vol 13, 1970; pp. 47-49 Algorithm # 368) was found to give good accuracy on a fairly wide range of functions.

Furthermore, Stehfest's algorithm was easier to implement than some of the comparable algorithms. In this paper, we have used Stehfest's algorithm for approximating the probability distribution function of S, given the individual probability distribution function's: $f_{X_i}(x)$, $i = 1, 2, \cdots, n$.

Our preliminary work involves the sum of two independent (not identical) random variables. In the two numerical examples presented at the end, we used Monte Carlo simulation to test our result.

Stehfest's Algorithm

Given $L_S(z)$: Laplace transform of S.

If $f_S(t)$ is the probability distribution function of S, then

$$f_S(t) \approx \frac{\ln 2}{t} \sum_{i=1}^{N} V_i L_S\left(\frac{\ln 2}{t} i\right),$$

where N is even and

$$V_i = (-1)^{N/2+i} \sum_{K=\left[\frac{i+1}{2}\right]}^{M_{in}(i,N/2)} \frac{k^{N/2+1}(2k)!}{(N/2-k)!\,k!\,(k-1)!\,(i-k)!\,(2k-i)!}$$

5. FURTHER CONSIDERATIONS

In many cases, there is a non-zero probability that a no claim occurs for one of the risks. (See: Insurance Risk Models by Panjer and Willmot). For this case, let $q_i = \Pr(X_i > 0)$. Then

$$1 - q_i = \Pr(X_i = 0)$$

In this case, the probability distribution function of X_i is

$$F_{X_i}(x) = (1 - q_i) + q_i F_{Y_i}(x)$$

where

$$y_i = x_i | x_i > 0$$

and F_{Y_i} is the claim size distribution. Then

$$L_{X_i}(z) = (1 - q_i) + q_i L_{Y_i}(z)$$

and

$$L_S(z) = \prod_{i=1}^{n} \left(p_i + q_i L_{Y_i}(z) \right)$$

where $p_i = 1 - q_i$.

For this situation, the discrete case is easy to handle, and the case where the Y_i's are continuous, requires numerical methods.

If the claims distribution is not known, then the probability distribution function of each X_i may be estimated by using a non parametric density estimator. Then each $M_{X_i}(t)$ is estimated and the product $M_S(t)$ of these is numerically inverted to obtain the probability distribution function of S.

6. NUMERICAL EXAMPLES

(a) Let X_1 have a chi-squared distribution with 1 degree of freedom and let X_2 have an exponential distribution with parameter $\theta = 1$. The distribution of the sum $S = X_1 + X_2$ cannot be obtained analytically.

(b) Let X_1 have an exponential distribution with $\theta = 1$ and let X_2 have a gamma distribution with $\alpha = 2$, $\theta = 2$. Then the sum $S = X_1 + X_2$ has a gamma distribution with $\alpha = 3$, $\theta = 2$.

In both cases, we used Monte Carlo simulation with $N = 100{,}000$ and Stehfest's algorithm for the numerical inversion of the Laplace transform, and then computed the mean and variance using a numerical quadrature. The following table gives the comparisons between the different methods.

Comparisons of Mean and Variance

Example	Mean			Variance		
	True	MCS	NILT	True	MCS	NILT
(a)	2	1.9965	1.9759	3	3.0135	3.1145
(b)	6	6.0104	5.9918	12	12.0147	11.8602

MCS: Monte Carlo Simulation

NILT: Numerical Inversion of the Laplace Transform

The following table shows the accuracy of the method when compared with the true probability distribution of the sum in Example (b).

Comparison of the True Density and the Approximated Density

s_i	$f(s_i)$	$\tilde{f}(s_i)$
1	0.037908	0.037908
2	0.091970	0.091970
3	0.125510	0.125511
4	0.135336	0.135335
5	0.128258	0.128258
6	0.112020	0.112021
7	0.092481	0.092479
8	0.073266	0.073263
9	0.056238	0.056239
10	0.042115	0.042112
11	0.030904	0.030906
12	0.022305	0.022309
13	0.015889	0.015880
14	0.011166	0.011171
15	0.007772	0.007778
16	0.005366	0.005367
17	0.003683	0.003675
18	0.002489	0.002499
19	0.001687	0.001689
20	0.001135	0.001135
21	0.000751	0.000759
22	0.000503	0.000505
23	0.000327	0.000335
24	0.000215	0.000221
25	0.000130	0.000146

$f(s_i)$: Probability density function of S

$\tilde{f}(s_i)$: Probability density function of S approximated by numerically inverting the Laplace transform

7. CONCLUSIONS

Inversion of the Laplace transform (moment generating funciton) is shown to be a better approach than the existing recursive method for find the distribution of sum of several independent random variables. In the continuous case, in many situations, numerical inversion needs to be used to compute the probability distribution function of the sum. We have shown by computer simulation that the numerical inversion formula of Stehfest gives good accuracy.

REFERENCES

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