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An Optimal Model for Asset Liability Management

Lijia Guo
The Ohio State University
231 W. 18th Avenue
Columbus, OH 43210
guo@math.ohio-state.edu

Abstract

This paper addresses the stochastic modeling for managing asset liability process. We start with developing a jump-diffusion process for evaluating of the liabilities of the insurance company in general. We then formulate the ALM process into a stochastic control problem. With this approach, we present a Bellman-Dreyfus Fundamental type formula for ALM process in terms of the solution of a system of algebraic equations and partial differential equations.

Keywords
Jump-diffusion process; optimal portfolio selection; stochastic control; heat equation; inverse problem.

1. Introduction

The theory and techniques of ALM are of concern of both insurance industry leaders and academician. With new products, unstable financial markets, and the competitive nature of the industry, an ALM process become critical to the profitability, and more importantly, the solvency of the insurers. For years, the actuaries and investment professionals have been fascinated by the state-of-the-art ALM modeling. A thorough description of the precise theory in general is beyond the scope of this proposal. However, the following examples illustrate the history of ideas on which the methods are based.

1.(1950's) **Redington's theory of immunization** (Redington, 1952). The notion of equating the mean term of assets with the mean term of liabilities has been used since then by a number of insurance companies worldwide. Markowitz (1959) presented the variance minimization approach.

2.(1970's and 1980's) **Generalized theory of immunization.** Starting from early 1970's, the Redington's theory of immunization has been extended to handle more complicated situation. Fisher and Weil (1971) relaxed Redington's assumption of flat yield curves and tested their model empirically. Shiu (1987) extended the Fisher-Weil immunization theorem to more general case where the interest rate shocks are functions of time. Meanwhile, the concept of immunization within the framework of a stochastic model for the interest rate is examined by Boyle (1978), Wilkie (1987), Page (1989) and many others.

3.(1990's) **Stochastic Modeling.** Key-Rate (multivariate) immunization is provided by Reitano (1991) and Ho (1990) where the term structure is partitioned in maturity segments. Janssen (1993 and 1994), Anthony (1994) and Smink (1994) have adopted stochastic method to model the ALM process from different perspectives. In actuarial practice, the stochastic approach for ALM processes has also been applied, see Correnti and Sweeney (1994).

In general, maximizing surplus return while minimizing risk is the most important objective of an ALM process. An effective ALM process is contingent upon reliable cash flow estimation on the liability side. One is naturally led to the following question:

> **Question** *How one can make optimal investment decision on assets to match the future liabilities and the profit goal?*

2. Modeling ALM Process

In this research, the death process is assumed to be independent of the processes ruling in the financial market. Furthermore, the insurance company is assumed to behave as risk neutral concerning the mortality risk. Under above assumptions, the main source of risk is due to the instability of the financial markets. Therefore, one could use the same type of the stochastic model used in modern finance theory to model the liabilities.

It is assumed that the stochastic volatility term structure describes the behavior of the short rate $r(t)$ by a diffusion process

$$dr(t) = \alpha(r,t)dt + \sigma(r,t)dB, \tag{2.1}$$

where α is the instantaneous mean of the interest rate, σ^2 is the instantaneous diffusion variance of the interest rate, and $B(t)$ is a standard Brownian motion.

We also assume that, in financial markets, there are n sources of uncertainty modeled by the components of the standard n-dimensional Wiener process

$Z = (Z(t) = (Z_1(t), \cdots, Z_m(t)), 0 \leq t \leq T)$. By letting $Z_1(t) = B(t)$ we assume that uncertainty of the short rate is modeled by the first component of the Z.

2.1 Liabilities

Since an effective ALM model start with reliable valuation of the liability, in this part of the project, we would start with modeling of the liabilities of an insurance company during the time period of $[0, T]$.

The valuation techniques for liabilities should vary due to the nature of the liabilities for different line of business. For example, Albizzati and Geman (1994) presented a valuation formula of a European surrender option in life insurance policies in the context of stochastic interest rates while Lee and D'Arcy (1989) studied the variable universal life insurance using the basic economic concepts of marginal and average rates of return.

In general, we denote the liabilities for the m business lines of the insurance company by $L_i(t), (i = 1, 2, \cdots, l)$ and suppose that $L_i(t)$ are governed by the following stochastic differential equation:

$$dL_i(t) = \mu_{L,i}(r,t)\, L_i(t)dt + \sum_{k=1}^{m} \sigma_{L,i,k}(r,t)\, L_i(t)dZ_k + J(\nu,\gamma)dq(\lambda), \quad 0 \le t \le T, \tag{2.2}$$

where $\mu_{L,i}$ is the instantaneous expected liabilities for the i^{th} line of business; $\sigma^2_{L,i,k}$ is the instantaneous variance with which the k^{th} source of uncertainty affects $L_i(t)$, conditional on no extreme event; J = the jump magnitude of the liabilities if the extreme event occurs, the distribution of J is Normal with mean ν and varianceγ; and q = a Poisson process that is independent of z with parameter λ defined as the mean number of the extreme event per unit time.

Special hedge strategy, for example, reinsurance and insurance derivatives could be used in the event of extreme case, see Cox and Schwebach (1992), Niehaus and Mann (1992), and Guo (1995).

For the illustrative purpose, we now consider $J = 0$ and $l = 1$. Hence $L(t)$ follows

$$dL(t) = \mu_L(r,t)L(t)dt + \sum_{k=1}^{m} \sigma_{L,k}(r,t)\, L(t)dZ_k, \quad 0 \le t \le T. \tag{2.3}$$

2.2 Trading strategies

Let us consider now an asset-liability manager with initial assets A_0, who invests the assets in the various securities.

We shall deal exclusively with a financial market in which n securities (risky or risk-free) can be traded continuously. Markets are frictionless and short-sales of assets are permitted. Asset payoffs are random variables which are elements of a space of contingent claims. The price for one share of the j^{th} security is modeled by

$$dP_j(t) = \alpha_j(r,t)P_j(t)dt + \sum_{k=1}^{m} \sigma_{j,k}(r,t)P_j(t)dZ_k, \quad j = 1,2,\cdots,n. \tag{2.4}$$

where $\sigma_{j,k}$ gives the instantaneous intensity with which the k^{th} source of uncertainty affects the price of the j^{th} security.

For example, if the j^{th} security is a pure discount bond (zero-coupon bond) maturing for the value of 1 at time s, $t \le s$. then (Beekman and Shiu, 1988)

$$dP_j = \left(\frac{\partial P}{\partial t} + \alpha_j \frac{\partial P}{\partial r} + \frac{1}{2}\sigma_{j,1}^2 \frac{\partial^2 P}{\partial r^2}\right) dt + \sigma_{j,1} P_j dZ_1. \tag{2.5}$$

We shall denote by $A(t)$ the assets at time t, by $x_j(t)$ the proportion of the $A(t)$ invested in the j^{th} security at time t $(1 \le j \le n)$.

Since short-sales of assets are permitted, one has the following short-sale constrains:

$$x_j \ge 0, j = m+1, \cdots, n; m \ge 0 \tag{2.6}$$

We shall now consider the surplus process S, $(S = S(t), 0 \le t \le T)$ defined as follows:

$$S(t) = A(t) - L(t);\ S(0) = S_0,\ S(T) = S_T. \tag{2.7}$$

Consider a period model with periods of length Δt, where all income is generated by capital gains, and the assets $A(t)$, the liability $L(t)$ and $P_j(t)$ are known at the beginning of period $[t, t+\Delta t]$. Let $N_j(t)$ be the number of shares of assets j purchased and held during period $[t, t+\Delta t]$ and $u(t)$ be the amount of cash/profit reserve per unit time during period $[t, t+\Delta t]$.

At the beginning of the period $[t, t+\Delta t]$,

$$A(t) = \sum_{j=1}^{n} N_j(t - \Delta t)\, P_j(t). \tag{2.8}$$

The amount of cash reserve for the period, $u(t)\, dt$, and the new portfolio, $N_j(t)$, are simultaneously chosen, and if it is assumed that all trades are made at current prices, then we have that

$$-\,u(t)\Delta t = \sum_{j=1}^{n} (N_j(t) - N_j(t - \Delta t))\, P_j(t). \tag{2.9}$$

Incrementing (2.8) and (2.9) by Δt we have that

$$-\,u(t+\Delta t)\Delta t = \sum_{j=1}^{n} (N_j(t+\Delta t) - N_j(t))\, P_j(t+\Delta t)$$

$$= \sum_{j=1}^{n} [N_j(t+\Delta t) - N_j(t)]\,[P_j(t+\Delta t) - P_j(t)]$$

$$+ \sum_{j=1}^{n} (N_j(t+\Delta t) - N_j(t))\, P_j(t) \qquad (2.10)$$

and

$$A(t+\Delta t) = \sum_{j-1}^{n} N_j(t)\, P_j(t+\Delta t). \qquad (2.11)$$

Taking the limits as $\Delta t \to 0$, we have

$$-u(t)dt = \sum_{j=1}^{n} dN_j(t)\, dP_j(t) \; + \; \sum_{j=1}^{n} dN_j(t)\, P_j(t) \qquad (2.12)$$

and

$$A(t) = \sum_{j=1}^{n} N_j(t)\, P_j(t). \qquad (2.13)$$

Using Itô's lemma and differentiate the above equation to get

$$\begin{aligned} dA &= \sum_{j=1}^{n} N_j(t)\, dP_j \; + \; \sum_{j=1}^{n} dN_j\, P_j \; + \; \sum_{j=1}^{n} dN_j\, dP_j \\ &= \sum_{j=1}^{n} N_j(t)\, dP_j \; - \; u(t)dt\,. \end{aligned} \qquad (2.14)$$

Notice that $x_j(t) \equiv N_j(t)P_j(t)/A(t)$ and from equation (2.4) and (2.7), we have the following equation for the surplus $S(t)$:

$$\begin{aligned} dS \; &= \; \sum_{j=1}^{n} x_j\, \alpha_j\, Sdt \; - \; (\mu_L L \; - \; \sum_{j=1}^{n} x_j\, \alpha_j)\, Ldt \; - \; udt \\ &+ \sum_{k=1}^{m} [\sum_{j=1}^{n} x_j \sigma_{j,k}(S+L) \; - \; \sigma_{L,k}\, L\,]\, dZ_k. \end{aligned} \qquad (2.15)$$

2.3 ALM model

The ALM process is now formulated as the problem of choosing optimal portfolio selection and cash reserve rules, $x(t) = \{x_j(t), j = 1, 2, \cdots, n\}$ and $u(t)$, for an Asset/Liability manager over the period of $[0, T]$, satisfying

$$max_{\{u,x\}} \; E_0[\int_0^T U(u(t),t)dt + F(S_T,T)] \tag{2.16}$$

subject to the budget constraint (2.15), and

$$u(t) \geq 0; \; S(t) > 0; \; S(0) = S_0 > 0. \tag{2.17}$$

In equation (2.16), $U(\cdot)$ is assumed to be a strictly concave utility function. $F(\cdot,\cdot)$ is assumed to be a strictly concave and continuously differentiable on $(0,\infty)$ for all $t \in [0,T)$, and E_0 is the conditional expectation operator, given $S(0) = S_0$ as known.

3. Method

We now derive a Bellman-Dreyfus type formula (see Merton, 1990) for solving the problem (2.16) – (2.17).

Without loss generality, we let $m = 1$ and consider

$$dL(t) = \mu_L(r,t)L(t)dt + \sigma_L(r,t)\,L(t)dZ_L, \; 0 \leq t \leq T, \tag{3.1}$$

$$dP_j(t) = \alpha_j(r,t)P_j(t)dt + \sigma_j(r,t)P_j(t)dZ_j, \; j = 1,2,\cdots,n; \tag{3.2}$$

and the equation (2.15) becomes

$$dS = [\sum_{j=1}^n x_j\,\alpha_j\,(S+L) - \mu_L L - u]\,dt + \sum_{j=1}^n x_j\sigma_j\,(S+L)dZ_j - \sigma_L\,LdZ_L. \tag{3.3}$$

Define

$$\mathcal{J}(S,t) \equiv max\; E_t[\int_t^T U(u(s),s)ds + F(S_T,T)] \tag{3.4}$$

where, as before, E_t is the conditional expectation operator, conditional on $S(t) = S$ and

$$\mathcal{D}(\mathcal{J}) \equiv \frac{\partial \mathcal{J}}{\partial t} + [\sum_{j=1}^n x_j\,\alpha_j\,(S+L) - \mu_L L - u]\frac{\partial \mathcal{J}}{\partial S} + \frac{1}{2}\sigma_L^2 L^2 \frac{\partial^2 \mathcal{J}}{\partial S^2}$$
$$+\frac{1}{2}[\sum_{i=1}^n\sum_{j=1}^n x_i x_j \sigma_{ij}\,(S+L)^2 - \sum_{j=1}^n x_j\sigma_L L(S+L)]\frac{\partial^2 \mathcal{J}}{\partial S^2}. \tag{3.5}$$

Where $\Omega = [\sigma_{i,j}]_{n\times n}$ $(\sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j)$ is the variance-covariance matrix.

It is assumed that no assets can be expressed as a linear combination of the other assets. so Ω is nonsingular. Under our model assumption, there exist a set of optimal rules, $\bar{x}$ and $\bar{u}$ satisfying

$$\mathcal{D}(\mathcal{J}(\bar{u},\bar{x};S,t)) + U(\bar{u},t) = max_{\{u,x\}} \{\mathcal{D}(\mathcal{J}(u,x;S,t)) + U(\bar{u},t)\} = 0 \tag{3.6}$$

subject to

$$\mathcal{J}(\bar{u},\bar{x};S,T) = F(S_T,T) \tag{3.7}$$

and

$$\sum_{j=1}^{n} \bar{x}_j(t) = 1 \tag{3.8}$$

for $t \in [0,T]$.

Equation (3.6) gives the stochastic Bellman equation for the optimal problem (2.16). Let λ denote the Lagrangian multiplier, $\bar{x}$ and $\bar{u}$ satisfy the first-order optimal conditions

$$U_u(\bar{u},t) - \mathcal{J}_S(\bar{u},\bar{x};S,T) = 0, \tag{3.9}$$

and

$$\begin{aligned} -\lambda + \mathcal{J}_{SS}[\sum_{j=1}^{n} \bar{x}_j\sigma_{k,j}(S+L)^2 + \bar{x}_k\sigma_L L(S+L)] \\ = -\mathcal{J}_S\alpha_k(S+L) \qquad k = 1,2,\cdots,n. \end{aligned} \tag{3.10}$$

From equation (3.8), (3.9) and (3.10), we now solve explicit for $\bar{x}$ and $\bar{u}$ as functions of $\mathcal{J}_S$, $\mathcal{J}_{SS}$, S, and t.

Let G denote the inverse function of U_u $(G \equiv (U_u)^{-1})$. Then from (3.9),

$$\bar{u} = G(\mathcal{J}_S,t). \tag{3.11}$$

To solve for the $\bar{x}$, define

$$\Gamma \equiv (\Omega + \frac{\sigma_L L}{S+L}I)^{-1} \tag{3.12}$$

and

$$\alpha \equiv (\alpha_1,\cdots,\alpha_n)^T, \qquad e \equiv (1,1,\cdots,1)^T. \tag{3.13}$$

Eliminating λ from (3.10) using (3.8), the solution for $\bar{x}$ can be written as

$$\bar{x} = y(\mathcal{J}_S, S, t) \equiv \mathcal{J}_S(S+L)\Gamma(I - \frac{1}{e^T\Gamma e} ee^T\Gamma)\alpha - \frac{1}{e^T\Gamma e}\Gamma e. \quad (3.14)$$

Then one can substitute $\bar{x}$ and $\bar{u}$ in equation (3.6) which now becomes the following fundamental partial differential equation (PDE) for $\mathcal{J}$.

$$\mathcal{J}_t + f(\mathcal{J}_S, S, t)\mathcal{J}_S + g(\mathcal{J}_S, S, t)\mathcal{J}_{SS} = 0 \quad (3.15)$$

subject to the boundary condition

$$h(\mathcal{J}_S, S, T) = F(S_T, T). \quad (3.16)$$

Where

$$f(\mathcal{J}_S, S, t) = (S+L)\,y(\mathcal{J}_S, S, t)^T\alpha - \mu_L L - G(\mathcal{J}_S), \quad (3.17)$$

$$\begin{aligned} g(\mathcal{J}_S, S, t) = \frac{1}{2}[\sigma_L^2 L^2 &+ (S+L)^2(y(\mathcal{J}_S, S, t)^T\Omega y(\mathcal{J}_S, S, t))] \\ &- \frac{1}{2}\sigma_L L(S+L)e^T y(\mathcal{J}_S, S, t)], \end{aligned} \quad (3.18)$$

and

$$h(\mathcal{J}_S, S, T) = \mathcal{J}(G(\mathcal{J}_S, T), y(\mathcal{J}_S, S, t)^T\alpha; S, T). \quad (3.19)$$

Once the above fundamental PDE (3.15) – (3.16) is solved for $\mathcal{J}$, we then derive the optimal rules as functions of S and t from the equation (3.11) and (3.14).

4. Concluding Remarks

This study has developed a stochastic model for the Asset-Liability management (ALM) process. The main result is presenting the solution of general ALM process by solving the fundamental PDE associated with a stochastic control problem.

There are still some interesting theoretical open questions. For example, we could derive the close-form solution for the PDE. Furthermore, we are also quite silent on the ruin analysis for the ALM process. We hope that further research in this area sheds some light on these issues.

We remark that the numerical solution for the fundamental PDE is more desirable than the analytical solution formula in order for the method to be actually implemented in actuarial practice. We plan to develop an explicit finite difference algorithm and computer software for the *theoretical* model solution.

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