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RUIN MODELING FOR COMPOUND NONSTATIONARY POISSON PROCESSES WITH PERIODIC CLAIM INTENSITY RATES

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Abstract

Aggregated claims are modeled by a compound nonstationary Poisson process with periodic intensity. It is natural to describe claims with such processes when the claim intensity shows a clear seasonal dependence.

Some general properties of such processes are briefly discussed as well as the possible shapes of the intensity function. A decomposition of the accumulated claims as a sum of independent compound Poisson sums is shown. Some results analogous to those of the classical model are established. An imbedded discrete risk model is introduced. It is shown that ruin probabilities and maximal losses for the nonstationary risk model find lower bounds with the imbedded discrete model.

Keywords: Nonstationary claim process, periodic intensity function, compound risk processes, ruin probability, imbedded discrete model, Lundberg bounds on ruin probabilities, seasonality.

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1. Introduction

The question of time periodicity for random processes is not new in Risk Theory . It is usually discussed in the context of the nonstationarity of nonhomogeneous processes. When time plays an essential role in the behavior of the observed processes, an approach in terms of stochastic differential equations can be used (see e.g. Garrido, 1989). When the process possesses apparent discontinuities, nonstationary Poisson processes can give an adequate representation. This approach is well developed in reliability theory (see e.g. Block et al., 1992 and 1985; Baxter, 1982; Beichelt, 1991). In the case of time periodic behaviors one can expect that differential equations with periodic coefficient functions could find solutions processes that would serve as valid models. This case is discussed by Gnedenko and Makarov, 1971. They have shown that under some conditions Markov chains in continuous time with periodic intensities have periodic stationary probabilities. For the nonstationary Poisson process (that we will denote NPP in the sequel) periodicity in time is conveniently described by the periodic behavior of its intensity (or failure rate) function $\lambda(t)$.

The name failure rate function comes from reliability theory. Under a *minimal repair policy* (Block et al. 1992; Baxter 1982) the failed unit is replaced by one of the same type and same age. The rate of the resulting process counting the number of failures is exactly equal to the intensity function $\lambda(t)$ of the failure time distribution. In Chukova et al. (1993) we have used this relation to show characterization properties of the NPP with periodic failure rate. On the basis of these properties we consider here some modeling problems in Risk Theory for the claims process and the time to ruin. A discussion of some simple statistical aspects such as the identification of the characteristics of an NPP is also included. We follow the description and results of Cinlar(1974) in this subject.

2. Preliminary Results

We consider the risk problems from the point of view of an insurance company. A general insurance contract is issued for a limited time period, say one year. During this period the insurance company will pay admissible claims to the policyholder; when no claim is made (i.e. no events occurred or they were not reported), then no payment is issued. At the end of the contract period the policyholder may renew the policy, effectively buying new coverage for the next year. We assume automatic renewals of the insurance contracts on the part of the policyholders and a constant average number of policyholders within the portfolio.

Let $\lambda(t)$ be the intensity of claims generated by a given policyholder at time t, $t \geq 0$. When seasonal conditions affect the insured risk (e.g. automobile or fire insurance, hurricanes) it is natural to assume that $\lambda(t)$ is a periodic function with a one year period. This corresponds to the case where the chances for an insurance event to occur will depend on the season, but will otherwise be identical from year to year. Moreover, if no insurance event is observed during a given year, it does not affect the chances that an event will occur during the next year.

We fix the beginning of the period. The dependence of $\lambda(t)$ on the time parameter t, describes the claims intensity. For instance, the form

(1) $$\lambda(t) = \lambda t^{p-1}(1-t)^{q-1} \quad \text{and} \quad \lambda(n+t) = \lambda(t) ,$$

for $\lambda>0$; $p>0$; $q>0$; $n\in\mathbb{N}$ and $t\in[0,1)$, similar to that of the beta distribution, seems flexible enough to include many possible shapes of claims intensity in a given year.

It is possible to have multiple claims generated by the same policyholder, as described above for minimal repairs in reliability theory. Also, many policyholders will have no claims during the period, but we assume that they renew their insurance for the next period. A NPP model is natural under such assumptions.

Let $\lambda_k(t)$ be the claims intensity for the kth individual policyholder. If there are K policyholders at the beginning of the year, we know from the theory of NPP's that the portfolio claims intensity for all K individual claim processes is

(2) $$\lambda(t) = \lambda_1(t) + \lambda_2(t) + \ldots + \lambda_K(t)$$

(see Çinlar, 1974) and that the portfolio claims process is also NPP. Further, if each $\lambda_k(t)$ is periodic, the same is true of $\lambda(t)$. In particular, if each individual claims intensity takes the form in (1), then the portfolio intensity takes the same form, namely

(3) $$\lambda(t) = K\lambda t^{p-1}(1-t)^{q-1} \quad \text{for } t\in[0,1)$$

which is suitable for statistical inference.

In conjunction, it might be natural to assume that claim sizes also depend on time. Consider a claim C_t occurring at time t, with a distribution depending on t. Therefore, the total claims in the time interval $[0,t)$ will be represented by the process

$$S_t = \int_0^t \lambda(u)C_u du \tag{4}$$

The process in (4) will not be investigated here in its full generality, but it follows that for $\lambda(t)$ periodic S_t is also a periodic process.

In the sequel we will refer to the following result of Chukova et al. (1993), concerning the NPP with periodic intensity function.

Let $\{N_t,\ t\geq 0\}$ be an NPP with intensity function $\lambda(t)$ and hazard function

$$\Lambda(t) = \int_0^t \lambda(u)du \tag{5}$$

Denote by $N_{[\tau,\tau+t)}$ the random number of events of the NPP over the interval $[\tau,\tau+t)$, $\tau\geq 0$, $t\geq 0$, i.e. $\{N_{[\tau,\tau+t)},\ t\geq 0\} = \{N_{\tau+t} - N_\tau,\ t\geq 0\}$.

Theorem 1: (i) If $\lambda(t)$ is a periodic function (with an arbitrary period, say $b>0$), then its hazard function $\Lambda(t)$ has the following property

$$\Lambda(t) = [t/b]\Lambda(b) + \Lambda(t-[t/b]b), \quad t\geq 0, \tag{6}$$

where $[.]$ denotes the "integer part" function;

(ii) Under the conditions of (i)

$$P\{N_{[nb,nb+t)} = m\} = P\{N_t = m\}, \qquad m = 0,1,2,\dots$$

for any integer $n\geq 0$ and any $t\geq 0$. Moreover, the random variables $N_{[nb,nb+t)}$ and $N_{[0,nb)}$ are mutually independent;

(iii) A NPP $\{N_t,\ t\geq 0\}$ has periodic intensity function $\lambda(t)$ with period $b>0$ if and only if the random numbers of failures $N_{[0,b)}$ and $N_{[b,b+t)}$ are independent and distributed as N_b and N_t, respectively;

(iv) If $\{N_t, t\geq 0\}$ is a NPP with periodic intensity function of period $b>0$, then for any $t\geq 0$ the random variable N_t can be represented in the form

$$N_t = M_1 + M_2 + \ldots + M_{[t/b]} + N_{[0,t-[t/b]b)},$$

where all terms are independent Poisson random variables, M_i being distributed as $N_{[0,b)}$, for $i=1,2,\ldots,[t/b]$.

3. Applications to Risk Theory

The properties of NPP's with periodic failure rates given in Theorem 1 suggest possible applications in risk theory. At first we will consider the modeling of the claims counting process. An application to the ruin problem follows.

3.1. The number of claims in an arbitrary time period

Assume that insurance is bought for a known term $[0,b)$, e.g $b=1$ for the usual one-year coverage. The number of claims recorded in an arbitrary period $[\tau,\tau+t) \subseteq [0,b)$, for a portfolio of K similar insurance policies, is denoted $N_{[\tau,\tau+t)}$ and has a Poisson distribution with parameter $\Lambda(\tau+t)-\Lambda(\tau) = \int_\tau^{\tau+t} \lambda(u)du$. Thus

$$(7) \quad P\{N_{[\tau,\tau+t)} = m\} = \frac{\{\int_\tau^{\tau+t} \lambda(u)du\}^m}{m!} \exp\{-\int_\tau^{\tau+t} \lambda(u)du\}, \quad m=0,1,2,\ldots$$

Let us further assume that $\lambda(t)$ takes the form in (3) for $t\in[0,b)$, i.e. the following periodic function of period b :

$$(8) \qquad \lambda(t) = \frac{K\lambda t^{p-1}(b-t)^{q-1}}{b^{p+q-2}} \quad \text{and} \quad \lambda(nb+t) = \lambda(t)$$

for $n\in\mathbb{N}$ and $t\in[0,b)$. We consider a few special cases of interest of the random variable in (7).

First the simple case $\tau=0$ and $t=b$ for $\lambda(t)$ as in (8). We get for N_b, the total number of claims over the first period, that

$$(9) \qquad P\{N_b = m\} = \frac{\{K\lambda b\ \beta(p,q)\}^m}{m!} \exp\{-K\lambda b\ \beta(p,q)\}\ ,\ m=0,1,2,\ldots$$

where

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 u^{p-1}(1-u)^{q-1}du$$

is the usual beta function for $p,q\geq 0$. The expected number of claims during the period is equal to its variance and is given by

$$\mathrm{E}(N_b) = K\lambda b\ \beta(p,q)$$

Here λ, p and q are parameters of the distribution and can be estimated from annual data records. In (8) K denotes the total number of policyholders over the period which is assumed known and constant.

Models with a random K can also be defined as immediate extensions and are similar to classical contagion models. For instance, if the number of policyholders over the year is a random variable J with known distribution, then its probability generating function (p.g.f.) is

$$P_J(z) = \mathrm{E}(z^J) = \sum_{k\geq 0} P\{J = k\}\ z^k$$

The total number of claims by time t is a random variable K_t, represented by a random sum of individual claim counting processes

$$K_t = N_1 + N_2 + \ldots + N_J$$

and the p.g.f. of the process K_t is given by

$$\mathrm{E}(z^{K_t}) = P_J(e^{\Lambda(t)(z-1)})$$

where $\Lambda(t)$ is given in (5), or in (6) for a fixed value of b. Hence, the expected number of claims over an interval $[0,t)$ is equal to

$$\mathrm{E}(K_t) = \Lambda(t)\mathrm{E}(J)$$

and its variance is

$$Var(K_t) = \Lambda(t)\{E(J) + \Lambda(t)Var(J)\}$$

If $P\{J = K\} = 1$ for a certain integer K, we obtain the previous result, where K_t is a superposition of K independent NPP's each with intensity function $\lambda(t)$.

Finally, when $\lambda(t)$ is as in (8), we can consider the number of claims during any arbitrary period $[\tau,\tau+t)$ within the τ-th year. Its expectation is given by:

(10) $$E(N_{[\tau,\tau+t)}) = K\lambda b\ \beta(p,q;t)$$

where $\beta(p,q;t) = \int_0^t u^{p-1}(1-u)^{q-1}du$ is the incomplete beta function. Numerical values of (10) can be obtained by computer and/or with statistical tables of the beta distribution, for p and q known.

3.2 Compound NPP claims processes with periodic intensity

Let C_i denote the i-th claim amount. Now assume C_i does not depend on the claim occurrence time and that $\{C_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables with c.d.f. $F(x) = P\{C_i < x\}$. Then the total claims for time period $[0,t)$ can be written as

(11) $$S_t = \sum_{n=0}^{N_{[0,t)}} C_n$$

where $N_{[0,t)}$ is assumed to be a NPP with the properties established in Theorem 1 and $C_0 \equiv 0$. From Theorem 1-(iv) we can re-write (11) in the form

(12) $$S_t = S_1 + S_2 + \dots + S_{[t/b]} + S_{[0,t-[t/b]b)} ,$$

where the S_i's are i.i.d. random variables equally distributed to

(13) $$S_b = \sum_{n=0}^{N_{[0,b)}} C_n$$

and $N_{[0,b)}$ is a Poisson r.v. with parameter $\Lambda(b) = \int_0^b \lambda(u)du$. Thus, the moment generating function (m.g.f.) of each S_i is given by

(14) $$E\{exp(tS_b)\} = exp(\Lambda(b)\{M_C(t)-1\})$$

where $M_C(t)$ is the m.g.f. of a single claim amount C_n. By contrast, $S_{[0,t-[t/b]b)}$ in (12) is independent of the previous terms and is equivalent to the random sum

(15) $$S_{[0,t-[t/b]b)} = C_1 + C_2 + \ldots + C_{N_{[0,t-[t/b]b)}}$$

where $N_{[0,y)}$ is the random number of events of that part of the original NPP claim-counting process, for which $y \in [0,b)$.

Moments of S_t can be obtained from (14) and (15). Exact premiums can thus be calculated with compound NPP's as easily as in the classical compound Poisson case. The introduction of time periodicity does not seem to add any difficulties in rate making. We will now see how it affects the solution of the ruin problem.

3.3 Ruin probabilities

3.3.1 The general NPP case

Consider the ruin probability in the case of a NPP with known claim intensity $\lambda(t)$. Let us define the *initial age* of the process as its age $\tau \geq 0$ when observation starts at time $t=0$. The number of claims is counted over age intervals $[\tau, \tau+t)$, for $t \geq 0$, and is denoted by $N_{[\tau,\tau+t)}$. The distribution of $N_{[\tau,\tau+t)}$ is given by (7) and can be expressed in a more compact form in terms of the hazard function $\Lambda(t)$. Thus the time of occurrence of the first claim follows the law

(16) $$g_\tau(t)dt = P\{N_{[\tau,\tau+u)} = 0 \text{ for all } u\le t \ , \ N_{[\tau+t,\tau+t+dt)} = 1\}$$
$$= \lambda(\tau+t)exp(-\{\Lambda(\tau+t)-\Lambda(\tau)\})dt$$

which defines the density g_τ of this time to the first claim. As in the previous section we assume that the claim severities $\{C_n\}_{n=1}^{\infty}$ are i.i.d. with common distribution F and density f. Now, denote by $R_\tau(u,t)$ the surplus process at time t when the initial reserve (at time 0) is u, with a claims process of initial age τ. In the classical, constant and continuous premium environment we have

(17) $$R_\tau(u,t) = u+ct-\sum_{n=0}^{N_{[\tau,\tau+t)}} C_n \quad , \quad \tau\ge 0 \ , \ t\ge 0$$

Define $T_\tau(u) = inf\ \{\ t\ge 0\ ;\ R_\tau(u,t)<0\ \}$, i.e. the time to ruin under the above conditions, and introduce the following notation

(18) $$\psi_\tau(u,t,s)dtds = P\{T_\tau(u) = t \ , \ R_\tau(u,t) = -s\}, \quad s>0$$

for the joint probability density that ruin occurs at time t and that its severity equals s. We can now prove the following result.

Lemma 1: The density $\psi_\tau(u,t,s)$ is solution of the following recursive functional equation:

(19) $$\psi(u,t,s) = \lambda(\tau+t)exp(-\{\Lambda(\tau+t)-\Lambda(\tau)\})\ f(u+ct+s) +$$
$$+\int_0^t \lambda(\tau+v)exp(-\{\Lambda(\tau+t)-\Lambda(\tau)\}) \int_0^{u+cv} \psi_{\tau+v}(u+cv-y,t-v,s)dF(y)dv$$

Proof: Conditioning on the first occurrence of a claim we distinguish two cases:

(i) it occurs exactly at time t and it is of size $u+ct+s$, causing ruin at that instant with a severity of s ,

(ii) it occurs at some prior time v , where $o\le v<t$, it does not cause ruin. Then, starting with an initial amount of $u+cv-C_1\ge 0$ and

with an initial age of $\tau+v$, ruin occurs in exactly $t-v$ years, that is $T_{\tau+v}(u+cv-C_1) \in [t-v,\ t-v+dt)$. Then (19) follows from (16) and by conditioning on the possible values of C_1 and its occurrence times. □

We define

(20) $$g_\tau(u,t)dt = P\{T_\tau(u) \in (t,t+dt]\} = \int_0^\infty \psi_\tau(u,t,s)ds$$

the marginal density of the time to ruin $T_\tau(u)$. It is obtained from the joint density $\psi_\tau(u,t,s)$ by integrating over all possible ruin severities $s\geq 0$. Therefore, the following results holds.

Corollary 1: The density $g_\tau(u,t)$ is solution of the following functional equation

(21) $$g_\tau(u,t) = \lambda(\tau+t)exp(-\{\Lambda(\tau+t)-\Lambda(\tau)\})\ \{1-F(u+ct)\} + \int_0^t \lambda(\tau+v)exp(-\{\Lambda(\tau+v)-\Lambda(\tau)\}) \int_0^{u+cv} g_{\tau+v}(u+cv-y,t-v)dF(y)dv$$

From the density of $T_\tau(u)$ one can find the marginal c.d.f. of the time to ruin,

(22) $$\psi_\tau(u,z) \overset{def}{=} P\{T_\tau(u) < z\} = \int_0^z g_\tau(u,t)dt,$$

which is also the probability of ruin within z years when the initial reserve is u and the initial age of the claim process is τ. The main conclusion of the above discussion rests in Theorem 2.

Theorem 2: The probability of ruin $\psi_\tau(u,z)$ in (22) satisfies the following integral equation

(23) $$\psi_\tau(u,z) = \int_0^z \lambda(\tau+t)exp(-\{\Lambda(\tau+t)-\Lambda(\tau)\})\ \{1-F(u+ct)\}dt + \int_0^z \lambda(\tau+t)exp(-\{\Lambda(\tau+t)-\Lambda(\tau)\}) \int_0^{u+ct} \psi_{\tau+t}(u+ct-y,z-t)dF(y)dt$$

Proof: According to (22) the probability of ruin can be obtained by integrating (21) over the interval $[0,z]$. The first term of the right hand side in (23) is clear from (21). The second term is obtained after a variable change in the integration in (21), i.e.

$$\int_0^z \left(\int_0^t \lambda(\tau+v)exp(-(\Lambda(\tau+v)-\Lambda(\tau))) \int_0^{u+cv} g_{\tau+v}(u+cv-y,t-v)dF(y)dvdt\right.$$

$$= \int_0^z \int_0^{u+cv} \lambda(\tau+v)exp(-(\Lambda(\tau+v)-\Lambda(\tau)))\left(\int_0^z g_{\tau+v}(u+cv-y,t-v)dt\right)dF(y)dv$$

$$= \int_0^z \lambda(\tau+v)exp(-(\Lambda(\tau+t)-\Lambda(\tau))) \int_0^{u+cv} \psi_{\tau+v}(u+cv-y,z-v)dF(y)dv \qquad \square$$

Corollary 2: The probability of ultimate ruin

$$\psi_\tau(u) = \lim_{z\to\infty} \psi_\tau(u,z)$$

in the case of a NPP claim process starting at age τ, satisfies the following integral equation:

$$(24) \quad \psi_\tau(u) = \int_0^\infty \lambda(\tau+t)exp(-(\Lambda(\tau+t)-\Lambda(\tau))) \int_0^\infty \psi_{\tau+t}(u+ct-y)dF(y)dt$$

where we assume $\psi_\tau(u)=1$ for $u<0$, $\tau\geq 0$.

Proof: Equation (24) follows after taking the limit $z \longrightarrow \infty$ in (23) and using the uniform integrability of the function in the second term of the right hand side. □

If $\lambda(t) = \lambda$ is constant then (24) coincides with the classical result of Gerber(1979), p.114.

Let us now consider the probability of ruin $\psi_0(u,z)$. By analogy with Lundberg's theory on the adjustment coefficient and bounds on ruin probabilities, let us denote by R_z a positive solution, if it exists, of the following equation

$$(25) \qquad M_C(R) \int_0^z \lambda(t)e^{-\Lambda(t)}e^{-Rct}dt = 1$$

where it is assumed that the moment generating function (m.g.f.) $M_C(R) = \int_0^\infty e^{Ry} dF(y)$ exists for some $R > 0$. Then the following holds.

Theorem 3: The probability of ruin $\psi_0(u,z)$ satisfies the inequality

$$\psi_0(u,z) \le exp(-R_0 u)$$

where R_0 is defined as the lowest upper bound of all positive solutions of (25).

Proof: First we observe that positive solutions R_z of (25) exist and these decrease in $z>0$. Equation (25) can be re-written in the form

$$\int_0^z \lambda(t)e^{-\Lambda(t)}e^{-Rct}dt = 1/M_C(R) \tag{26}$$

The right hand side of (26) is decreasing in R , from 1 to 0, over the interval $[0,R_0)$. The left hand side in (26) is positive and less than 1, it is increasing in z and also positive for $R \in [0,R_0)$. Therefore, solutions R_z of (26) exist, are decreasing in z and thus belong to $[0,R_0)$.

The rest of the proof is by induction as in the classical case (e.g. refer to Gerber,1979, pp.119-120). Denote by $\psi_\tau^{(n)}(u,t)$ the probability that ruin occurs with or before the n-th claim, within the age interval $(\tau,\tau+t]$, when the initial surplus is u ($n = 1,2,\ldots,$ $0\le\tau\le\tau+t\le z$). By induction with respect to n we see that

$$\psi_\tau^{(n)}(u,t) \le exp(-R_0 u) \text{ , } \tau \text{ and } \tau+t\in(0,z] \tag{27}$$

This inequality implies that cf Theorem 3, since $\psi_0(u,z) = \lim_{n\to\infty} \psi_0^{(n)}(u,z)$, with the understanding that $\psi_0^{(o)}(u,t) = 1$ for $u<0$, $\psi_0^{(o)}(u,t) = 0$ for $u\ge0$ and, hence, that $\psi_0^{(o)}(u,t) \le exp(-R_0 u)$ for all u . Moreover, the same considerations also lead to

$$\psi_{\tau}^{(o)}(u,t) \le exp(-R_t u) \le exp(-R_z u) \text{ , for } \tau \text{ and } \tau+t \in (0,z]$$

Assume (27) is true for some n=1,2,... and then apply the law of total probability, conditioning on the time and the amount of the first claim within $(0,z]$. The induction assumption gives

$$\begin{aligned}\psi_0^{(n)}(u,z) &= \int_0^z \lambda(t)e^{-\Lambda(t)} \int_0^\infty \psi_t^{(n-1)}(u+ct-y,z-t)dF(y)dt \\ &\le \int_0^z \lambda(t)e^{-\Lambda(t)} \int_0^\infty exp\{-R_z(u+ct-y)\}dF(y)dt \\ &= exp(-R_z u)\ \{\int_0^z \lambda(t)e^{-\Lambda(t)}e^{-R_z ct}dt\}M_C(R_z) \\ &= exp(-R_z u)\end{aligned}$$

□

Corollary 2: The probability $\psi_0(u)$ of ultimate ruin satisfies

$$\psi_0(u) \le exp(-R_\infty u) \text{ for all } u$$

where R_∞ is the positive solution of the equation

$$M_C(R) \int_0^\infty \lambda(t)e^{-\Lambda(t)}e^{-Rct}dt = 1$$

Proof: It follows from Theorem 3 and (25), since $\psi_0(u) = \lim_{z\to\infty} \psi_0(u,z)$ and $R_\infty = \lim_{z\to\infty} R_z$. □

The result of Theorem 3 can be used even in the case of compound Poisson claim processes to improve the upper bounds for the probability of ruin on a finite interval. In that case equation (25) reduces to

$$\frac{\lambda M_C(R)}{\lambda + cR}\{1 - exp(-\lambda z-cRz)\} = 1$$

We have $R_z > R_\infty$, for $0<z<\infty$, and therefore the upper bound $exp(-R_z u)$ can be smaller than $exp(-R_\infty u)$.

3.3.2. The periodic NPP case

Now we turn to the case where the intensity function $\lambda(t)$ is periodic with period $b>0$. Then $\lambda(nb+t) = \lambda(t)$ for all $t \geq 0$ and for any integer $n=1,2,...$

From (17) and from Theorem 1-(iv) we obtain the following representation for the surplus process $R_0(t,u)$:

$$R_0(t,u) = u + ct - \sum_{n=0}^{N_{[0,t)}} C_n \tag{28}$$

$$\stackrel{\mathcal{D}}{=} u + c\{[t/b]b + (t - [t/b]b)\} - \sum_{k=1}^{[t/b]} \sum_{n=0}^{M_k} C_n^{(k)} - \sum_{n=0}^{N_{[0,t-[t/b]b)}} C_n^{([t/b]+1)}$$

$$= u + \sum_{k=1}^{[t/b]} \{cb - \sum_{n=0}^{M_k} C_n^{(k)}\} + \{cy - \sum_{n=0}^{N_{[0,y)}} C_n^{([t/b]+1)}\}$$

where $y = t - [t/b]b \in [0,b)$, the $\{C_n^{(k)}\}_{n=0}^{\infty}$, $k=1,2,...$ are independent sequences of i.i.d. random variables distributed as a single claim amount C_n, and M_k (respectively $N_{[0,y)}$) are i.i.d. Poisson random variables with parameter $\Lambda(b)$ (resp. $\Lambda(y)$), independent of $\{C_n^{(k)}\}_{n=0}^{\infty}$.

We introduce the new random variables

$$U_k = cb - \sum_{n=0}^{M_k} C_n^{(k)}, \quad k = 1,2,... \tag{29}$$

Then $\{U_k\}_{k=1}^{\infty}$ are i.i.d. and their common m.g.f. is $M_U(t) = \mathrm{E}(e^{tU}) = e^{tcb}exp(\Lambda(b)\{M_C(t)-1\})$. Consider now the random walk (k,S_k), $k=0,1,2,...$, where $U_0=u$ and

$$S_k = U_0 + U_1 + U_2 + ... + U_k, \quad k=0,1,2,... \tag{30}$$

Denote by $\tilde{T}$ its first passage time to a negative value, i.e.

(31) $$\tilde{T}(u) = \inf\{k\geq 0\ ;\ S_k<0\}.$$

The process S_k denotes the insurer's surplus at time k, k=0,1,2,... in a discrete time collective risk model (see for instance Bowers et al., 1986, pp.354-358). Then u is the initial surplus, cb is the amount of premiums received each period and

$$W_k = \sum_{n=0}^{M_k} C_n^{(k)}$$

is the sum of the claims in period k. As it is shown above, $W_1, W_2, \ldots$ are i.i.d. random variables which have compound Poisson distribution $F_W(y)$ with parameter $\Lambda(b)$. Moreover, $\tilde{T}(u)$ in (31) is the time to ruin for the discrete risk model; we say, it is an "imbedded discrete risk model" into the original continuous risk model with periodic NPP of claims.

Let $T_0(u) = \inf\{t\geq 0;\ R_0(t,u)<0\}$ be the time to ruin of our original risk process with periodic intensity function $\lambda(t)$ of period $b>0$. Then the following is true.

Theorem 4: The time to ruin distribution satisfies the following inequality

(32) $$P\{T_0(u)>t\} \leq P\{\tilde{T}(u)>[t/b]\}$$

where $\tilde{T}(u)$ is defined in (31).

Proof: We have the following obvious relations, obtained on the basis of representation (28) for the surplus process (for convenience we set $k=[t/b]$):

$$P\{T_0(u)>t\} = P\{R_0(s,u)\geq 0 \text{ for all } s\in[0,t]\}$$

$$\leq P\{R_0(s,u)\geq 0 \text{ for } s=nb,\ n=1,\ldots,k \text{ and for } s\in[kb,t]\}$$

$$= P\{S_1\geq 0,\ S_2\geq 0,\ \ldots,\ S_k\geq 0,\ R_0(s,u+S_k)\geq 0 \text{ for all } s\in[0,t-kb]\}$$

$= P\{\tilde{T}(u)>k,\ R_0(s,u+S_k)\geq 0 \text{ for all } s\in[0,t-kb]\} \leq P\{\tilde{T}(u)>k\}.$ □

Therefore, a lower bound for the ruin probability $\psi_0(u,t)$ is given by

$$\psi_0(u,t) \geq \tilde{\psi}(u,[t/b]) = \sum_{n=1}^{[t/b]} P\{\tilde{T}(u)=n\}. \tag{33}$$

Here $\tilde{\psi}(u,k) = 1 - P\{\tilde{T}(u)>k\}$ is the probability that the random walk (n,S_n) hits $(-\infty,0)$ within k moves, if $S_0 = u$. There are general results for random walks in $\mathbb{R}$ (e.g. Feller, 1966, chapter 12) one can use to derive explicit forms for the distribution of $\tilde{T}(u)$ and thus for the probability of ruin. For instance the p.g.f. of $\tilde{T}(u)$ is seen to be

$$P_{\tilde{T}}(s) = \sum_{n=1}^{\infty} P\{\tilde{T}(u)=n\}s^n = 1 - \exp\left(\sum_{n=1}^{[t/b]} \frac{s^n}{n} P\{S_n<-u\}\right)$$

Moreover, the inequality

$$\mathbf{E}(U_k) = cb - \Lambda(b)\mathbf{E}(C_n) > 0$$

implies that $P\{\tilde{T}(u)<\infty\} < 1$, i.e. this is a necessary and sufficient condition for the probability of ultimate ruin of the imbedded discrete risk model to be less than one. Relation (32) shows that the r.v. $T_0(u)$ is in some sense stochastically smaller then $\tilde{T}(u)$.

The probabilities $\tilde{\psi}(u,k)$ satisfy the following recurrent equations:

$$\tilde{\psi}(u,k+1) = 1 - F_W(u+cb) + \int_0^{u+cb} \tilde{\psi}(u+cb-y,k)dF_W(y)$$

with $\tilde{\psi}(u,0)=0$ for $u\geq 0$.

The inequality in (33) remains valid also for the probability of ultimate ruin. This is seen after taking the limit as $t\to\infty$ (and therefore as $[t/b] \to \infty$).

Referring to Theorem 12.2 of Bowers et al.(1986) we can also conclude that the probability of ultimate ruin $\psi(u) = \lim_{t \to \infty} \psi(u,t)$ satisfies the following inequality

$$\psi(u) \geq \frac{exp(-\tilde{R}u)}{E(exp(-\tilde{R}S_{\tilde{T}(u)}) \mid \tilde{T}(u)<\infty)}$$

where $\tilde{R}$ is the positive solution of the equation

$$\Lambda(b)(M_C(R)-1) - cRb = 1$$

Finally we consider the maximal loss random variable

$$L = \sup_{t \geq 0} (S_t - ct),$$

where S_t , given by (11), is the total claims for time interval $[0,t]$. Denote by $\tilde{L}$ the maximal loss for the imbedded discrete model, i.e.

$$\tilde{L} = \sup_{k \geq 1} (W_1+W_2+...+W_k - kcb).$$

Then the following is true.

Theorem 5: The maximal loss L in the continuous time model with an NPP claim process and periodic intensity is stochastically larger than the maximal loss $\tilde{L}$ for the imbedded discrete risk model.

Proof: It follows from the following inequality,

$$\sup_{t \geq 0} (S_t - ct) \geq \sup_{k \geq 0} (S_{kb} - ckb),$$

and from (12) which gives $S_{kb} = W_1+W_2+...+W_k$. □

As in the classical case, the imbedded discrete model is a useful tool for comparing the ruin probabilities and maximal losses of continuous risk models where the claim process is described by an NPP with periodic intensity.

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