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Dynamic Spanning of Contingent Claims

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Abstract

Most of the literature on the arbitrage-free pricing of contingent claims places its primary emphasis on methods of computing the prices of contingent claims. The underlying economic principle that permits one to calculate the price of a contingent claim within the arbitrage-free framework is that the contingent claim may be replicated through an appropriate investment portfolio of risky assets and the risk-free asset. Although these replicating strategies are implicit in the analysis, they are not required to compute the price of a contingent claim. Consequently, in most of this literature the replicating strategies are not explicitly discussed. In this paper we discuss the link between the price of a contingent claim and its replicating strategies. We compute the replicating strategies for some contingent claims when the price of stocks are modelled using either the Wiener process or the Poisson process. These replicating strategies are essential knowledge if one wishes to implement a dynamic trading strategy. It is hoped that this paper will provide the reader who is interested in the theory of arbitrage-free pricing of contingent claims with an opportunity to peruse the underlying dynamic trading component of the theory. The notion of Arrow-Debreu security is reviewed and

applied. The lookback option, an exotic option of recent practical and theoretical interest, is examined.

1. Introduction

The main focus of the literature dealing with the valuation of contingent claims is on characterising the price of a contingent claim. This characterisation may be in the form of an abstract expression, a numerical procedure, or a closed formula. Although the underlying theory that supports these characterisations of price involves the notion that there exist investment strategies that replicate the payoffs of the contingent claim, these investment strategies do not usually receive much attention in the literature. One reason for this is that in practice one is seeking to price a contingent claim that is traded on an exchange, such as an option traded on the Chicago Board Options Exchange (CBOE). In such an instance, one is buying and selling options and is interested in getting an assessment of the value of the option and not in synthetically reproducing the payoffs of the option by managing a portfolio of assets. However, if one wants to fully appreciate the concepts that underlie the theory of contingent claims valuation then an understanding of the role that these replicating portfolios play is important. Moreover, in order to dynamically manage an investment portfolio to some desired terminal wealth distribution, an investor will require the appropriate portfolio mix throughout the investment horizon.

In this paper we discuss the relation of contingent claims valuation formulas to the replicating strategies that synthetically produce these contingent claims. In the case of the Wiener process model we provide a simple relation for a wide class of contingent claims. In the case of the Poisson process model we determine the replicating strategies completely. In theory, a knowledge of these replicating strategies will allow an investor to customise the payoffs that he receives at the end of his investment horizon. However, in practice there are market frictions that make it impossible for a small investor to carry out such a procedure, to say nothing of the fact that the assumed model for the stock price dynamics may be inaccurate. Nevertheless, the intention of this paper is to offer an accessible treatment of the relationship between the valuation of contingent claims and the portfolios that replicate these claims so that interested readers may enhance their understanding of the theory of contingent claims valuation.

Although a knowledge of how to calculate the investment strategy that replicates any terminal payoff distribution enables an investor to generate the terminal payoff scenario of his choice, the terminal payoff distribution that an investor chooses will depend on his tolerance for risk. Ultimately, the analysis of such portfolio selection lies in the province of expected utility theory. We do not attempt to discuss this here. Instead, we discuss the investment strategies that an investor can employ to obtain a specified choice of terminal payoffs. We note that for the class of diffusion models the expected utility analysis has been elegantly merged with the contingent claims valuation framework by Cox and Huang(1989.a).

2. A General Model

In this section we outline a general model for securities trading as set forth in Harrison and Pliska (1981). We attempt only to convey the flavour of their framework here and we therefore confine ourselves to the conceptual aspects of their model. The reader who wishes to also study the important technical components of the theory may consult Harrison and Pliska (1981), Duffie (1992), or Müller (1985).

Trading will take place over the time interval $[0, T]$. The securities that are available for trading consist of a risk-free asset, denoted B, and n stocks, denoted $S_1, \ldots, S_n$. An investor may take positions in any of these assets. The value of the risk-free asset is assumed to accumulate at force of interest δ and is taken to be given by $B(t) = \exp(\delta t)$. Consequently, money invested in the risk-free asset amounts to placing cash on deposit at force of interest δ. The number of shares of stock j held by an investor at time t is denoted by $\theta_j(t)$. The number of units of the risk-free asset held by an investor at time t is denoted by $\phi(t)$. The vector $(\phi, \theta_1, \ldots, \theta_n)$ is referred to as an *investment strategy*. The value of an investor's portfolio at any point in time is then

$$\phi(t)B(t) + \theta_1(t)S_1(t) + \theta_2(t)S_2(t) + \cdots + \theta_n(t)S_n(t)\,.$$

For ease of notation we will denote this quantity by $V(t)$.

What happens to an investor's portfolio when the asset prices change? If at time t an investor is holding $\phi(t)$ units of the risk-free asset and $\theta_j(t)$ shares of stock j then over the next instant the value of his portfolio changes by

$$\phi(t)[B(t+dt) - B(t)] + \theta_1(t)[S_1(t+dt) - S_1(t)] + \cdots + \theta_n(t)[S_n(t+dt) - S_n(t)]\,.$$

If we add up[1] all of these changes in the investor's portfolio from the beginning of the trading interval to time t then the change in the value of his portfolio over [0, t] due to capital gains and losses is

$$\int_0^t \phi(u)dB(u) + \int_0^t \theta_1(u)dS_1(u) + \cdots + \int_0^t \theta_n(u)dS_n(u)\,.$$

If all changes in the investor's portfolio are due to capital gains and losses then the value of his portfolio at time t is

$$V(0) + \int_0^t \phi(u)dB(u) + \int_0^t \theta_1(u)dS_1(u) + \cdots + \int_0^t \theta_n(u)dS_n(u)\,.$$

[1] How one adds up these changes depends on the nature of the stock price behaviour that is assumed. For the case of the Wiener process one must use the technique of stochastic integration. For the case of the Poisson process one can use the ordinary Riemann-Stieltjes integral. In any case, the type of limiting procedure that is employed does not affect our interpretation of the investor's holdings of stock and risk-free asset for a particular investment strategy.

An investment strategy for which all changes in value are the result of capital gains and losses is called *self-financing*. Thus a self-financing investment strategy is an investment strategy which at all times $0 \le t \le T$ satisfies the relation

$$V(t) = V(0) + \int_0^t \phi(u)dB(u) + \int_0^t \theta_1(u)dS_1(u) + \cdots + \int_0^t \theta_n(u)dS_n(u) .$$

One understands this relation to say that an investor begins with a total investment of V(0) and that any subsequent changes in the value of his portfolio are due to capital gains and losses in his investment portfolio.

A *contingent claim* is a random payment that is received at time T. For instance, a contingent claim could be $S_2(T)$, the value of the second stock at the terminal date T or it might be the larger of zero and $S_2(T) - 40$, the payoff from a European call option written on the second stock with a strike price of $40. We will say that a contingent claim X is *spanned* if there is a self-financing investment strategy which replicates the payoffs from X at time T, i.e. V(T) = X. When a claim is spanned its *price* is defined to be the initial cost of the self-financing investment strategy that replicates that claim's payoff, which we have denoted by V(0). This is a sensible definition since any other price would give rise to an arbitrage opportunity. More generally, the price of a spanned claim at time t is seen to be equal to V(t). We emphasise that for a given model there is no reason why a claim should be spanned. This issue must be addressed in the context of each particular model that is employed. Claims that are not spanned cannot be priced by the reasoning that we have described here.

For the Wiener and Poisson models [sections 3 and 4], the prices of contingent claims can be obtained by means of the transformed probability measure, for which

$$S_i(0) = e^{-\delta t}E[S_i(t)] ,$$

where δ is the risk-free force of interest. Specifically, the value of a contingent claim at time t is given by

$$e^{-\delta(T-t)}E_t[X], \tag{1}$$

where the subscript t serves to denote that all information available up to time t is taken into consideration in computing the expectation. Intuitively, one says that the value of the claim is the expected discounted value of the implied payments. Although we are not going to justify this characterisation of prices, we will employ it in what follows. The reader may consult Gerber and Shiu (1993) for many practical applications of this result.

3. Stock Prices Modelled using the Wiener Process

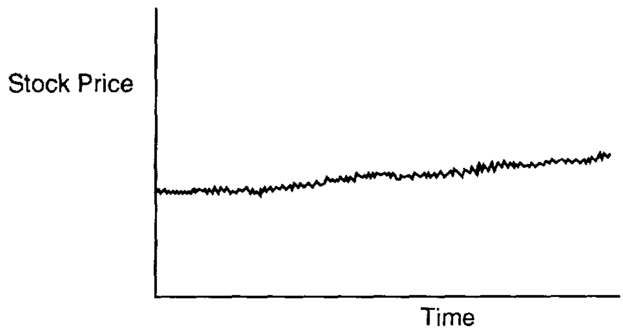
In this section we will discuss the replication of contingent claims within the workhorse model of option pricing, the case of stock prices modelled using the Wiener process. The one-dimensional version of this model provided the setting for the famous analysis of Black and Scholes (1973).

Let $(X_1(t), \ldots, X_n(t))$ denote an n-dimensional Wiener process with nonsingular covariance matrix $\Sigma = \{\sigma_{ij}\}$ per unit time. Stock prices are defined by the equation

$$S_i(t) = S_i(0)\exp[X_i(t)].$$

Thus the stock prices are continuous non-negative processes and the movements of each stock are permitted to be correlated with the movements of some of the other stocks. The usual interpretation of this model is that each stock has an expected earnings rate which is subject to random fluctuations due to uncontrollable economic variables. In consequence, the price of each stock will tend to follow a deterministic growth pattern with ongoing random fluctuations from this growth pattern. A picture of a typical stock price history for this model is shown in figure 1.

figure 1



One class of contingent claims that are ubiquitous in practice are those contingent claims for which the random payment at time T is some function of the stock prices at that time. Mathematically, such contingent claims are of the form

$$X = \varphi(S_1(T), \ldots, S_n(T)) .$$

Among these contingent claims are European call and put options on a stock. As we have noted, the price of a contingent claim is given by an expected discounted value [equation (1)]. For the contingent claim that we are considering, the expression in equation (1) is a function of time and the current stock prices only. This follows from the independent increments property of the Wiener process and the special form of the contingent claim. Therefore, let us denote the price of this claim at time t by $V(S_1(t), \ldots, S_n(t), T-t)$. It follows that the process

$$\{e^{-\delta t}V(S_1(t), \ldots, S_n(t), T-t)\}$$

is a martingale for $0 \le t \le T$. Indeed, one need only note that

$$e^{-\delta t}V(S_1(t), \ldots, S_n(t), T-t)) = e^{-\delta T}E_t[\varphi(S_1(T), \ldots, S_n(T))], \tag{2}$$

and recall one of the basic examples of a martingale as noted in Gerber (1979, method 1, page 35). We now establish that the investment strategy that replicates the payoff from this contingent claim may be expressed in terms of our function $V(S_1(t), \ldots, S_n(t), T-t)$. Consequently, a knowledge of the price of this claim at each point in time provides us with the information that we require to synthetically produce this claim by managing a portfolio of stocks and the risk-free asset.

Theorem 1 The self-financing investment strategy that replicates the contingent claim $X = \varphi(S_1(T), \ldots, S_n(T))$ may be computed from the function V. The replicating strategy is

$$\theta_i(t) = \frac{\partial V}{\partial s_i}, \quad i = 1, \ldots, n$$

$$\phi(t) = \left[V - \sum_{i=1}^{n} \frac{\partial V}{\partial s_i} S_i(t)\right] / e^{\delta t}.$$

Remarks: (i) The above expression for the replicating strategy involves an abbreviation of notation which would otherwise be written as

$$\theta_i(t) = \frac{\partial V(S_1(t), \dots, S_n(t), T-t)}{\partial s_i}, \quad i = 1, \dots, n$$

$$\phi(t) = \left[V(S_1(t), \dots, S_n(t), T-t) - \sum_{i=1}^{n} \frac{\partial V(S_1(t), \dots, S_n(t), T-t)}{\partial s_i} S_i(t) \right] / e^{\delta t}.$$

(ii) We emphasise that the amount of stock and risk-free asset held in the investment portfolio at any point in time are random variables that depends on the current stock prices. This is not surprising since one would expect that the investor will be adjusting his portfolio in response to changes in the stock prices.

(iii) Theorem 1 says that a knowledge of the price function for a contingent claim is sufficient to determine the replicating strategies for that contingent claim.

Proof: For brevity, we denote $V(S_1(t), \dots, S_n(t), T-t)$ by V_t. By Itô's lemma[2]

$$\begin{aligned} d(e^{-\delta t}V_t) &= -\delta e^{-\delta t}V_t dt + e^{-\delta t} dV_t \\ &= -\delta e^{-\delta t}V_t dt + e^{-\delta t}\left[-\frac{\partial V}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial V}{\partial s_i} dS_i + QVdt \right], \end{aligned}$$

where

$$QV = \tfrac{1}{2}\mathbf{S'HS}, \text{ with } h_{ij} = \sigma_{ij}\frac{\partial^2 V}{\partial s_i \partial s_j}, \text{ and } \mathbf{S} = (S_1, \dots, S_n).$$

Since $d(e^{-\delta t}S_i(t)) = -\delta e^{-\delta t}S_i(t)dt + e^{-\delta t}dS_i(t)$, we may rearrange this expression as

[2]The usual rules of calculus apply except for the additional term QV.

$$d(e^{-\delta t}V_t) = e^{-\delta t}\left[-\delta V_t - \frac{\partial V}{\partial t} + QV_t + \sum_{i=1}^{n}\frac{\partial V}{\partial s_i}\delta S_i\right]dt + \sum_{i=1}^{n}\frac{\partial V}{\partial s_i}d(e^{-\delta t}S_i). \tag{3}$$

As $e^{-\delta t}V_t$ is a martingale [equation (2)] and as $\{e^{-\delta t}S_i(t)\}$ are also martingales for each i [page 5], it follows that the sum of the dt terms must vanish. Therefore,

$$d(e^{-\delta t}V_t) = \sum_{i=1}^{n}\frac{\partial V}{\partial s_i}d(e^{-\delta t}S_i).$$

Substituting $d(e^{-\delta t}S_i(t)) = -\delta e^{-\delta t}S_i(t)dt + e^{-\delta t}dS_i(t)$, doing the same for V_t, cancelling $e^{-\delta t}$, and rearranging terms it follows that

$$dV_t = e^{-\delta t}\left[V_t - \sum_{i=1}^{n}\frac{\partial V}{\partial s_i}S_i\right]dB(t) + \sum_{i=1}^{n}\frac{\partial V}{\partial s_i}dS_i.$$

In integrated form this expression is

$$V_t = V_0 + \int_0^t e^{-\delta t}[V_t - \sum_{i=1}^{n}\frac{\partial V}{\partial s_i}S_i]dB(t) + \sum_{i=1}^{n}\int_0^t \frac{\partial V}{\partial s_i}dS_i.$$

Therefore, the investment strategy $(\phi, \theta_1, \dots, \theta_n)$ is a self-financing strategy that replicates the contingent claim. QED.

We illustrate Theorem 1 by exhibiting the replicating strategies for a European call option with strike price equal to K in the Black-Scholes model. As is well known[3], the price of this option at time t is given by

[3]An elegant derivation may be found in Gerber and Shiu (1993, (3.1.3)).

$$S(t)\Phi\left(\frac{-\kappa+(\delta+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)-e^{-\delta\tau}K\Phi\left(\frac{-\kappa+(\delta-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right),$$

where Φ is the standard normal distribution function, $\kappa := \ln[K/S(t)]$, and $\tau := T-t$. Since the terminal payoff for this contingent claim is $X = \mathrm{Max}[0, S(T) - K]$, Theorem 1 is applicable. Calculating $\partial V/\partial S$, we find that the replicating strategy is

$$\theta(t) = \Phi\left(\frac{-\kappa+(\delta+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right)$$

$$\phi(t) = -e^{-\delta T}K\Phi\left(\frac{-\kappa+(\delta-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right).$$

In some contexts, θ is referred to as the *delta* of the contingent claim [Hull (1989), pages 186-194].

Let us examine the case when the contingent claim has a payoff function which is homogeneous of order 1. Mathematically then, we are considering a payoff function φ which satisfies

$$\varphi(\lambda s_1, \dots, \lambda s_n) = \lambda\varphi(s_1, \dots, s_n)$$

for all $\lambda > 0$. It follows from equation (2) that $V(s_1, \dots, s_n, T-t)$ has this same property. Therefore, by Euler's theorem [Olmsted (1961), page 272] we see that

$$V(s_1, \dots, s_n, T-t) = \frac{\partial V}{\partial s_1}s_1 + \cdots + \frac{\partial V}{\partial s_n}s_n.$$

Consequently, we see from Theorem 1 that the replicating strategy for any contingent claim that is homogeneous of order 1 is of the form

$$\theta_i(t) = \frac{\partial V}{\partial s_i}, \quad i = 1, \dots, n$$

$$\phi(t) = 0.$$

In particular, when replicating the payoff from such a contingent claim an investor will invest all of his capital in the stocks and nothing will be placed in the risk-free asset. Examples of contingent claims that are homogeneous of order 1 include

$$\varphi(s_1, \dots, s_n) = \mathrm{Max}[s_1, \dots, s_n]$$

$$\varphi(s_1, \dots, s_n) = (s_1 s_2 \cdots s_n)^{1/n}.$$

The first of these two examples appears in Margrabe (1979) and Johnson (1987).

Homogeneous payoff functions of order 1 result in a particular form for V. Let **s** denote the vector $(s_1, \dots, s_n)$. In general [noted after equation (3)], the price function V will satisfy the partial differential equation

$$-\delta V_t - \frac{\partial V}{\partial t} + \tfrac{1}{2}\mathbf{s'Hs} + \sum_{i=1}^{n} \frac{\partial V}{\partial s_i}\delta s_i = 0, \tag{4}$$

which together with the boundary condition $V(s_1, \dots, s_n, 0) = \varphi(s_1, \dots, s_n)$ determines the function V uniquely. In the case of our homogeneous payoff function equation (4) becomes

$$-\frac{\partial V}{\partial t} + \tfrac{1}{2}\mathbf{s'Hs} = 0$$

which together with the boundary condition $V(s_1, \dots, s_n, 0) = \varphi(s_1, \dots, s_n)$ again determines the function V uniquely. This implies that V is independent of δ. One may interpret this result as reflecting the fact that the risk-free asset is not needed to generate such a contingent claim. Gerber and Shiu (1993, (7.6)) is one example of this result.

A subclass of the contingent claims that are covered by Theorem 1 is the family of European call options on powers of the stock price, with strike price K. These European call options have terminal payoff $X = \mathrm{Max}[0, S^{\alpha}(T) - K]$ for $\alpha > 0$. This family of options is sometimes employed by practitioners because of its flexibility in setting up terminal payoff distributions and because the option prices [and replicating strategies] may be expressed as convenient closed formulas. Indeed, a straightforward application of the techniques from Gerber and Shiu (1993, III.1) establishes that the price of this option at time t is

$$S^{\alpha}(t)\exp[(\alpha-1)[\delta+\alpha\sigma^2/2]\tau]\Phi(\xi + \alpha\sigma\sqrt{\tau}) - e^{-\delta\tau}K\Phi(\xi) ,$$

where $\tau := T-t$ and $\xi := \dfrac{-\ln[K/S^{\alpha}(t)] + \alpha[\delta - \sigma^2/2]\tau}{\alpha\sigma\sqrt{\tau}}$. We now apply Theorem 1. Upon calculating $\partial V/\partial S$ we find that the appropriate replicating strategy is

$$\theta(t) = \alpha S^{\alpha-1}(t)\exp[(\alpha-1)[\delta + \alpha\sigma^2/2]\tau]\Phi(\xi + \alpha\sigma\sqrt{\tau})$$

$$\phi(t) = e^{-\delta T}[(1-\alpha)S^{\alpha}(t)\exp(\alpha[\delta + (\alpha-1)\sigma^2/2]\tau)\Phi(\xi + \alpha\sigma\sqrt{\tau}) - K\Phi(\xi)] .$$

For $\alpha = 1$ the valuation formula and the replicating strategies reduce to the Black-Scholes case. It is interesting to note that for all $\alpha > 0$ the replicating portfolio is always long in the stock, just as is the case for the Black-Scholes formula, but if $0 < \alpha < 1$ then when the stock price is sufficiently high the replicating portfolio will be long in the risk-free asset.

This is in contrast to the Black-Scholes case in which the replicating portfolio is always short in the risk-free asset.

4. Stock Prices Modelled using a Poisson Process

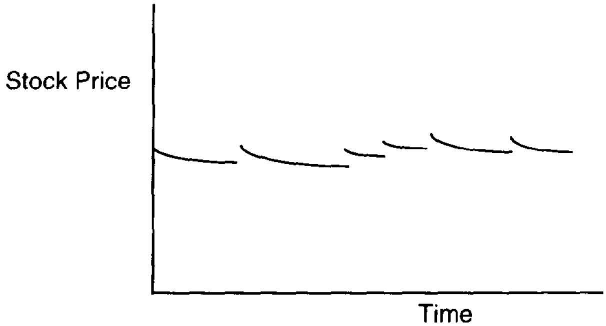
In this section we discuss the replication of contingent claims when the stock price is modelled using the Poisson process. Although a multi-dimensional counterpart to the Wiener process model may be defined, we will restrict our attention to a one-dimensional model. The Poisson model provides an interesting setting for the study of dynamic spanning and because of the countable state space, the Poisson model admits a more direct analysis than is possible for the Wiener process model. Our approach to obtaining the replicating strategies for contingent claims will take the characterisation of price as an expected discounted value as a natural guide to discovering the replicating strategies. This approach is comparable to the technique that was used in the previous section. However, some differences in the analysis are necessary because of the jump nature of the stock price process for the Poisson model. Nevertheless, if the reader compares the analysis of this section with that of the previous section it will be seen that the same basic ideas are common to both analyses.

Let $\{N(t)\}$ denote a Poisson process and let $X(t) = kN(t) - ct$, where k and c are constants. The stock price is defined by the equation

$$S(t) = S(0)\exp[X(t)].$$

The stock price process is a non-negative process which experiences occasional jumps as well as an ongoing deterministic drift. A picture of a typical stock price history for this model is shown in figure 2.

figure 2



Let us recall the notion of an *Arrow-Debreu security*. In general, an Arrow-Debreu security is a contingent claim which pays a unit in one and only one state of the world at time T and nothing in the other states [Ingersoll (1987)]. For the Poisson model, the states of the world at time T are indexed by the number of jumps in the stock price over the trading interval [0, T]. The Arrow-Debreu securities constitute a basis for all terminal payoff distributions. Consequently, a knowledge of the replicating strategies for each Arrow-Debreu security is sufficient to enable us to replicate the payoffs from a general contingent claim. In the following, we will obtain the replicating strategy for each Arrow-Debreu security. There is no counterpart to the Arrow-Debreu security for the Wiener process model. Arrow-Debreu securities play a fundamental role in finite state space models such as the binomial model.

Let $V^n(t)$ denote the price at time t of the Arrow-Debreu security for the state consisting of n jumps over [0, T]. $V^n(t)$ may be computed in accordance with equation (1). For the essential details of this calculation we refer the reader to Gerber and Shiu (1993, III.2). If $N(t) > n$ then $V^n(t) = 0$ since the number of jumps is non-decreasing and so the state $\{N(T) = n\}$ cannot be attained once $N(t) > n$. Otherwise, one has

$$V^n(t) = e^{-\delta(T-t)}\exp(-\lambda(T-t))\,[\lambda(T-t)]^{n-N(t)}\frac{1}{[n-N(t)]!}, \quad N(t) \le n \tag{5}$$

where $\lambda := (c+\delta)/(e^k-1)$. We note that this expression is the discounted adjusted probability that there are exactly $n-N(t)$ jumps in the stock price in the time remaining over the investment horizon.

We now establish a result that will take the place of Theorem 1 in our analysis. This result is usually referred to as a change of numeraire. The importance of this result for our analysis is that it affords a technique whereby we can determine the replicating strategies for each of our Arrow-Debreu securities. Define $S^*(t) := e^{-\delta t}S(t)$.

Theorem 2 Let θ be a process for which

$$e^{-\delta t}V^n(t) = V^n(0) + \int_0^t \theta(u)dS^*(u), \quad 0 \le t \le T \tag{6}$$

and define

$$\phi(t) := [V^n(t) - \theta(t)S(t)]\,/\,e^{\delta t}.$$

Then one has the relation

$$\phi(t)B(t) + \theta(t)S(t) = \phi(0)B(0) + \theta(0)S(0) + \int_0^t \phi(u)dB(u) + \int_0^t \theta(u)dS(u), \quad 0 \le t \le T.$$

Proof: For $0 \le t \le T$ we have $V^n(t) = e^{\delta t}V^n(0) + e^{\delta t}\int_0^t \theta(u)dS^*(u)$.

After integrating by parts and substituting from equation (6) we obtain

$$e^{\delta t}\int_0^t \theta(u)dS^*(u) = \int_0^t [\int_0^s \theta(u)dS^*(u)]de^{\delta s} + \int_0^t e^{\delta s}d[\int_0^s \theta(u)dS^*(u)]$$

$$= \int_0^t [e^{-\delta u}V^n(u) - V^n(0)]de^{\delta u} + \int_0^t e^{\delta u}\theta(u)dS^*(u)\,.$$

Integration by parts shows that $S^*(t) = -S(0) + \int_0^t e^{-\delta u}dS(u) + \int_0^t S(u)de^{-\delta u}$ and thus

$$e^{\delta t}\int_0^t \theta(u)dS^*(u) = \int_0^t \delta V^n(u)du - V^n(0)(e^{\delta t} - 1) + \int_0^t e^{\delta u}\theta(u)[e^{-\delta u}dS(u) + S(u)de^{-\delta u}]$$

$$= -V^n(0)(e^{\delta t} - 1) + \int_0^t \delta[V^n(u) - \theta(u)S(u)]du + \int_0^t \theta(u)dS(u)$$

$$= -V^n(0)(e^{\delta t} - 1) + \int_0^t [(V^n(u) - \theta(u)S(u))/e^{\delta u}]dB(u) + \int_0^t \theta(u)dS(u)\,.$$

Consequently, $V^n(t) = V^n(0) + \int_0^t \phi(u)dB(u) + \int_0^t \theta(u)dS(u)\,.$ QED.

We point out that although Theorem 2 is stated in terms of the price process for an Arrow-Debreu security, the proof applies for any contingent claim. Furthermore, although the proof of Theorem 2 for the Poisson model depends only on the real analysis tools that can be found in Apostol (1974), the same proof works for the general model of securities trading [Harrison and Pliska (1981)]. Theorem 2 tells us that in order to determine the replicating strategy for the Arrow-Debreu security we need only determine a process θ such that equation (6) is satisfied. We now combine this with our knowledge of $V^n(t)$ from equation (5) to obtain the replicating strategies for each Arrow-Debreu security.

Substituting for $V^n(t)$ using equation (5), the relation expressed in equation (6) for the Arrow-Debreu security paying a unit in state $\{N(t) = n\}$ becomes

$$
\begin{aligned}
&e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^{n-N(t)}\,\frac{1}{[n-N(t)]!} \\
&\qquad = e^{-\delta T}\exp(-\lambda T)\,[\lambda T]^{n}\,\frac{1}{n!}+\int_0^t \theta(u)dS^*(u)\,,
\end{aligned}
\tag{7}
$$

for $0 \leq t \leq T$. Therefore, if we can determine a strategy θ that satisfies equation (7) for $0 \leq t \leq T$, then Theorem 2 says that we will have determined the replicating strategy for our Arrow-Debreu security[4].

In order to handle the jump nature of the stock prices process, we require some additional notation. To this end we define the random time

$$T_j := \inf\{t \mid N(t) = j\}\,.$$

Furthermore, for a general process Z we define $Z(t-)$ by evaluating the sample path of the process as a left hand limit, i.e.

$$Z(t-) := \lim_{s \to t-} Z(s)\,.$$

Let us give a heuristic discussion of how one might now arrive at the replicating strategies. To begin with, one notices that between jump times $dS^*(t) = -(c+\delta)S^*(t)dt$. Also, for $0 \leq t < T_1$ equation (7) becomes

[4]To be precise, there is a technical point that we have left by the wayside for the moment. It is the problem of doubling strategies. We will come back to this at the end of the section.

$$e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^n \frac{1}{n!}$$

$$= e^{-\delta T}\exp(-\lambda T)\,[\lambda T]^n \frac{1}{n!} + \int_0^t \theta(u)dS^*(u)\,. \qquad (8)$$

This suggests that θ should be such that

$$(c+\delta)S^*(t)\theta(t) = -\frac{d}{dt}\,[e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^n \frac{1}{n!}]\,. \qquad (9)$$

Indeed, the fundamental theorem of calculus confirms that this definition of θ will render equation (8) valid on $0 \le t < T_1$. Equation (9) may be rewritten as

$$\theta(t) = \frac{-1}{(c+\delta)S^*(t)}\frac{d}{dt}\,[e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^n \frac{1}{n!}]\,. \qquad (9')$$

A problem with this definition of θ arises concerning the jump time T_1. Indeed, as our definition stands, $\theta(T_1)$ would be able to anticipate the jump that is about to occur at time T_1 because $S(T_1)$ appears in this definition of $\theta(T_1)$. Since this information is not available to the investor in the instants before the jump actually occurs, this definition of θ will not do for the jump time T_1. However, since $S(t) = S(t-)$ for $0 \le t < T_1$ one may remedy the situation with the definition:

$$\theta(t) = \frac{-1}{(c+\delta)S^*(t-)}\frac{d}{dt}\,[e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^n \frac{1}{n!}]\,. \qquad (10)$$

With this definition of θ both sides of equation (7) will be equal at the jump time T_1. Indeed, at the jump time T_1 the left hand side of equation (7) is

$$e^{-\delta T}\exp(-\lambda(T-T_1))\,[\lambda(T-T_1)]^{n-1} \frac{1}{[n-1]!}$$

and the right hand side of equation (7) is

$$e^{-\delta T}\exp(-\lambda(T-T_1))\,[\lambda(T-T_1)]^n \frac{1}{n!} + \theta(T_1)[S^*(T_1)-S^*(T_1-)]\,.$$

Using the fact that $S^*(T_1) - S^*(T_1-) = S^*(T_1-)[e^k - 1]$ and the definition of $\theta(T_1)$ [equation (10)] one finds that these two expressions are equal. One can proceed in this fashion for the next stochastic interval and so forth.

Theorem 3 The replicating strategy for the Arrow-Debreu security which pays a unit in the state $\{N(T) = n\}$, to be denoted by (ϕ^n, θ^n), is given by

$$\theta^n(t) = \begin{cases} \dfrac{-1}{(c+\delta)S^*(t-)}\dfrac{d}{dt}\left[e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^n \dfrac{1}{n!}\right],\ 0 \le t \le T_1 \\[2ex] \dfrac{-1}{(c+\delta)S^*(t-)}\dfrac{d}{dt}\left[e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^{n-j} \dfrac{1}{[n-j]!}\right],\ T_j < t \le T_{j+1},\ j = 1, \dots, n \\[2ex] 0 \ \text{ for } \ t > T_{n+1} \end{cases}$$

and

$$\phi^n(t) = [V^n(t) - \theta^n(t)S(t)]\,/\,e^{\delta t}\,.$$

Proof: By Theorem 2 it is sufficient to establish that equation (7) holds for $0 \le t \le T$. We have already shown that equation (7) holds for $0 \le t \le T_1$. We proceed by induction. Suppose that equation (7) holds for $0 \le t \le T_j$ where $j \le n-1$. By the induction hypothesis

$$e^{-\delta T}\exp(-\lambda(T-T_j))\,[\lambda(T-T_j)]^{n-j}\frac{1}{[n-j]!}$$

$$= e^{-\delta T}\exp(-\lambda T)\,[\lambda T]^{n}\frac{1}{n!}+\int_0^{T_j}\theta^n(u)dS^*(u)\,. \qquad (*)$$

Consider a time t with $T_j < t < T_{j+1}$. Bearing in mind equation $(*)$, we see then that for this particular time t, the requirement of equation (7) is that we have the equality

$$e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^{n-j}\frac{1}{[n-j]!}$$

$$= e^{-\delta T}\exp(-\lambda(T-T_j))\,[\lambda(T-T_j)]^{n-j}\frac{1}{[n-j]!}+\int_{T_j}^{t}\theta^n(u)dS^*(u)\,.$$

Since $dS^*(t) = -(c+\delta)S^*(t)dt$ between jump times, it is clear that this equality holds. At the jump time T_{j+1} the left hand side of this equation is equal to

$$e^{-\delta T}\exp(-\lambda(T-t))\,[\lambda(T-t)]^{n-j-1}\frac{1}{[n-j-1]!}\,,$$

and the right hand side of this equation is equal to

$$e^{-\delta T}\exp(-\lambda(T-T_{j+1}))\,[\lambda(T-T_{j+1})]^{n-j}\frac{1}{[n-j]!}+\theta^n(T_{j+1})[S^*(T_{j+1})-S^*(T_{j+1}-)]\,.$$

Using the fact that $S^*(T_{j+1}) - S^*(T_{j+1}-) = S^*(T_{j+1}-)[e^k - 1]$ and the formula for $\theta^n(T_{j+1})$ we find that this expression for the right hand side of the equation is equal to the left hand side of the equation. Thus equation (7) holds for $0 \le t \le T_n$. To complete the proof we must check that equation (7) holds for $T_n < t \le T_{n+1}$. The same argument applies except that when we check equality at the jump time T_{n+1}, the left hand side of the equation is equal to 0. QED.

Remark: The reader may ask whether ϕ^n [the strategy for the risk-free asset] defined in Theorem 3 anticipates the jump times of the stock price process as was the case for the provisional definition of θ [the strategy for the stock] that was made in equation (9')? The answer to this question is no. In fact, the reader may check that

$$V^n(t) = V^n(t-) + \theta^n(t)[S(t) - S(t-)]$$

and since $\theta^n(t)S(t) = \theta^n(t)S(t-) + \theta^n(t)[S(t) - S(t-)]$ one finds that

$$\phi^n(t) = [V^n(t) - \theta^n(t)S(t)] / e^{\delta t} = [V^n(t-) - \theta^n(t)S(t-)] / e^{\delta t}.$$

Since the right hand side of this equation is part of the information that is available to the investor in the instants before any jump in stock price, we see that the strategy (ϕ^n, θ^n) can be implemented by the investor.

Within this model, the terminal payoffs for a general contingent claim can be expressed in the form

$$X = \sum_{n=0}^{\infty} a_n 1_{\{N(T)=n\}}, \tag{11}$$

for some sequence of non-negative state contingent payoff amounts $\{a_n\}$ and where $1_{\{N(T)=n\}}$ denotes the indicator function for the state $\{N(T) = n\}$. The price of this general contingent claim at time t is equal to the weighted sum of the prices of the Arrow-Debreu securities at time t

$$\sum_{n=0}^{\infty} a_n V^n(t),$$

and the associated replicating strategy (ϕ, θ) is

$$\theta(t) = \sum_{n=0}^{\infty} a_n \theta^n(t) \qquad \phi(t) = \sum_{n=0}^{\infty} a_n \phi^n(t).$$

This information is sufficient to price and replicate any contingent claim.

Let us illustrate this result for a European call option on the stock with strike price K. The terminal payoff of the option is Max[0, S(T) – K]. Let n_t denote the least integer n such that $n \geq [\ln[K / S(t)] + c(T-t)]/k$. The option finishes up in the money if and only if $S(T) \geq K$ and this inequality holds if and only if

$$N(T) \geq \frac{\ln[K / S(0)] + cT}{k}.$$

Therefore, the option's state contingent payoffs are

$$a_n = \begin{cases} S(0)\exp(kn - cT) - K, & n \geq n_0 \\ 0 \quad \text{otherwise} \end{cases}$$

Consequently, the representation corresponding to equation (11) for the terminal payoffs of the option is

$$\sum_{n=n_0}^{\infty} [S(0)\exp(kn - cT) - K] 1_{\{N(T)=n\}}.$$

It follows that the price of the option at time t is

$$\sum_{n=n_0}^{\infty} [S(0)\exp(kn - cT) - K] V^n(t), \qquad (12)$$

and the replicating strategies are

$$\theta(t) = \sum_{n=n_0}^{\infty} [S(0)\exp(kn - cT) - K]\theta^n(t)$$

$$\phi(t) = \sum_{n=n_0}^{\infty} [S(0)\exp(kn - cT) - K]\phi^n(t) .$$

Since $V^n(t) = 0$ for $n < n_t$, equation (12) may be rewritten as

$$\sum_{n=n_t}^{\infty} [S(0)\exp(kn - cT) - K]V^n(t) .$$

It is straightforward to confirm that this formula is equivalent to that given in Gerber and Shiu (1993, (3.2.6)).

A technical point that we have hitherto ignored is the regularity conditions that must be imposed on the replicating strategies within our model. In fact, without some such regularity conditions the model will admit doubling strategies [Harrison and Kreps (1979), page 400] and will not be arbitrage-free. Consider a gambler playing roulette in a casino. He adopts the strategy of betting \$1 on red and subsequently doubles his bet until he wins. This generates a certain profit of \$1. However, this strategy is not for the faint of heart for the gambler will have to bet arbitrarily large amounts of money to make his certain \$1 profit. Of course, the casino would never tolerate such a gambler and in practice the gambler cannot implement such a strategy because of table limits set by the casino. The regularity condition that must be imposed in models of continuous securities trading to avoid a similar doubling strategy is analogous to the table limits set by casinos. The regularity condition serves to impose a restriction on the size of the position that an investor can take in the traded assets. With such regularity conditions in place, the model is arbitrage-free.

5. The Lookback Option

The lookback option is an example of an option for which Theorem 1 does not apply but for which the replicating strategies can be computed explicitly and are analogous in form to those that appear in Theorem 1. Apparently, this option is sold through several brokerage houses in the United States and Europe. We will discuss the European version of this option which was analysed by Goldman, Sosin, and Gatto (1979). Recently, there has been an interest in the American version of this option over an infinite time horizon [Shepp and Shiryaev (1993), Duffie and Harrison (1993), Gerber and Shiu (1994)], the so-called Russian option.

The lookback option is a contingent claim for which the terminal payoff is equal to the maximum of the stock price over the investment horizon[5] [0, T]. This is an example from a class of contingent claims which are sometimes referred to as path dependent. The general characterisation of price in equation (1) covers path dependent and path independent contingent claims alike. However, as a practical matter the computation of the price function for path dependent claims is more difficult than it is for path independent claims.

Assume that the stock price follows the one-dimensional version of the Wiener process model of section 3. The price of the lookback option can be derived using the technique of Gerber and Shiu (1993, III.1). However, the actual calculations are somewhat involved. The reader who wishes to carry out this calculation will require a boundary crossing probability result that may be found in Beekman (1974, (7) page 127), Karatzas and Shreve (1988, (3.41), page 265), Shepp(1979), and Park and Schuurmann (1976). Define

$$M(t) := \mathrm{Max}\{\, S(u) \mid 0 \le u \le t \,\}\,.$$

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[5] If the stock price is continuous then the definition of the payoff as a maximum over [0, T] is well defined.

In this notation, the terminal payoff from the lookback option is M(T). Upon carrying out the lengthy computation that we have described, one finds that the price of the lookback option is a function of S(t), M(t), and T–t [Goldman, Sosin, and Gatto (1979, (10) page 1116)]. Let us denote this price function by V(S(t), M(t), T–t). Then we have

$$V(S(t), M(t), \tau) = M(t)e^{-\delta\tau}[\Phi(\frac{\alpha - \beta\tau}{\sigma\sqrt{\tau}}) - \frac{\sigma^2}{2\delta}e^{2\alpha\beta/\sigma^2}\Phi(\frac{-\alpha - \beta\tau}{\sigma\sqrt{\tau}})]$$

$$+ S(t)[1 + \frac{\sigma^2}{2\delta}][1 - \Phi(\frac{\alpha - (\beta + \sigma^2)\tau}{\sigma\sqrt{\tau}})], \tag{13}$$

where Φ is the standard normal distribution function, $\alpha := \ln[M(t)/S(t)]$, $\beta := \delta - \sigma^2/2$, and $\tau := T–t$.

We now show that the replicating strategies for this option are of the same form as those that are given in Theorem 1, despite the fact that the lookback option does not satisfy the hypothesis of Theorem 1.

Theorem 4 The self-financing investment strategy that replicates the contingent claim $X = \text{Max}\{ S(u) \mid 0 \le u \le 1 \}$ may be computed from the function V [equation (13)]. The replicating strategy is

$$\theta(t) = \frac{\partial V}{\partial s} \quad \text{and} \quad \phi(t) = \left[V - \sum_{i=1}^{n} \frac{\partial V}{\partial s} S(t) \right] / e^{\delta t}.$$

Proof: For brevity, we denote V(S(t), M(t), T–t)by V_t. By Itô's lemma[6]

[6]The process M is nondecreasing and is thus a process of finite variation. Consequently, the quadratic variation terms which involve M are equal to zero.

$$d(e^{-\delta t}V_t) = -\delta e^{-\delta t}V_t dt + e^{-\delta t}dV_t$$

$$= -\delta e^{-\delta t}V_t dt + e^{-\delta t}[-\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial s}dS + \frac{\partial V}{\partial m}dM + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial s^2}dt].$$

Since $d(e^{-\delta t}S(t)) = -\delta e^{-\delta t}S(t)dt + e^{-\delta t}dS(t)$, we may rearrange this expression as

$$d(e^{-\delta t}V_t) = e^{-\delta t}[-\delta V_t - \frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial s^2} + \frac{\partial V}{\partial s}\delta S]dt + \frac{\partial V}{\partial m}dM + \frac{\partial V}{\partial s}d(e^{-\delta t}S)$$

As $e^{-\delta t}V_t$ and $e^{-\delta t}S(t)$ are martingales it follows that the finite variation terms must vanish[7]. Therefore,

$$d(e^{-\delta t}V_t) = \frac{\partial V}{\partial s}d(e^{-\delta t}S).$$

One now carries out the same calculation that was done in the proof of Theorem 1 to find that

$$V_t = V_0 + \int_0^t e^{-\delta t}[V_t - \frac{\partial V}{\partial s}S]dB(t) + \int_0^t \frac{\partial V}{\partial s}dS. \text{ QED.}$$

6. Concluding Remarks

In this paper we have analysed the replicating strategies for several different contingent claims. There are many other types of contingent claims for which a similar analyses may be carried out. Among these are the exotic options. Although the theory which tells us that these exotic options are dynamically spanned and what their prices are

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[7]The process $\int_0^t \frac{\partial V}{\partial m}dM$ is of finite variation [Apostol (1974), page 161].

is no different, this abstract theory does not provide a technique for explicitly computing the option's price or its replicating strategies. Consequently, the analysis of exotic options is a whole problem in itself. Examples of exotic options include the Asian option, the barrier option, and the lookback option. The Asian option is discussed in Geman and Yor (1993) and Kemna and Vorst (1990). A simple analysis for a special case of the Asian option may be carried out as in Bergman (1985). Barrier options are treated in Rubinstein and Reiner (1991). Boyle and Lee (1993) is an interesting paper which uses barrier options. Goldman, Sosin, and Shepp (1979) discuss certain market timing options. The reader may wish to peruse Bhattacharya and Constantinides (1989) for more material on option pricing theory. The partial differential equation that appears in equation (4) is treated in Jarrow and Rudd (1983), Müller (1985), Hull (1989), and Duffie (1992).

Option pricing for models in which the stock price is permitted to jump may be found in Merton (1976), Naik and Lee (1990), Naik (1990), and Colwell and Elliott (1993). More information concerning Poisson models can be found in Elliott and Kopp (1990) and Cutland, Kopp, and Willinger (1993). A very general analysis of asset pricing, including jump processes, is carried out in Back (1991). If a model is employed which permits the stock price to jump too much then an arbitrary contingent claim will not be dynamically spanned and one cannot use the theory outlined in section 2 for pricing. In such cases a utility based approach must be adopted or some other method, such as that of Föllmer and Sondermann (1986), might be used[8]. For instance, if the stock price involves a gamma process [Garman (1985, pages 856-857), Gerber and Shiu (1993, IV.1)] then a general contingent claim cannot be dynamically spanned and some other technique must be used to associate a price to the contingent claim. It is interesting to note that although it is generally accepted that the gamma process model is incomplete, there appears to be available no formal proof of this fact.

[8]An insightful paper relating to the approach of Föllmer and Sondermann is Dybvig (1992).

As we have mentioned, the choice of which portfolio investment strategy to follow is a problem separate from the determination of the investment strategies that one requires to synthetically produce a particular terminal payoff distribution. One type of portfolio investment strategy that is used by practitioners is constant proportion portfolio insurance. Constant proportion portfolio insurance is discussed in Perold and Sharpe (1988) and Black and Perold (1992). Cox and Huang (1989.a) is an important paper which merges portfolio optimisation with option pricing theory. An accessible description of the ideas in this paper may be found in Cox and Huang (1989.b). A treatment is also provided in Duffie (1992). Benninga and Blume (1985) discuss one version of a portfolio insurance scheme involving the purchase of a risky asset and a put option on that asset. Rubinstein (1988) discusses some aspects of portfolio insurance and the 1987 market crash. Tilley (1988) contains a general discussion of portfolio insurance. Dybvig (1988) is an interesting paper which points out some flaws in several dynamic portfolio strategies.

The reader who wants a source for stochastic calculus will find Arnold (1974) to be a friendly reference. The mathematically determined reader will find Karatzas and Shreve (1988) an excellent reference as well as Chung and Williams (1990). Stochastic differential equations with Poisson components may be found in Gihman and Skorohod (1972), Elliott (1982), and Kushner (1967).

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