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NONEXPONENTIAL BOUNDS ON THE TAILS OF COMPOUND DISTRIBUTIONS

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Random sum models with compound distributions are used extensively in modelling of insurance risks. Unfortunately, the compound distributions themselves are awkward to evaluate. Consequently, various numerical and analytical approximative techniques have been used. In this talk we derive various upper and lower bounds on the tails of compound distributions. Some notions in reliability theory are used.

Consider a sequence of i.i.d. positive random variables $X_1, X_2, \cdots, X_n, \cdots$ with distribution function $F(x)$ and a counting random variable N independent of X_i, with

$$Pr(N = n) = p_n, \quad n = 0, 1, 2, \cdots. \tag{1}$$

Let

$$S = X_1 + X_2 + \cdots + X_N \tag{2}$$

We are interested in estimating the tail probability

$$\psi(x) = Pr(S > x), \quad x \geq 0, \tag{3}$$

which has applications in many disciplines.

Example 1 Total Claim Amounts Distribution in Group Insurance

Let N be the number of claims for a particular period and X_n be the nth claim amount. Then $S = X_1 + X_2 + \cdots + X_N$ is the total claim amounts. $\psi(x) = Pr(S > x)$ is the probability of the total claim amount greater than x. Furthermore, $\int_x^\infty \psi(s)ds$ is the stop-loss premium.

Example 2 Infinite Ruin Probability

Let N_t be the number of claims by time t, whix is assumed to be a Poisson process with parameter λ. $S_t = X_1 + X_2 + \cdots + X_{N_t}$ is thus the total claimamounts by time t. Denote u to be the initial surplus at time 0 and u_t to be the surplus at t. We have

$$u_t = u + (1+\theta)\lambda t E(X_1) - S_t$$

where θ is the relative security loading.

The ruin probability therefore is

$$\psi(u) = \Pr(u_t < 0, \quad \text{for} \quad \text{some} \quad t \geq 0)$$

. It is well known that it can be expressed as

$$\psi(u) = \Pr(\tilde{S} > u)$$

where $\tilde{S} = Y_1 + Y_2 + \cdots + Y_{\tilde{N}}$, Y_n are i.i.d. with density function $\frac{1-F(x)}{E(X_1)}$, and $\tilde{N}$ is a geometric distribution with $\Pr(\tilde{N} = n) = \frac{\theta}{1+\theta}(\frac{1}{1+\theta})^n$.

The tail probability can be written as

$$\psi(x) = \sum_{n=1}^{\infty} p_n \overline{F}^{*(n)}(x),$$

where $F^{*(n)}(x) = Pr(X_1 + X_2 + \cdots + X_n \leq x)$ is defined to be the distribution function of the n-fold convolution. Its computation involves infinite many convolutions of distributions and it is unwieldy numerically except for a few cases. Derivation of parametric upper and lower bounds for the tail probabilities of compound distributions would give us qualitative and quantitative insight into these distributions.

The classical Cramer-Lundberg bound gives a simple exponential upper bound for a class of compound geometric distributions. More precisely, if $Pr(N = n) = (1-p)p^n$ and there is κ such that

$$p^{-1} = \int_0^\infty e^{\kappa y} dF(y),$$

then,

$$\psi(x) \leq e^{-\kappa x}.$$

We now introduce some definitions that will be used to derive our main results. A distribution function $B(x)$ is said to be new worse than used(NWU)[new better than used(NBU)] if

$$\overline{B}(x)\overline{B}(y) \leq [\geq] \overline{B}(x+y), \quad \text{for} \quad x > 0, \quad y > 0.$$

Many common used distributions are either NWU or NBU. For example, Exponential, Gamma(α, λ) with $\alpha > 1$, Pareto, and Weibull(α) with $\alpha \leq 1$ are NWU; whilst Exponential, Gamma$((\alpha, \lambda)$ with $\alpha < 1$, and Weibull(α) with $\alpha \geq 1$ are NBU.

Our idea is to use the tail of a new worse than used distribution as upper bound and the tail of a new better than used distribution as lower bound.

We now state our main results:

Let $a_n = \sum_{m=n+1}^{\infty} p_m$.

Theorem 1 If

(i) there exists $0 < \phi < 1$ such that

$$a_n \leq \phi a_{n-1}, \quad n = 1, 2, \cdots, \quad (\star)$$

(ii) $B(x)$ is NWU and satisfies

$$\int_0^\infty \{\overline{B}(y)\}^{-1} dF(y) = \phi^{-1},$$

(iii) there is $\Delta(x) > 0$ for all $x \geq 0$ such that

$$[\Delta(x)]^{-1} \leq \inf_{0 \leq z \leq x, \overline{F}(z) > 0} \frac{\int_z^\infty [\overline{B}(x - z + y)]^{-1} dF(y)}{\overline{F}(z)},$$

then

$$\psi(x) \leq \frac{1 - p_0}{\phi} \Delta(x).$$

Remark: $(\star)$ is satisfied by many distributions including Geometric, Poisson, Binomial and Negative Binomial.

Theorem 2 If

(i) there exists ϕ such that

$$a_n \geq \phi a_{n-1}, \quad n = 1, 2, \cdots, \quad (\star\star)$$

(ii) $B(x)$ is NBU and satisfies

$$\int_0^\infty \{\overline{B}(y)\}^{-1} dF(y) = \phi^{-1},$$

(iii) there is $\Delta(x) > 0$ for all $x \geq 0$ such that

$$[\Delta(x)]^{-1} \geq \sup_{0 \leq z \leq x, \overline{F}(z) > 0} \frac{\int_z^\infty [\overline{B}(x - z + y)]^{-1} dF(y)}{\overline{F}(z)},$$

then

$$\psi(x) \geq \frac{1-p_0}{\phi}\Delta(x).$$

It is easy to see that Theorem 2 is similar to Theorem 1 except the inequalities are reversed. Hence, hereafter we state results on upper bound only. The results on lower bound can be derived accordingly.

Corollary 1 If κ satisfies

$$\phi^{-1} = \int_0^\infty e^{\kappa y} dF(y)$$

and

$$\delta^{-1} = \inf_{0 \leq z, \overline{F}(z) > 0} \frac{\int_z^\infty e^{\kappa y} dF(y)}{e^{\kappa z}\overline{F}(z)}$$

then,

$$\psi(x) \leq \frac{1-p_0}{\phi}\delta e^{-\kappa x}.$$

Easy to see $\delta \leq 1$. Hence,

$$\psi(x) \leq \frac{1-p_0}{\phi}e^{-\kappa x}.$$

Corollary 2

$$\psi(x) \leq \frac{1-p_0}{\phi}\overline{B}(x).$$

Corollary 3 If $F(x)$ is NWU, then

$$\psi(x) \leq \frac{1-p_0}{\phi}\{\int_0^\infty [\overline{B}(x+y)]^{-1} dF(y)\}^{-1}.$$

Corollary 4 If $[\overline{B}(x)]^{-1}$ is a convex function in x, then

$$\psi(x) \leq \frac{1-p_0}{\phi}\overline{B}(x + \inf_{0 \leq z \leq x} r_F(z)),$$

where

$$r_F(z) = \frac{\int_z^\infty (y-z) dF(y)}{\overline{F}(z)}$$

is the mean residual lifetime of F.

Some applications are now given.

1. Compound Geometric Distributions

$$Pr(N = n) = (1-\rho)\rho^n, \quad n = 0, 1, 2, \cdots.$$

$$\rho^{-1} = \int_0^\infty e^{\kappa x} dF(x).$$

Then,

$$\psi(x) \le \delta e^{-\kappa x},$$

where

$$\delta^{-1} = \inf_{0 \le z} \frac{\int_z^\infty e^{\kappa y} dF(y)}{e^{\kappa z}\overline{F}(z)},$$

If $F(x)$ is NWUC,

$$\psi(x) \le \rho e^{-\kappa x},$$

a refinement of the Lundberg bound.

2. Distributions with a Finite Number of Moments

$$\int_0^\infty x^j dF(x) < \infty, \quad j \le m.$$

$$\int_0^\infty e^{\kappa x} dF(x) = \infty, \quad \text{for any} \quad \kappa > 0.$$

Let

$$\overline{B}(x) = (1 + \kappa x)^{-m}, \quad \kappa > 0, \ x \ge 0$$

be the tail of a Pareto distribution.

Choose $\kappa > 0$ such that

$$\phi^{-1} = \int_0^\infty (1 + \kappa x)^m dF(x).$$

$$\psi(x) \le \frac{1 - p_0}{\phi}(1 + \kappa x)^{-m}.$$

When $m = 1$,

$$\psi(x) \le \frac{(1 - p_0)E(X_1)}{\phi E(X_1) + (1 - \phi)x}.$$

3. Generalized Inverse Gaussian Disribution

Assume that X_1 has the density

$$f(x) \sim K x^{\lambda - 1} e^{-\mu x}, x \ge 0, x \to \infty$$

with $\mu \ge 0, \lambda < 0$. If $E(e^{\mu X_1}) < \phi^{-1}$, the best exponential upper bound is of the form $Ce^{-\mu x}$.

Choose

$$\overline{B}(x) = (1 + \kappa x)^{-m} e^{-\mu x},$$

where $0 < m < -\lambda$ and $\kappa > 0$ satisfying

$$\phi^{-1} = \int_0^\infty (1 + \kappa x)^m e^{\mu x} f(x) dx.$$

$$\psi(x) \leq \frac{1 - p_0}{\phi}(1 + \kappa x)^{-m} e^{-\mu x}.$$

4. Ruin Probability for Compound Poisson Claims Process Perturbed by Diffusion

$$u_t = u + (1 + \theta)\lambda t E(X_1) - S_t + W_t,$$

where W_t is a Wiener process with infinitesimal drift 0 and infinitesimal variance $2D$.

$$\psi(u) = \Pr(u_t < 0, \quad \text{for some} \quad t \geq 0).$$

$$\psi(u) \leq \{1 - \frac{c(u)}{\phi_A}\} e^{-\xi u} + \frac{c(u)}{\phi_A} e^{-\kappa u}$$

where

$$\int_0^\infty e^{\kappa x} dH_1 * H_2(x) = (\frac{\theta}{1 + \theta})^{-1},$$

H_1 is the exponential distribution with parameter $\xi = (1+\theta)\lambda E(X_1)/D$, H_2 has the density function $\frac{1-F(x)}{E(X_1)}$, $\phi_A = \frac{\xi - \kappa}{\xi}$, and

$$[c(x)]^{-1} = \inf_{0 \leq z \leq x} \frac{\int_z^\infty e^{\kappa y} dH_1 * H_2(y)}{e^{\kappa z} \overline{H_1 * H_2}(z)}.$$

We now give a variation of Theorem 1. More applications are given based on this variation.

Theorem 3 If $H_x(y) = 1 - \overline{H}_x(y)$ is a df satisfying $H_x(0) = 0$ and

$$\overline{H}_x(y) \leq \inf_{0 \leq z \leq x} \overline{F}(z + y)/\overline{F}(z), \quad y \geq 0,$$

then

$$\psi(x) \leq \frac{1 - p_0}{\phi} \left\{ \int_0^\infty \frac{dH_x(y)}{\overline{B}(x + y)} \right\}^{-1}, \quad x \geq 0.$$

The result comes from the relation

$$\int_z^\infty \frac{\left\{\overline{B}(x + y - z)\right\}^{-1}}{\overline{F}(z)} dF(y)$$

$$= \frac{1}{\overline{B}(x)} + \int_0^\infty \frac{\overline{F}(y+z)}{\overline{F}(z)} d\left\{\overline{B}(y+x)\right\}^{-1}$$

Corollary 5 If $F(y)$ is IFR then

$$\psi(x) \leq \frac{1-p_0}{\phi} \left\{ \int_x^\infty \frac{dF(y)}{\overline{B}(y)} \right\}^{-1} \overline{F}(x), x \geq 0.$$

5. Burr Distribution

Suppose that

$$\overline{F}(y) = \left(\frac{\mu^2}{\mu^2 + y^2} \right)^\alpha \ , \ x \geq 0 \ ,$$

here $\mu > 0, \alpha > 0$. Moments exists up to order 2α.

Choose $\overline{B}(x) = (1 + \kappa x)^{-r}$, where $0 < r < 2\alpha$ and $\kappa > 0$ satisfies

$$\frac{1}{\phi} = \int_0^\infty (1 + \kappa y)^r dF(y) \ .$$

And then choose

$$\overline{H}_x(y) = \left\{ \frac{2\mu}{y + \sqrt{y^2 + 4\mu^2}} \right\}^{2\alpha} \ , \ y \geq 0 \ .$$

Then,

$$\psi(x) \leq \frac{1-p_0}{\phi} \left\{ \int_0^\infty (1 + \kappa x + \kappa y)^r dH_x(y) \right\}^{-1}$$

6. Mixture of Weibull Distributions

Let

$$\overline{F}(y) = \int_\Theta \overline{F}(y|\theta) g(\theta) d\theta,$$

where $\overline{F}(y|\theta) = e^{-\lambda y^\theta}$ and $0 < \theta_1 \leq \theta \leq \theta_2 < \infty$.

Choose

$$\overline{H}_1(y) = e^{-\lambda[(x+y)^{\theta_2} - x^{\theta_2}]},$$

$$\overline{H}_2(y) = \begin{cases} e^{-\lambda y^{\theta_1}}, & y < 1 \\ e^{-\lambda y}, & y \geq 1. \end{cases} ,$$

and

$$\overline{H}_x(y) = \min\{\overline{H}_1(y),\ \overline{H}_2(y)\}.$$

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