

**Valuation of a Catastrophe Insurance Futures Contract
using Compound Poisson Claim Assumptions**

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Abstract

In 1993, the Chicago Board of Trade introduced a futures contract on a financial index that reflects the insurance claims emerging from catastrophes in a portfolio of policies. This article presents a valuation formula for the contract, under the assumption that catastrophes emerge as a Poisson process over time and that the claims from each catastrophe emerge as a Compound Poisson process. The formula is developed by taking conditional expectations with respect to a history of information that includes knowledge about the time of the catastrophes.

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As with any document preparation system, it is possible with \TeX to create the ugliest output from the soundest macros and styles, so please attribute the typographical æsthetics of this paper to the above-mentioned people and the typographical blunders to me.

— K. B.

1 Introduction

Futures contracts written on the loss ratios of preselected pools of catastrophe insurance contracts—aptly named *catastrophe insurance futures*—represent one of a number of new insurance-based financial instruments recently introduced, together with their respective put and call options, by the Chicago Board of Trade (CBOT). The valuation of such futures contracts, and the options written on them, might very well be of interest to insurers wishing to use the instruments to hedge against unexpected claims (see [4] and [6]) or, indeed, to speculators wishing to participate in the insurance industry in the sense these instruments allow (see [3]). However, valuation of the contracts and associated options is rather difficult without some model of the behaviour of the futures contract value.

In this working paper, we attempt to develop such a model with the eventual goal of producing an expression for the valuation of the catastrophe insurance futures contract. In the earliest sections of the paper, the aggregate claims process on which the futures contract is based is modelled as the sum of the aggregate claims processes resulting from a random sequence of catastrophic events. After the development of a number of useful statistical results in the middle sections, the paper concludes with the valuation of the futures contract under the assumptions that the occurrence of catastrophes is governed by a Poisson counting process and that the aggregate claims processes associated with the individual catastrophes are independent compound Poisson processes with identical claim frequency parameters and claim amount distributions.

While the real world is unlikely to provide us with a collection of homogeneous natural disasters each resulting in a Poisson claims counting process, the present result serves as an example of a plausible plan of attack for the development of more “realistic” models for the valuation of catastrophe insurance futures contracts. Additionally, the statistical results should be applicable in the analysis of these other models.

The remainder of this section describes a typical catastrophe insurance futures contract and explains how its settlement price is calculated. The CBOT publication [1] describes

the technical aspects of catastrophe insurance futures contracts in some detail, and [3] also contains a good description¹, so it should suffice to provide a brief summary of the salient features of the contract here. Prior to the start of trading on a new catastrophe insurance futures contract, the CBOT defines and fixes a pool of insurance contracts and a time period (generally a quarter of the year) on which the new futures contract will be based. Information concerning the insurers involved, the demographic breakdown of the insurance pool, the weights the different types and locations of claims have been assigned, and the best estimate of the total premium of the pool are provided to the investing public before trading commences. Following the start of trading, claims resulting from catastrophic events that occur within the assigned time period are reported by the insurance companies to the Insurance Services Office (ISO). Following the end of the assigned time period, the ISO allows additional time for insureds to report losses to the insurance companies (and for the insurance companies to report the losses to the ISO). This additional time period is typically three months. On the settlement date, the ISO calculates the total aggregate claims that were reported within the allowed time, and the contracts are settled at a price equal to \$25,000 times the ratio of aggregate catastrophic losses to the original estimate of the total premium. Since all weights and estimated premium levels are fixed and known at the start of trading, the variation of the futures contract settlement price is due wholly to changes in the value of aggregate claims.

The December, 1993 catastrophe futures contract provides an example of the typical time periods involved [1]. The December, 1993 contract began trading at the start of 1993. At that time, the demographic breakdown of the premium pool (and an estimate of its size) was provided by the CBOT to the investing public. This contract covered losses on catastrophic events occurring between July and September, 1993. The ISO allowed insurers to report claims resulting from these events up to the end of December, 1993. The last trading day is

¹Please note that the description of catastrophe insurance futures contracts found in [3] is not totally accurate, since some aspects of the contract were changed prior to the contract's introduction but following the publication of this article.

April 5, 1994 when the contract settlement price will be calculated and the contracts settled.

2 Terminology

For the futures contract under consideration, we take $t = 0$ to be the time at which trading commences. We denote the start of the claims collection period by $Q \geq 0$. This is also the earliest time at which catastrophes relevant to the given contract may occur. The latest date at which relevant catastrophes may occur is denoted R with $Q < R$. Claims on the contract premium pool that result from the catastrophic events between times Q and R may be reported up to and including the end of the claims collection period at time S where $R \leq S$. Claims reported to insuring companies after this date are not included in the calculation of the settlement price of the contract. We assume all claims reported to insurance companies by time S will be reported to the investing public by the settlement time $T \geq S$. At time T , all trading on the contracts ceases, the settlement price is calculated, and the contracts are settled on a cash basis.

To avoid burdening our discussion with unnecessary details, we assume that the catastrophe insurance futures contract is written on a unit estimated premium, and we ignore the \$25,000 multiplier in the settlement price calculation. We write Y_t as the value at time $t \geq 0$ of the catastrophe futures contract based on a unit premium. We denote by J_t the set of information available to the investing public at time t . Let $\{X_t\}_{t \geq 0}$ be the stochastic process representing the total nominal aggregate claims on the unit premium pool resulting from catastrophes occurring between times Q and R that are reported to insurance companies no later than time S . Then X_t represents the aggregate claims from this pool reported to insuring companies as of time t . We see that, since no claims are reported before time Q and no claims reported after time S are included in the process $\{X_t\}$, we must necessarily have

$$\forall t < Q, \quad X_t = 0; \tag{1}$$

$$\forall t > S, \quad X_t = X_S. \quad (2)$$

We note that Y_T , the value of the futures contract on the settlement date, will be equal to the settlement price, which is simply the total nominal aggregate claim amount on that date $Y_T = X_T$. Let δ_t be the force of interest at each time $t \geq 0$. If we assume investors are risk-neutral and rational then the value to any individual investor of a futures contract at time $t \leq T$ will be its expected value at the settlement date T conditioned on the available information \mathcal{J}_t and discounted back to the present time. Hence, we have the following for $t \leq T$

$$\begin{aligned} Y_t &= e^{-\int_t^T \delta_r dr} \cdot E[Y_T | \mathcal{J}_t] \\ &= e^{-\int_t^T \delta_r dr} \cdot E[X_T | \mathcal{J}_t]. \end{aligned} \quad (3)$$

In the special case where $\delta_t = \delta$ is constant for all $t \geq 0$, we have

$$Y_t = e^{-\delta(T-t)} \cdot E[X_T | \mathcal{J}_t]. \quad (4)$$

To further characterize the aggregate claims process $\{X_t\}$, we consider the counting process $\{N_t\}_{t \geq 0}$ for catastrophes occurring between times Q and R . We note that $\forall t < Q$, $N_t = 0$ and $\forall t > R$, $N_t = N_R$. We will denote by $T_{(i)}$ for $1 \leq i \leq N_R$ the ordered times of the catastrophes so that $Q \leq T_{(1)} \leq T_{(2)} \leq \dots \leq T_{(N_R)} \leq R$.

With each catastrophe i , we will associate the pair $(\{M_{i,t}\}_{t \geq 0}, \{X_{i,t}\}_{t \geq 0})$ where $\{M_{i,t}\}$ is the counting process for all reported claims on the contract premium pool resulting from this catastrophe and $\{X_{i,t}\}$ is the associated aggregate claims process.² We note that, for

²Note that the processes $\{M_{i,t}\}$ and $\{X_{i,t}\}$ include *all* claims on the premium pool under consideration that result from catastrophe i and are eventually reported to the insurance companies, not merely those claims that are reported to the insurance companies by time S and included in the value of the futures contract.

each $i = 1, 2, \dots, N_R$, we have $M_{i,t} = X_{i,t} = 0$ for all $t < T_{(i)}$. Hence, we have

$$\forall 0 \leq t \leq S, \quad X_t = \sum_{i=1}^{N_R} X_{i,t}; \quad (5)$$

$$\forall t > S, \quad X_t = \sum_{i=1}^{N_R} X_{i,S}. \quad (6)$$

For $1 \leq i \leq N_R$ and all $j \geq 1$, let us denote by $S_{i,j}$ the time relative to $T_{(i)}$ of the reporting of the j th time-ordered claim resulting from catastrophe i such that $0 \leq S_{i,1} \leq S_{i,2} \leq \dots \leq S_{i,j} \leq S_{i,j+1} \leq \dots$ and with $B_{i,j}$ its nominal amount.³ Then, for any catastrophe i with $1 \leq i \leq N_R$, we may define the stochastic processes $\{M_{i,t}\}$ and $\{X_{i,t}\}$ in terms of $T_{(i)}$, $S_{i,j}$, and $B_{i,j}$ as follows

$$M_{i,t} = \sum_{j=1}^{\infty} I(T_{(i)} + S_{i,j} \leq t) \quad (7)$$

$$X_{i,t} = \sum_{j=1}^{\infty} B_{i,j} I(T_{(i)} + S_{i,j} \leq t). \quad (8)$$

Additionally, it may be useful, at times, to refer to the claims processes for a particular catastrophe with respect to a time scale originating at the catastrophe time. In this spirit, we will define, for each $i = 1, 2, \dots, N_R$, the stochastic processes $\{M'_{i,\tau}\}_{\tau \geq 0}$ and $\{X'_{i,\tau}\}_{\tau \geq 0}$ where $M'_{i,\tau} = M_{i,T_{(i)}+\tau}$ and $X'_{i,\tau} = X_{i,T_{(i)}+\tau}$. We may construct analogues of equations (7) and (8) for these processes as follows

$$M'_{i,\tau} = \sum_{j=1}^{\infty} I(S_{i,j} \leq \tau) \quad (9)$$

$$X'_{i,\tau} = \sum_{j=1}^{\infty} B_{i,j} I(S_{i,j} \leq \tau). \quad (10)$$

³In the case where a catastrophe i results in only a finite number of claims (i.e., where $M_{i,\infty} = \lim_{t \rightarrow \infty} M_{i,t} < +\infty$), $S_{i,j}$ and $B_{i,j}$ will be defined as above only for claims $j \leq M_{i,\infty}$. In this case, for all $j > M_{i,\infty}$, we will take $B_{i,j} = 0$ and $S_{i,j} = +\infty$ with probability 1, and the inequality $S_{i,1} \leq S_{i,2} \leq \dots \leq S_{i,M_{i,\infty}}$ will hold.

As defined above, the stochastic process $\{X_t\}$ represents the nominal aggregate claims reported to the insurance companies. However, the claims reported to the insurance companies are not immediately known to the investing public. In actuality, these claims are reported by the insurance companies to the Insurance Services Office (ISO), which releases information about the stochastic process $\{X_t\}$ to the investing public essentially at its own discretion. We will denote by $\{X_t^*\}$ the *publicized aggregate claims process*. This process represents actual aggregate claims as reported to the investing public by the ISO at time t . We will also assume that the ISO disseminates aggregate claims data it receives from insurance companies to the general public in such a way that the publicized claims process $\{X_t^*\}$ may be written

$$X_t^* = X_{\gamma(t)} = \sum_{i=1}^{N_R} X_{i,\gamma(t)} \quad (11)$$

where $\gamma(t)$ is, for all $0 \leq t \leq T$, a nondecreasing function that satisfies $\gamma(t) \leq t$, $\gamma(0) = 0$, and $\gamma(T) = S$.⁴ Here, the function $\gamma(\cdot)$ itself is assumed to be completely known to the investing public at time $t = 0$. We will call this function $\gamma(\cdot)$ the *aggregate claims publication schedule*.

In this paper, we deal with information sets in the formal sense. They are considered to be σ -fields of events over some sample space Ω . If \mathcal{F}_1 and \mathcal{F}_2 are two σ -fields of events, we adopt the usual convention of writing $\mathcal{F}_1 \vee \mathcal{F}_2$ for the minimal σ -field containing both of the

⁴This specification allows for a larger degree of generality than might immediately be supposed. For example, if we take $\eta \geq 0$ and let

$$\gamma(t) = \begin{cases} 0 & \text{for } 0 \leq t < \eta; \\ t - \eta & \text{for } \eta \leq t < S + \eta; \\ S & \text{for } S + \eta \leq t \leq T \end{cases}$$

then, intuitively, the ISO is assumed to provide claims data in aggregate on a continuous basis, but after a constant time lag η . Similarly, if we take $Q \leq t_0 \leq S$ and some $t'_0 \geq t_0$, we may let

$$\gamma(t) = \begin{cases} 0 & \text{for } 0 \leq t < t'_0; \\ t_0 & \text{for } t'_0 \leq t < T; \\ S & \text{for } t = T. \end{cases}$$

Intuitively, the only information the ISO releases in this case is the value of the aggregate claims reported to the insurance companies as of time t_0 , where this information is released to the public at time $t'_0 \geq t_0$.

sets \mathcal{F}_1 and \mathcal{F}_2 , and $\mathcal{F}_1 \wedge \mathcal{F}_2$ for their intersection (which is necessarily the maximal σ -field contained in each of the two sets). If $\{X_1, X_2, \dots, X_n\}$ is a set of random variables (or, indeed, entire stochastic processes), we write $\sigma\{X_1, X_2, \dots, X_n\}$ for the σ -field generated by this set.

Finally, please note that we rigorously define constructs of the following form

$$t_1 \leq t \leq t_2 \tag{12}$$

as shorthand for the equivalent logical construct

$$(t_1 \leq t) \wedge (t \leq t_2) \tag{13}$$

where \wedge is the logical “and” operator. In particular, when we write that “ $P(t)$ is true for $t_1 \leq t \leq t_2$ ”, we are not asserting that $t_1 \leq t_2$. When the reverse holds—when $t_1 > t_2$ —this statement is merely making no claim about the truth of proposition $P(t)$ for *any* values of t since no value of t satisfies (13) under these conditions.

3 The Futures Value in More Detail

For the moment, we assume investors are risk neutral and rational and the force of interest is a constant δ . We note that, by equation (2), $X_T = X_S$ since $T > S$. Hence, we may apply equations (4) and (5) to obtain the following expression for the futures value Y_t based on a unit premium

$$\begin{aligned} Y_t &= e^{-\delta(T-t)} \cdot E[X_T | \mathcal{J}_t] \\ &= e^{-\delta(T-t)} \cdot E[X_S | \mathcal{J}_t] \\ &= e^{-\delta(T-t)} \cdot E[X_{1,S} + X_{2,S} + \dots + X_{N_R,S} | \mathcal{J}_t]. \end{aligned} \tag{14}$$

We might divide the expression of Y_t^* up on the basis of the information likely available to potential investors at time t . If we assume N_t is \mathcal{J}_t -measurable, then one possible expression is given by

$$\begin{aligned}
 e^{\delta(T-t)}Y_t^* = & \ E\{X_t^* \mid \mathcal{J}_t\} + \left\{ \left(\sum_{i=1}^{N_t} E\{X_{i,t} \mid \mathcal{J}_t\} \right) - E\{X_t^* \mid \mathcal{J}_t\} \right\} \\
 & + \sum_{i=1}^{N_t} E\{X_{i,S} - X_{i,t} \mid \mathcal{J}_t\} + E\left[\sum_{i=N_t+1}^{N_R} X_{i,S} \mid \mathcal{J}_t \right]
 \end{aligned} \tag{15}$$

where, on the right hand side, the first term represents claims reported to the insurance companies by claimants and publicized by the ISO. the second term represents the expectation of claims reported to the insurance companies that have not been reported to the public, the third term represents the expectation of claims incurred as a result of known catastrophes but not yet reported to insurers, and the final term represents the expectation of claims resulting from future catastrophes.

Since it might be reasonable to suppose that the investing public's information set \mathcal{J}_t is composed of information on the occurrence of all catastrophes to date as well as aggregate claim information publicized by the ISO (i.e., that $\mathcal{J}_t = \sigma\{X_r^*, N_r \text{ for } r \leq t\}$), this expression for Y_t^* can be a useful tool for the calculation of futures prices under our model.

4 Results Concerning Conditional Expectation

In this section, we develop a number of useful statistical results dealing with conditional expectations. The most important of the results, Theorem 1, will be a very valuable tool in section 6. Intuitively, this theorem allows us to remove extraneous information from a conditional expectation when we can establish a form of independence between the information sets and random variables involved that is weaker than full mutual independence.

We begin by stating the following (possibly intuitively obvious) lemma which is found as Theorem 9.1.3 in Chung [2].

Lemma 1 *Let \mathcal{A} be a σ -field of events. Then, for any \mathcal{A} -measurable random variable W and any random variable Z , we have*

$$E[WZ | \mathcal{A}] = W E[Z | \mathcal{A}] \tag{16}$$

provided both sides of the equation exist.

Proof. Omitted. See proof of Theorem 9.1.3 in [2]. \square

We now introduce a lemma proved by Chung in [2] which we employ both in the proof of Theorem 1 and several times in following sections.

Lemma 2 *Let $\mathcal{F}_1, \mathcal{F}_2$, and \mathcal{F}_3 be σ -fields of events such that $\mathcal{F}_1 \vee \mathcal{F}_2$ is independent of \mathcal{F}_3 . Then, for an \mathcal{F}_1 -measurable random variable Z , we have*

$$E[Z | \mathcal{F}_2 \vee \mathcal{F}_3] = E[Z | \mathcal{F}_2]. \tag{17}$$

Proof. Omitted. See [2, p. 308]. \square

We will also require Chung's Theorem 9.2.1, which we state here.

Lemma 3 *Let A be an arbitrary index set, and let $\{\mathcal{F}_\alpha$ for $\alpha \in A\}$ be a collection of σ -subfields of some σ -field \mathcal{F} of events. For each $\alpha \in A$, let $\mathcal{F}^{(\alpha)}$ denote the smallest σ -field containing all \mathcal{F}_β with $\beta \in A \setminus \{\alpha\}$. Then, the sets \mathcal{F}_α are conditionally independent relative to some σ -subfield $\mathcal{G} \subseteq \mathcal{F}$ if and only if, for each α and $E_\alpha \in \mathcal{F}_\alpha$, we have*

$$E[I(E_\alpha) | \mathcal{F}^{(\alpha)} \vee \mathcal{G}] = E[I(E_\alpha) | \mathcal{G}]. \tag{18}$$

Proof. Omitted. See the proof of Theorem 9.2.1 in [2]. \square

We prove one final, trivial lemma.

Lemma 4 Let \mathcal{A} , \mathcal{B} , and \mathcal{D} be σ -fields of events such that $\mathcal{D} \subseteq \mathcal{A} \vee \mathcal{B}$. If Z is a random variable such that

$$E[Z | \mathcal{A} \vee \mathcal{B}] = E[Z | \mathcal{B}] \quad (19)$$

then

$$E[Z | \mathcal{B} \vee \mathcal{D}] = E[Z | \mathcal{B}]. \quad (20)$$

Proof. We see that the following holds

$$\begin{aligned} E[Z | \mathcal{B} \vee \mathcal{D}] &= E\{E[Z | \mathcal{A} \vee \mathcal{B}] | \mathcal{B} \vee \mathcal{D}\} \\ &= E\{E[Z | \mathcal{B}] | \mathcal{B} \vee \mathcal{D}\} \\ &= E[Z | \mathcal{B}] \end{aligned} \quad (21)$$

and the lemma is proved. \square

Finally, we prove our main theorem, which will become an invaluable tool in section 6.

Theorem 1 Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be σ -fields of events such that \mathcal{A} and $\mathcal{B} \vee \mathcal{C}$ are independent. Suppose that \mathcal{D} is a σ -field such that $\mathcal{D} \subseteq \mathcal{A} \vee \mathcal{B}$. Let Z be an $\mathcal{A} \vee \mathcal{B}$ -measurable random variable. Then

$$E[Z | \mathcal{B} \vee \mathcal{C} \vee \mathcal{D}] = E[Z | \mathcal{B} \vee \mathcal{D}]. \quad (22)$$

Proof. Let $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in \mathcal{C}$, and $E \in \mathcal{B} \vee \mathcal{D}$ be arbitrary events. Then,

$$\begin{aligned} E[I(A)I(B)I(C)I(E)] &= E\{E[I(A)I(B)I(C)I(E) | \mathcal{A} \vee \mathcal{B}]\} \\ &= E\{I(A)I(B)I(E)E[I(C) | \mathcal{A} \vee \mathcal{B}]\} \end{aligned} \quad (23)$$

by Lemma 1. Let $\mathcal{F}_1 = \mathcal{C}$, $\mathcal{F}_2 = \mathcal{B}$, and $\mathcal{F}_3 = \mathcal{A}$. Then, by Lemma 2, we have $E[I(C) | \mathcal{A} \vee \mathcal{B}] = E[I(C) | \mathcal{B}]$. Moreover, by Lemma 4, we have $E[I(C) | \mathcal{A} \vee \mathcal{B}] = E[I(C) | \mathcal{B} \vee \mathcal{D}]$.

Therefore, we have the following

$$\begin{aligned}
 & E[I(A)I(B)I(C)I(E)] \\
 &= E\{I(A)I(B)I(E)E\{I(C) \mid \mathcal{B} \vee \mathcal{D}\}\} \\
 &= E\{E\{I(A)I(B)I(E)E\{I(C) \mid \mathcal{B} \vee \mathcal{D}\} \mid \mathcal{B} \vee \mathcal{D}\}\} \tag{24} \\
 &= E\{I(B)I(E)E\{I(A) \mid \mathcal{B} \vee \mathcal{D}\}E\{I(C) \mid \mathcal{B} \vee \mathcal{D}\}\} \\
 &= E\{I(E)\{E\{I(A)I(B) \mid \mathcal{B} \vee \mathcal{D}\}E\{I(C) \mid \mathcal{B} \vee \mathcal{D}\}\}\}.
 \end{aligned}$$

As $E\{I(A)I(B) \mid \mathcal{B} \vee \mathcal{D}\}E\{I(C) \mid \mathcal{B} \vee \mathcal{D}\}$ is $\mathcal{B} \vee \mathcal{D}$ -measurable, we have by the definition of conditional expectation that

$$E\{I(A)I(B)I(C) \mid \mathcal{B} \vee \mathcal{D}\} = E\{I(A)I(B) \mid \mathcal{B} \vee \mathcal{D}\}E\{I(C) \mid \mathcal{B} \vee \mathcal{D}\} \tag{25}$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $C \in \mathcal{C}$. Therefore, $\mathcal{A} \vee \mathcal{B}$ and \mathcal{C} are conditionally independent with respect to $\mathcal{B} \vee \mathcal{D}$ by definition.

By Lemma 3 applied over $A = \{1, 2\}$ with $\mathcal{F}_1 = \mathcal{A} \vee \mathcal{B}$, $\mathcal{F}_2 = \mathcal{C}$, and $\mathcal{G} = \mathcal{B} \vee \mathcal{D}$, we have, for all $E \in \mathcal{A} \vee \mathcal{B}$

$$E\{I(E) \mid \mathcal{B} \vee \mathcal{C} \vee \mathcal{D}\} = E\{I(E) \mid \mathcal{B} \vee \mathcal{D}\}. \tag{26}$$

Having established that the theorem holds for $Z = I(E)$, we may generalize it to all $\mathcal{A} \vee \mathcal{B}$ -measurable simple functions Z and, from there, to all $\mathcal{A} \vee \mathcal{B}$ -measurable random variables by a monotone convergence argument. \square

5 Distribution of Interevent Times Conditioned on Number of Events

In this section, we will examine the distribution of the interevent times of a Poisson counting process $\{M_t\}_{t \geq 0}$ for events occurring between time $t = t_1$ and time $t = t_2$ for some $0 \leq t_1 \leq t_2$ when conditioned on the numbers of events M_{t_1} and M_{t_2} known to have occurred at times

t_1 and t_2 respectively. This result will be a useful device for the valuation of the futures contract in section 6, where we assume the catastrophes follow a Poisson counting process.

We will begin by establishing the fact that we need consider only the case where $t_1 = 0$ without loss of generality. Take $0 \leq t_1 \leq t_2$, let $\{M_t\}_{t \geq 0}$ be a Poisson counting process, and define the process $\{M'_t\}_{t \geq 0}$ by $M'_t = M_{t+t_1} - M_{t_1}$. Let \mathcal{F}_1 be the information set generated by $\{M'_t\}_{t \geq 0}$. Note that \mathcal{F}_1 is also the σ -field generated by all increments of $\{M_t\}_{t \geq 0}$ of the form $M_{r_2} - M_{r_1}$, with $t_1 \leq r_1 \leq r_2$. Let \mathcal{F}_2 be the information set generated by $M_{t_2} - M_{t_1}$, and let \mathcal{F}_3 be the information set generated by M_{t_1} alone. Note that, as the process $\{M_t\}_{t \geq 0}$ has independent increments [5, p. 27], we have $\mathcal{F}_1 \vee \mathcal{F}_2 = \mathcal{F}_1$ independent of \mathcal{F}_3 . By Lemma 2, we have for any \mathcal{F}_1 -measurable random variable Z , the following

$$\begin{aligned} E[Z \mid M_{t_1}, M_{t_2}] &= E[Z \mid M_{t_1}, M_{t_2} - M_{t_1}] \\ &= E[Z \mid M_{t_2} - M_{t_1}] \\ &= E[Z \mid M'_{t_2-t_1}]. \end{aligned} \tag{27}$$

Finally, note that as the increments of $\{M_t\}_{t \geq 0}$ are stationary [5, p. 27], the processes $\{M_t\}_{t \geq 0}$ and $\{M'_t\}_{t \geq 0}$ have identical (unconditional) distributions. Since Z is measurable with respect to $\{M'_t\}_{t \geq 0}$, we can evaluate expressions of the form $E[Z \mid M_{t_1}, M_{t_2}]$ by examining $E[Z \mid M'_{t_2-t_1}]$ where the random variable Z is considered with respect to the evolution of the Poisson counting process M'_t .

Having established this, we will examine the distribution of the interevent times in the case where the starting time $t_1 = 0$. Let $\{M'_t\}_{t \geq 0}$ be a Poisson counting process with parameter λ and ordered event times $0 < S_{(1)} < S_{(2)} < \dots < S_{(i)} < S_{(i+1)} < \dots$. We define the interevent times $V_1 = S_{(1)}$ and $V_i = S_{(i)} - S_{(i-1)}$ for all $i \geq 2$. We note that V_i are independent exponential random variables each with parameter λ [5, p. 28].

We will now examine the form of the conditional probability function of $M'_{\tau_0} = n$ given V_1, V_2, \dots, V_n for some $\tau_0 \geq 0$ and $n \geq 1$. The probability that M'_{τ_0} is equal to n ,

given V_1, V_2, \dots, V_n , is zero if $\sum_{i=1}^n V_i > \tau_0$ and precisely equal to the probability that $V_{n+1} > \tau_0 - V_1 - \dots - V_n$ if $\sum_{i=1}^n V_i \leq \tau_0$. Hence, as V_{n+1} has an exponential distribution independent of V_1, V_2, \dots, V_n , we have the following for the conditional probability function of $M'_{\tau_0} = n$ given V_i for $i = 1, 2, \dots, n$

$$g(n \mid V_1 = v_1, \dots, V_n = v_n) = \begin{cases} e^{-\lambda(\tau_0 - v_1 - \dots - v_n)} & \text{if } \sum_1^n v_i \leq \tau_0; \\ 0 & \text{if } \sum_1^n v_i > \tau_0. \end{cases} \quad (28)$$

Now, the joint marginal p.d.f. of V_1, V_2, \dots, V_n is given by

$$h(v_1, \dots, v_n) = \lambda^n e^{-\lambda \sum_1^n v_i} \quad \text{for } v_i \geq 0. \quad (29)$$

The resulting joint p.d.f. of $V_1, V_2, \dots, V_n, M'_{\tau_0}$ is given by

$$f(v_1, \dots, v_n, n) = \begin{cases} \lambda^n e^{-\lambda \tau_0} & \text{if } \sum_1^n v_i \leq \tau_0; \\ 0 & \text{if } \sum_1^n v_i > \tau_0. \end{cases} \quad (30)$$

Since the marginal p.d.f. of M'_{τ_0} is given by [5, p. 27]

$$g(n) = \frac{(\lambda \tau_0)^n}{n!} e^{-\lambda \tau_0} \quad (31)$$

we have the following conditional joint p.d.f. of V_1, \dots, V_n given $M'_{\tau_0} = n$

$$\begin{aligned} h(v_1, \dots, v_n \mid M'_{\tau_0} = n) &= \begin{cases} \frac{\lambda^n e^{-\lambda \tau_0}}{(\lambda \tau_0)^n e^{-\lambda \tau_0}} & \text{if } \sum_1^n v_i \leq \tau_0; \\ 0 & \text{if } \sum_1^n v_i > \tau_0 \end{cases} \\ &= \begin{cases} \frac{n!}{(\tau_0)^n} & \text{if } \sum_1^n v_i \leq \tau_0; \\ 0 & \text{if } \sum_1^n v_i > \tau_0. \end{cases} \end{aligned} \quad (32)$$

Note that this function is symmetric with respect to each ordering of the variables V_i .

Using this conditional joint p.d.f., we may calculate the expected value of an arbitrary V_i (say V_1) conditioned on $M'_{\tau_0} = n$. This expectation is given by the following expression.

$$\begin{aligned}
 E[V_1 | M'_{\tau_0} = n] &= \int_0^{\tau_0} \int_0^{\tau_0 - v_1} \dots \int_0^{\tau_0 - v_1 - \dots - v_{n-1}} v_1 \frac{n!}{(\tau_0)^n} dv_n \dots dv_1 \\
 &= \frac{n!}{(\tau_0)^n} \int_0^{\tau_0} v_1 \frac{(\tau_0 - v_1)^{n-1}}{(n-1)!} dv_1 \\
 &= \frac{n!}{(\tau_0)^n} \int_0^{\tau_0} (\tau_0 - y) \frac{y^{n-1}}{(n-1)!} dy \\
 &= \frac{n}{(\tau_0)^n} \left[\frac{(\tau_0)^{n+1}}{n} - \frac{(\tau_0)^{n+1}}{n+1} \right] \\
 &= \frac{\tau_0}{n+1}.
 \end{aligned} \tag{33}$$

Now, let us generalize this result as follows. Let $\{M_t\}_{t \geq 0}$ be a Poisson counting process with parameter λ and ordered event times $0 < T_{(1)} < T_{(2)} < \dots < T_{(i)} < T_{(i+1)} < \dots$, and take any times $0 \leq t_1 \leq t_2$. Define the interevent times measured from time t_1 by $W_1 = T_{(M_{t_1+1})} - t_1$ and $W_i = T_{(M_{t_1+i})} - T_{(M_{t_1+i-1})}$ for $i \geq 2$. If we define the stochastic process $\{M'_\tau\}_{\tau \geq 0}$ by $M'_\tau = M_{\tau+t_1} - M_{t_1}$ as above, we note that W_1 (or indeed any W_i) is measurable with respect to the information set \mathcal{F}_1 as previously defined. Hence, we may write

$$E[W_1 | M_{t_1}, M_{t_2}] = E[W_1 | M'_{t_2-t_1}] \tag{34}$$

from equation (27). However, when we consider W_1 with respect to the process $\{M'_\tau\}$, we see $W_1 = T_{(M_{t_1+1})} - t_1 = S_{(1)} = V_1$ in terms of the above notation. Since, $\{M'_\tau\}$ —like its counterpart $\{M_t\}$ —is a Poisson counting process with parameter λ , we can use the result constructed in equation (33), with $\tau_0 = t_2 - t_1 > 0$, to get the following

$$E[W_1 | M_{t_1}, M_{t_2}] = \frac{t_2 - t_1}{M_{t_2} - M_{t_1} + 1}. \tag{35}$$

Intuitively, the expected value of the interevent time conditioned on $M_{t_2} - M_{t_1}$ events having occurred in the time interval $[t_1, t_2]$ is $\frac{1}{M_{t_2} - M_{t_1} + 1}$ th the length of the time interval.

6 Valuation of Futures Contract under Poisson Claim

Process Assumptions

We consider Y_t , the value of our futures contract on a unit premium at time t . We assume that the catastrophe counting process $\{N_t\}_{Q \leq t \leq R}$ is a Poisson counting process with parameter λ_{cat} , the expected number of catastrophes per unit time. As above, we write $Q < T_{(1)} < T_{(2)} < \dots < T_{(N_R)} < R$ for the ordered times of the catastrophies between times Q and R , with the inequalities strict almost surely.

For each catastrophe $i = 1, 2, \dots, N_R$, we have an associated claims counting process $\{M_{i,t}\}_{t \geq 0}$ and aggregate claims process $\{X_{i,t}\}_{t \geq 0}$ representing the accumulated nominal value of claims on the contract premium pool (including those reported to the insurance companies after time S) resulting from catastrophe i . Let us define the stochastic processes $\{M'_{i,\tau}\}_{\tau \geq 0}$ and $\{X'_{i,\tau}\}_{\tau \geq 0}$ as above to represent the same claims counting and aggregate claims processes originating at the time $T_{(i)}$ of catastrophe i . We assume that the claims counting process $\{M'_{i,\tau}\}_{\tau \geq 0}$ is a Poisson counting process with parameter λ_{claim} , the expected number of claims per unit time resulting from any individual catastrophe. We assume the associated aggregate claims process $\{X'_{i,\tau}\}_{\tau \geq 0}$ is compound Poisson with individual claim amounts $B_{i,j}$ for $j \geq 1$ independent and identically distributed with c.d.f. F and mean μ .

Finally, we assume the aggregate claims processes $\{X'_{1,\tau}\}, \{X'_{2,\tau}\}, \dots, \{X'_{k,\tau}\}, \dots$ and the catastrophe counting process $\{N_t\}$ are mutually independent. Intuitively, this amounts to assuming that the aggregate claims process resulting from a particular catastrophe (and measured from the time of that catastrophe) is independent of the time of that catastrophe and independent of the occurrence of all other catastrophes and the evolution of their respective claims processes.

Our information set \mathcal{J}_t at time t is taken to include the times of all catastrophes that have occurred by time t and the claims process as reported to the general public by the ISO for all times prior to and including time t . This gives us the following explicit definition of our information set

$$\mathcal{J}_t = \sigma\{T_{(1)}, T_{(2)}, \dots, T_{(N_t)}, N_t, X_{\gamma(r)} \text{ for } 0 \leq r \leq t\} \quad (36)$$

for all $t \geq 0$. Recall that the aggregate claims publication schedule $\gamma(\cdot)$ is assumed to be completely known to the investing public at time $t = 0$.

We will now attempt to construct a useful, generalized expression for the value of the futures contract Y_t based on information available at that time. We wish to derive this expression from equation (15). Note that this equation expresses the quantity Y_t in terms of expectations of specific increments of the individual catastrophes' aggregate claims processes $\{X_{i,t}\}$. For this reason, let us fix some i between 1 and N_R inclusive and examine the general expectation $E[X_{i,t_2} - X_{i,t_1} | \mathcal{F}_{t_0}]$ for any $Q \leq t_0 \leq t_1 < t_2 \leq T$ where the information set \mathcal{F}_{t_0} is defined as follows

$$\mathcal{F}_{t_0} = \sigma\{T_{(1)}, T_{(2)}, \dots, T_{(N_R)}, N_R, X_{1,r}, X_{2,r}, \dots, X_{N_R,r} \text{ for } 0 \leq r \leq t_0\}. \quad (37)$$

Intuitively, the information set \mathcal{F}_{t_0} includes exact information concerning the occurrence times of *all* catastrophes of interest and the evolution of their individual aggregate claims processes up to a certain fixed time t_0 . While a potential investor could never be supposed to have such information at his or her disposal, this information set will prove to be a very convenient set on which to iterate.

Let \mathcal{A} be the information set generated by $\{X'_{i,r}\}_{r \geq 0}$, let \mathcal{B} be the information set generated by $T_{(i)}$, and let \mathcal{C} be the information set generated by N_R , $T_{(j)}$, and $\{X_{j,t}\}_{0 \leq t \leq t_0}$ for all $1 \leq j \leq N_R$ such that $j \neq i$. If we let $\mathcal{D} \subseteq \mathcal{A} \vee \mathcal{B}$ be the information set generated by $\{X'_{i,r}\}_{0 \leq r \leq t_0 - T_{(i)}}$ and $T_{(i)}$, then $\mathcal{F}_{t_0} = \mathcal{B} \vee \mathcal{C} \vee \mathcal{D}$. Additionally, as we have the indepen-

dence of \mathcal{A} and $\mathcal{B} \vee \mathcal{C}$ by hypothesis, we may apply Theorem 1 to the $\mathcal{A} \vee \mathcal{B}$ -measurable expression $X_{i,t_2} - X_{i,t_1}$ to get the following

$$E[X_{i,t_2} - X_{i,t_1} \mid \mathcal{F}_{t_0}] = E[X_{i,t_2} - X_{i,t_1} \mid \mathcal{D}]. \quad (38)$$

In the case where $t_0 \leq t_1 < t_2 \leq T_{(i)}$, we will have $E[X_{i,t_2} - X_{i,t_1} \mid \mathcal{F}_{t_0}] = 0$, for no claims will be realized on a catastrophe before that catastrophe occurs. Where $t_0 \leq t_1 \leq T_{(i)} < t_2$, we see that equation (38) implies

$$\begin{aligned} E[X_{i,t_2} - X_{i,t_1} \mid \mathcal{F}_{t_0}] &= E[X'_{i,t_2-T_{(i)}} - 0 \mid T_{(i)}] \\ &= \lambda_{claim} \mu(t_2 - T_{(i)}) \end{aligned} \quad (39)$$

from the distribution of the process $\{X'_{i,\tau}\}_{\tau \geq 0}$. In the case where either $t_0 \leq T_{(i)} < t_1 < t_2$ or $T_{(i)} < t_0 \leq t_1 < t_2$, we have the following

$$E[X_{i,t_2} - X_{i,t_1} \mid \mathcal{F}_{t_0}] = E[X'_{i,t_2-T_{(i)}} - X'_{i,t_1-T_{(i)}} \mid T_{(i)}, X'_{i,\tau} \text{ for } 0 \leq \tau \leq t_0 - T_{(i)}] \quad (40)$$

$$= E[X'_{i,t_2-T_{(i)}} - X'_{i,t_1-T_{(i)}} \mid T_{(i)}] \quad (41)$$

$$= \lambda_{claim} \mu(t_2 - t_1) \quad (42)$$

with equality (41) following from the independence of increments [5, p. 27]⁵ of the process $\{X'_{i,\tau}\}$ and equality (42) following from the distribution of $\{X'_{i,\tau}\}$.

⁵More exactly, equality (41) follows from the fact that the disjoint increments $X'_{i,t_2-T_{(i)}} - X'_{i,t_1-T_{(i)}}$ and $X'_{i,t_0-T_{(i)}} - X'_{i,T_{(i)}-T_{(i)}}$ are conditionally independent with respect to $T_{(i)}$ allowing an application of Lemma 3.

Combining the cases, we may produce the following general expression valid for any $Q \leq t_0 \leq t_1 < t_2 \leq T$

$$\begin{aligned} E[X_{i,t_2} - X_{i,t_1} | \mathcal{F}_{t_0}] &= \lambda_{claim} \mu \left\{ (t_2 - t_1) I(T_{(i)} < t_1) \right. \\ &\quad \left. + (t_2 - T_{(i)}) I(t_1 \leq T_{(i)} < t_2) \right\}. \end{aligned} \quad (43)$$

Having generated this result, it will now be possible to construct a general expression for the value of the futures contract for any time $Q \leq t \leq T$. Recall the generalized expression for Y_t given in equation (15). Using equation (11), we may rewrite this expression as follows

$$\begin{aligned} e^{\delta(T-t)} Y_t &= X_{\gamma(t)} + \sum_{i=1}^{N_t} E[X_{i,S} - X_{i,\gamma(t)} | \mathcal{J}_t] \\ &\quad + E\left[\sum_{i=N_t+1}^{N_R} X_{i,S} - X_{i,\gamma(t)} \middle| \mathcal{J}_t \right]. \end{aligned} \quad (44)$$

Noting that $\mathcal{J}_t \subseteq \mathcal{F}_{\gamma(t)}$ (because $\gamma(\cdot)$ is nondecreasing), we may iterate the terms on the right hand side of this expression to produce an explicit general expression for Y_t under the assumptions made in this section.

Intuitively, the second term on the right hand side of equation (44) represents the expectation of the aggregate claims on the contract premium pool (both those that have been reported to insurance companies and those that have not) that resulted from past catastrophies but have not yet been publicized by the ISO. Taking this term and iterating its

component expectations with respect to $\mathcal{F}_{\gamma(t)}$, we see the following holds

$$\begin{aligned} \sum_{i=1}^{N_t} \mathbb{E}[X_{i,S} - X_{i,\gamma(t)} \mid \mathcal{J}_t] &= \sum_{i=1}^{N_t} \mathbb{E}\left[\mathbb{E}[X_{i,S} - X_{i,\gamma(t)} \mid \mathcal{F}_{\gamma(t)}] \mid \mathcal{J}_t\right] \\ &= \lambda_{claim} \mu \mathbb{E}\left[N_{\gamma(t)}(S - \gamma(t)) + \sum_{i=N_{\gamma(t)}+1}^{N_t} (S - T_{(i)}) \mid \mathcal{J}_t\right] \quad (45) \\ &= \lambda_{claim} \mu \left\{ N_t(S - \gamma(t)) - \sum_{i=N_{\gamma(t)}+1}^{N_t} (T_{(i)} - \gamma(t)) \right\} \end{aligned}$$

from equation (43).

The third term on the right hand side of equation (44) can be interpreted intuitively as the expectation of aggregate claims on future catastrophes. We may iterate its component expectations with respect to $\mathcal{F}_{\gamma(t)}$ and apply equation (43) as follows

$$\mathbb{E}\left[\sum_{i=N_t+1}^{N_R} X_{i,S} - X_{i,\gamma(t)} \mid \mathcal{J}_t\right] = \mathbb{E}\left[\sum_{i=N_t+1}^{N_R} \mathbb{E}[X_{i,S} - X_{i,\gamma(t)} \mid \mathcal{F}_{\gamma(t)}] \mid \mathcal{J}_t\right] \quad (46)$$

$$= \lambda_{claim} \mu \mathbb{E}\left[\sum_{i=N_t+1}^{N_R} (S - T_{(i)}) \mid \mathcal{J}_t\right]. \quad (47)$$

Let us define the information set \mathcal{G}_t as follows

$$\mathcal{G}_t = \sigma\{T_{(1)}, T_{(2)}, \dots, T_{(N_t)}, N_t, X'_{i,\tau} \text{ for } \tau \geq 0 \text{ and } i = 1, 2, \dots, N_R\} \quad (48)$$

and note that $\mathcal{J}_t \subseteq \mathcal{G}_t$. If we let \mathcal{A} be the σ -field generated by $\{N_r\}_{r \geq 0}$, \mathcal{B} be the trivial σ -field, \mathcal{C} be the σ -field generated by $\{X'_{i,\tau}\}_{r \geq 0}$ for $i = 1, 2, \dots, N_R$, and $\mathcal{D} \subseteq \mathcal{A} \vee \mathcal{B}$ be the σ -field generated by $\{N_r\}_{0 \leq r \leq t}$, we see that $\mathcal{G}_t = \mathcal{B} \vee \mathcal{C} \vee \mathcal{D}$ and that \mathcal{A} and $\mathcal{B} \vee \mathcal{C} = \mathcal{C}$ are independent by hypothesis. We may therefore iterate the right hand side of equation (47) over \mathcal{G}_t and apply Theorem 1 to the $\mathcal{A} \vee \mathcal{B}$ -measurable random expression $\sum_{i=N_t+1}^{N_R} (S - T_{(i)})$

to obtain . . .

$$\begin{aligned} E\left[\sum_{i=\lambda_t+1}^{N_R} X_{i,S} - X_{i,\mathcal{N}(t)} \mid \mathcal{J}_t\right] \\ = \lambda_{\text{claim}} \mu E\left\{E\left[\sum_{i=N_t+1}^{N_R} (S - T_{(i)}) \mid \mathcal{S}_t\right] \mid \mathcal{J}_t\right\} \end{aligned} \quad (49)$$

$$= \lambda_{\text{claim}} \mu E\left\{E\left[\sum_{i=N_t+1}^{N_R} (S - T_{(i)}) \mid N_r \text{ for } 0 \leq r \leq t\right] \mid \mathcal{J}_t\right\} \quad (50)$$

$$= \lambda_{\text{claim}} \mu E\left\{E\left[\sum_{i=N_t+1}^{N_R} (S - T_{(i)}) \mid N_t\right] \mid \mathcal{J}_t\right\} \quad (51)$$

where (51) results from the the Poisson counting process $\{N_t\}_{0 \leq t \leq R}$ being a Markov process [5, p. 28].

In the case $t \leq R$, we may define the interevent times $W_1 = T_{(N_t+1)} - t$, $W_i = T_{(N_t+i)} - T_{(N_t+i-1)}$ for $2 \leq i \leq N_R - N_t$. Our analysis in Section 5 allows us to conclude that, for any $1 \leq i \leq N_R - N_t$, the following holds

$$E[W_i \mid N_t, N_R] = \frac{R - t}{N_R - N_t + 1} \quad (52)$$

since $\{N_t\}_{0 \leq t \leq R}$ has been assumed to be a Poisson counting process. Therefore, the following holds for $t \leq R$

$$\begin{aligned} E\left[\sum_{i=N_t+1}^{N_R} (S - T_{(i)}) \mid N_t, N_R\right] &= E\left[(N_R - N_t)(S - t) - (N_R - N_t)W_1 - (N_R - N_t - 1)W_2 \right. \\ &\quad \left. - \dots - W_{N_R - N_t} \mid N_t, N_R\right] \\ &= \left(S - \frac{R+t}{2}\right)(N_R - N_t). \end{aligned} \quad (53)$$

Finally, then, by iterating the innermost expectation of (51) with respect to the additional information N_R , we obtain the following

$$\begin{aligned}
 E\left[\sum_{i=N_{t+1}}^{N_R} X_{i,S} - X_{i,\gamma(t)} \mid \mathcal{J}_t\right] &= \lambda_{claim} \mu E\left\{\left(S - \frac{R+t}{2}\right) E[N_R - N_i \mid N_i] \mid \mathcal{J}_t\right\} \\
 &= \lambda_{claim} \mu E\left\{\left(S - \frac{R+t}{2}\right) \lambda_{cat}(R-t) \mid \mathcal{J}_t\right\} \quad (54) \\
 &= \lambda_{claim} \mu \left(S - \frac{R+t}{2}\right) \lambda_{cat}(R-t)
 \end{aligned}$$

for $t \leq R$. In the case that $t > R$, the expectation on the left hand side of equation (54) has value 0, since no additional relevant catastrophes occur after time R .

Substituting the expression for the second term (45) and for the third term (54) into equation (44), we get the following expression for the value of the futures contract Y_t for any time $Q \leq t \leq T$

$$\begin{aligned}
 e^{\delta(T-t)} Y_t &= X_{\gamma(t)} + \lambda_{claim} \mu \left\{ N_t(S - \gamma(t)) - \sum_{i=N_{\gamma(t)+1}}^{N_t} (T_{(i)} - \gamma(t)) \right. \\
 &\quad \left. + I(t \leq R) \cdot \left(S - \frac{R+t}{2}\right) \lambda_{cat}(R-t) \right\}. \quad (55)
 \end{aligned}$$

Note that the third term in the braces becomes 0 when $t > R$.

At the present time, the ISO provides the investing public with the value of the aggregate claims reported to insuring companies as of times R and S . The value of X_R is reported as soon as possible, but since some time is required to collect and summarize the information provided by participating insurers, this value is not actually reported until some time $\delta > 0$ has elapsed beyond time R . The value of X_S (which is actually the settlement price, since $Y_T = X_T = X_S$) is not reported until the settlement time T . The resulting aggregate

claims publication schedule $\gamma(\cdot)$ is given by

$$\gamma(t) = \begin{cases} 0 & \text{if } 0 \leq t < R + \delta; \\ R & \text{if } R + \delta \leq t < T; \\ S & \text{if } t = T. \end{cases} \quad (56)$$

for $0 \leq t \leq T$. Note that this function $\gamma(\cdot)$ is nondecreasing and satisfies the requirements $\gamma(0) = 0$, $\gamma(T) = S$, and $\gamma(t) \leq t$ for all t .

In this particular case, equation (55) gives the following expression for the value of the futures contract Y_t for any time $Q \leq t \leq T$

$$e^{\delta(T-t)}Y_t = \begin{cases} \lambda_{\text{claim}} \mu \left\{ N_t S - \sum_{i=1}^{N_t} T_{(i)} \right. \\ \quad \left. + \left(S - \frac{R+t}{2} \right) \lambda_{\text{cat}} (R-t) \right\} & \text{if } 0 \leq t < R; \\ \lambda_{\text{claim}} \mu \left\{ N_t S - \sum_{i=1}^{N_t} T_{(i)} \right\} & \text{if } R \leq t < R + \delta; \\ X_R + \lambda_{\text{claim}} \mu N_R (S - R) & \text{if } R + \delta \leq t < T; \\ X_S & \text{if } t = T. \end{cases} \quad (57)$$

7 Conclusion

The explicit goal of this paper has been the development of the expression for the value of the catastrophe insurance futures contract in equation (55) and, in particular, the result in equation (57). Implicitly, however, the creation of the general model of section 3 and the methods of attack on conditional expectation illustrated in section 6 and facilitated by the techniques of sections 4 and 5 may be more important results.

The eventual goal of our continued work in this area is the development of a valuation method that can be safely applied to the *options* written on the catastrophe insurance futures contracts. While there is a significant body of literature concerned with the valuation of options written on futures contracts, usually called simply *futures options*, this existing literature generally restricts itself to cases where the underlying asset is a stock or stock market index, making assumptions about the distribution of the underlying asset on this

basis. However, distribution assumptions that may be appropriate when modelling the behaviour of a stock price are wholly inappropriate when the underlying asset is a measure of the aggregate insurance claims resulting from a sequence of catastrophic events.

Whereas a much simpler model might provide reasonable approximations of the value of the catastrophe insurance futures contract for valuation in a practical setting, the nature of futures options generally makes options valuations techniques much more sensitive to the assumed probability distributions than the valuation of their underlying assets. In this regard, the development of a more complex and comprehensive model, as illustrated in this paper, seems worthwhile.

References

- [1] Chicago Board of Trade. CAT: Catastrophe insurance futures and options. Chicago, 1992.
- [2] Kai Lai Chung. *A Course in Probability Theory*. Academic Press, Inc., Toronto, 2nd edition, 1974.
- [3] Samuel H. Cox and Robert G. Schwebach. Insurance futures and hedging insurance price risk. *Journal of Risk and Insurance*, 59(4):628–644, 1992.
- [4] Stephen P. D'Arcy and Virginia G. France. Catastrophe futures: A better hedge for insurers. *Journal of Risk and Insurance*, 59(4):575–601, 1992.
- [5] Hans U. Gerber. *An Introduction to Mathematical Risk Theory*. Richard D. Irwin, Inc., Homewood, Illinois 60430, 1979. Monograph No. 8.
- [6] Greg Niehaus and Steven V. Mann. The trading of underwriting risk: An analysis of insurance futures contracts and reinsurance. *Journal of Risk and Insurance*, 59(4):601–627, 1992.