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Martingales and Ruin Probability

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Abstract

The classical Lundberg's inequality of insurance risk theory is proved by using a martingale inequality. The method is extended to more general case. A similar result to Willmot [5] is obtained.

Key Words: Martingale inequality, Lundberg's inequality, NWU, NBU, DFR, Ruin Probability.

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1 Introduction

In a series papers by Willmot and Lin (see [3], [4] and [5]), both exponential and nonexponential bounds for the tail probability of various compound distributions have been derived. In Willmot [5], it was suggested that nonexponential bounds for the ruin probability were difficult to obtain using martingale arguments. In this note, we show how to use martingale inequalities to obtain some upper bounds for the ruin probability.

In this section we first state the martingale inequality, and then use it to give a short proof of Lundberg's inequality.

Theorem 1.1. Let $X = (X_n)_{n\in T}$ be a sub-martingale. Then for every $\lambda > 0$ and $N \in T$,

$$\lambda P(\max_{0\le n\le N} X_n \ge \lambda) \le E(X_N : \max_{0\le n\le N} X_n \ge \lambda) \le E(X_N^+) \le E(|X_N|). \tag{1}$$

and

$$\lambda P(\min_{0\le n\le N} X_n \le -\lambda) \le -E(X_0) + E(X_N : \min_{0\le n\le N} X_n \ge -\lambda) \le E(|X_0|) + E(|X_N|). \tag{2}$$

Proof: See Dellacherie, C. and Meyer [1].

Theorem 1.2. Let $X = (X_n)_{n\in T}$ be a super-martingale. Then for every $\lambda > 0$ and $N \in T$,

$$\lambda P(\max_{0\le n\le N} X_n \ge \lambda) \le E[X_0] - E(X_N : \max_{0\le n\le N} X_n \le \lambda) \le E[X_0] + E(X_N^-) \tag{3}$$

and

$$\lambda P(\min_{0\le n\le N} X_n \le -\lambda) \le -E(X_N : \min_{0\le n\le N} X_n \le -\lambda) \le E(X_N^-). \tag{4}$$

Proof: See Dellacherie, C. and Meyer [1].

An example of application of above inequality is proof of the Lundberg inequality of risk theory.

Theorem 1.3. Suppose that $\{X_1, X_2, ...\}$ is a sequence of i.i.d. non-negative random variables, and $\{Y_1, Y_2, ...\}$ is also a sequence of i.i.d. non-negative random variables and independent of X. Let $S_n = \sum_{i=1}^n (Y_i - X_i)$, and the ruin probability

$$\psi(x) = P(\cup_{n=1}^\infty \{S_n > x\}), \quad x \ge 0 \tag{5}$$

Assume that $\kappa > 0$ satisfies

$$E(e^{\kappa Y})E(e^{-\kappa X}) = 1 \tag{6}$$

Then

$$\psi(x) \leq e^{-\kappa x} \tag{7}$$

Proof: Since

$$E(e^{\kappa Y})E(e^{-\kappa X}) = 1 \tag{8}$$

Then $Z_n = \prod_{i=1}^{n} e^{\kappa(Y_i - X_i)}$ is a martingale, therefore it is a sub-martingale, and from theorem 1.1, we have

$$\begin{aligned}
\psi(x) &= P(\cup_{n=1}^{\infty}\{S_n > x\}) = P(\lim_{n\to\infty} \cup_{i=1}^{n}\{S_i > x\}) = \lim_{n\to\infty} P(\cup_{i=1}^{n}\{S_i > x\}) \\
&= \lim_{n\to\infty} P(\max_{1\leq k\leq n}\{S_k > x\}) = \lim_{n\to\infty} P(\max_{1\leq k\leq n} e^{\kappa S_k} > e^{\kappa x}) \\
&= \lim_{n\to\infty} P(\max_{1\leq k\leq n} \prod_{i=1}^{k} e^{\kappa(Y_i - X_i)} > e^{\kappa x}) = \lim_{n\to\infty} P(\max_{1\leq k\leq n} Z_k > e^{\kappa x}) \\
&\leq \lim_{n\to\infty} e^{-\kappa x}E(Z_n) = e^{-\kappa x}
\end{aligned} \tag{9}$$

which is the Lundberg inequality in risk theory. In the above proof, we have used the fact that $E(Z_n) = E\prod_{i=1}^{n} e^{\kappa(Y_i - X_i)} = \prod_{i=1}^{n} E(e^{\kappa(Y_i - X_i)}) = 1$.

2 Main results

In this section, we will try to extend the method to give some general results. The main result is given as:

Theorem 2.1 Suppose that $B_1(x)$ is a NWU d.f., and $B_2(x)$ is a NBU d.f.

$$E\{\frac{1}{\bar{B}_1(Y)}\bar{B}_2(X)\} \leq 1 \tag{10}$$

and

$$\bar{B}_1(y - x) \geq \bar{B}_1(y)\{\bar{B}_2(x)\}^{-1} \quad \text{for} \quad y \geq x \tag{11}$$

Then

$$\psi(x) \leq \phi(x)\bar{B}_1(x) \tag{12}$$

where

$$\phi(x) = E\{\frac{\bar{B}_2(X)}{\bar{B}_1(Y)}\} - \lim_{N\to\infty}\int_{\{\max_{1\leq n\leq N}\prod_{i=1}^{n}\frac{\bar{B}_2(X_i)}{\bar{B}_1(Y_i)}\leq\frac{1}{\bar{B}_1(x)}\}}\prod_{i=1}^{N}\frac{\bar{B}_2(X_i)}{\bar{B}_1(Y_i)}dP \leq 1. \tag{13}$$

Proof:
Let $Z_n = \prod_{i=1}^{n}\{\frac{\bar{B}_2(Y_i)}{\bar{B}_1(Y_i)}\}$. From (10) we know that Z_n is a super-martingale. The ruin probability

$$\begin{aligned}\psi(x) &= P\{\cup_{n=1}^{\infty}(S_n > x)\} = P\{\cup_{n=1}^{\infty}(\sum_{i=1}^{n}Y_i - \sum_{i=1}^{n}X_i > x)\} \\ &= \lim_{N\to\infty}P\{\cup_{n=1}^{N}(\sum_{i=1}^{n}Y_i - \sum_{i=1}^{n}X_i > x)\}\end{aligned} \tag{14}$$

Since $x > 0$ we have that

$$\begin{aligned}\psi(x) &= \lim_{N\to\infty}P\{\cup_{n=1}^{N}((\sum_{i=1}^{n}Y_i - \sum_{i=1}^{n}X_i)^+ > x)\} \\ &\leq \lim_{N\to\infty}P\{\cup_{n=1}^{N}(\frac{1}{\bar{B}_1((\sum_{i=1}^{n}Y_i - \sum_{i=1}^{n}X_i)^+)} \geq \frac{1}{\bar{B}_1(x)})\} \\ &= \lim_{N\to\infty}P\{\max_{1\leq n\leq N}(\frac{1}{\bar{B}_1((\sum_{i=1}^{n}Y_i - \sum_{i=1}^{n}X_i)^+)} \geq \frac{1}{\bar{B}_1(x)})\} \\ &\leq \lim_{N\to\infty}P\{\max_{1\leq n\leq N}(\frac{\bar{B}_2(\sum_{i=1}^{n}X_i)}{\bar{B}_1(\sum_{i=1}^{n}Y_i)} \geq \frac{1}{\bar{B}_1(x)})\} \\ &\leq \lim_{N\to\infty}P\{\max_{1\leq n\leq N}\prod_{i=1}^{n}(\frac{\bar{B}_2(X_i)}{\bar{B}_1(Y_i)} \geq \frac{1}{\bar{B}_1(x)})\} \\ &\leq \phi(x)\bar{B}_1(x)\end{aligned} \tag{15}$$

A special case of the theorem is the following case:

Corollary 3.1. Suppose that $B(x)$ is a NWU d.f. and satisfies:

$$i). \qquad \bar{B}(y-x) \geq \bar{B}(y)e^{\mu x} \quad \text{for} \quad y \geq x \tag{16}$$

$$ii). \qquad E\{\frac{1}{\bar{B}(Y)}\}E(e^{-\mu X}) \leq 1 \tag{17}$$

The condition $i)$ is true if the failure rate of $\bar{B}$ satisfies $\mu_B(t) \geq \mu$.

Then

$$\psi(x) \leq \phi(x)\bar{B}(x) \tag{18}$$

where

$$\phi(x) = E\{\frac{1}{\bar{B}(Y)}e^{-\mu X}\} - \lim_{N\to\infty}\int_{\{\max_{1\leq n\leq N}\prod_{i=1}^{n}\frac{e^{-\mu X_i}}{\bar{B}(Y_i)}\leq\frac{1}{\bar{B}(x)}\}}\prod_{i=1}^{N}\frac{1}{\bar{B}(Y_i)}e^{-\mu X_i}dP \leq 1. \tag{19}$$

Proof: Let $\bar{B}_2(x) = e^{-\mu x}$, then $\bar{B}_2(x)$ is a NBU d.f. By Theorem 2.1, the result follows.

Next we will discusse the properties of $\phi(x)$. We have noticed that $\phi(x) \leq 1$, also from the expression of $\phi(x)$, we can easily see that $\phi(x)$ is a decreasing function of x. The following corollary says that in some special cases, we have that $\phi(x) \to 0$ as $x \to \infty$.

Corollary 2.2.

1). In Theorem 2.1, if $E\{\frac{\bar{B}_2(X)}{\bar{B}_1(Y)}\} \leq 1$ is replaced by $E\{\frac{\bar{B}_2(X)}{\bar{B}_1(Y)}\} = 1$

Then $\psi(x) \leq \phi(x)\bar{B}(x)$ and $\phi(x) \to 0$

2) In Corollary 2.1, if ii) is replaced by

$$ii') \qquad E\{\frac{1}{\bar{B}(Y)}\}E(e^{-\mu X}) = 1 \tag{20}$$

Then $\psi(x) \leq \phi(x)\bar{B}(x)$ and $\phi(x) \to 0$

Proof: The proof of 1) is same as the proof of 2), so we only prove 2).

Let $Z_n = \prod_{i=1}^{n}\{\frac{1}{\bar{B}(Y_i)}e^{-\mu X_i}\}$. From $ii')$ we know that Z_n is a martingale. By theorem 1.1, and using the same argument as in the proof of theorem 2.1, we have

$$\begin{aligned}\psi(x) &\leq \lim_{N\to\infty}\Big[\int_{\{\max_{1\leq n\leq N}\prod_{i=1}^{n}(\frac{1}{\bar{B}(Y_i)}e^{-\mu X_i}\geq\frac{1}{\bar{B}(x)}\}}\prod_{i=1}^{N}(\frac{1}{\bar{B}(Y_i)}e^{-\mu X_i})dP\Big]\bar{B}(x) \\ &= \phi(x)\bar{B}(x)\end{aligned} \tag{21}$$

Since Z_n is nonnegative, so

$$\sup_n E|Z_n| = \sup_n EZ_n = 1 \tag{22}$$

From Theorem 2.7 of Liptser and Shiryayev [2], we have that

$$\lim_{n\to\infty} Z_n = \prod_{i=1}^{\infty}\{\frac{1}{\bar{B}(Y_i)}e^{-\mu X_i}\} \tag{23}$$

exist almost surely, and also Z_n is a closed martingale, therefore Z_n is uniform integrable. Let $A_N = \{\max_{1\le n\le N}\prod_{i=1}^{n}(\frac{1}{\bar B(Y_i)}e^{-\mu X_i}) \ge \frac{1}{\bar B(x)}\}$, by Theorem 1.1, we have that $P(A_N) \le \phi(x)\bar B(x) \le \bar B(x) \to 0$ as $x\to\infty$, so

$$\phi(x) \le \sup_N \int_{A_N} Z_N dp \to 0 \quad \text{as} \quad x\to\infty \tag{24}$$

The following result is a special case of our main result. A version of the following result has been given by Willmot [5].

Corollary 2.3. Suppose that $B(x)$ is a DFR d.f. with failure rate $\mu_B(x) = -\frac{d}{dx}\ln \bar B(x)$, let $\mu = \lim_{x\to\infty}\mu_B(x) > 0$ and satisfies:

$$E\{\frac{1}{\bar B(Y)}\}E(e^{-\mu X}) \quad \le \quad 1 \tag{25}$$

Then

$$\psi(x) \le \phi(x)\bar B(x) \tag{26}$$

where $\phi(x)$ is given in Corollary 2.1.

Proof: Since $\bar B(x)$ is a DFR d.f., so $\bar B(x)$ is a NWU d.f., so we only need to check the condition i) of Corollary 2.1.

$$\begin{aligned}\bar B(y-x) &= e^{-\int_0^{y-x}\mu_B(t)dt} = e^{-\int_0^{y}\mu_B(t)dt+\int_{y-x}^{y}\mu_B(t)dt} \\ &= e^{-\int_0^{y}\mu_B(t)dt}e^{\int_{y-x}^{y}\mu_B(t)dt} \ge e^{-\int_0^{y}\mu_B(t)dt}e^{\int_{y-x}^{y}\mu dt} \\ &= \bar B(y)e^{\mu x}\end{aligned} \tag{27}$$

this is condition i) of Corollary 2.1. Therefore the result follows.

Example 2.1: Suppose that $\bar B(x) = (1+kx)^{-\alpha}e^{-\mu x}$ here $k>0$, from

$$(1+k(y-x))^{-\alpha}e^{-\mu(y-x)} \ge (1+ky)^{-\alpha}e^{-\mu y}e^{\mu x} \quad \text{for} \quad y \ge x \tag{28}$$

and by choosing α and k such that

$$\int_0^\infty (1+kx)^\alpha e^{\mu y} f_Y(y)dy = \frac{1}{E(e^{-\mu X})} \tag{29}$$

we know that the condition of Corollary 2.2 hold, so

$$\psi(x) \le \phi(x)(1+kx)^{-\alpha}e^{-\mu x} \tag{30}$$

and $\phi(x) \to 0$

For the inverse Gaussian case $f_Y(y) = My^{\lambda-1}e^{-\mu y-\frac{\beta}{y}}$, the α and k can be chosen by

$$\int_0^\infty (1+kx)^\alpha e^{\mu y} M y^{\lambda-1} e^{-\mu y-\frac{\beta}{y}} dy = \frac{1}{E(e^{-\mu X})} \tag{31}$$

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