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Variability of Pension Plan Cost Ratio and Funding Level

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Abstract

Depending on the actual experience of the pension fund, cost ratio (ratio of actual contributions to normal cost) and funding level can vary widely from year to year. After reviewing the related literature, the author brushes up on pension mathematics. She then goes on to the main subject of this paper: studying the variability of cost ratio and funding level, both using historical data and simulated futures. To do so, she focuses on three particularly simple types of pension plans. Some assumptions are varied so as to study their impact on the two variables of interest. Results are also presented for other sets of assumptions, which are deemed to be somewhat more realistic.

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Introduction

Of course, defined-benefit pension plans offer more protection to the participants than do defined-contribution ones. However, a price is attached: the sponsor must assume the financial risk. As a result, to ensure solvency, when the surplus falls below a certain level, he may be called on to make a special payment, in addition to the regular contributions. The opposite may also occur when the fund attains an unnecessarily high level of funding.

Hence, as noted by Chinery and McInerney (1994), for some plans, huge surpluses have accumulated during the late 1970's and early 1980's, owing to very favorable yields on the assets. These have raised concern among unions and legislators, leading to rules concerning its ownership. Lately, the situation has reversed. Low yields coupled with the downward revision of the valuation interest rates shrink, if not eliminate, the surplus then accumulated. This situation poses a new challenge to pension actuaries and to plan sponsors as well.

As will be seen in the next section, analytical results have been obtained for the moments of insurance and annuity contracts, with both mortality and interest rate as random variables. A few researchers have also studied the variability of pension plan contributions and fund level. However, to the author's knowledge, little has been done on the subject using historical data, and virtually nothing using Wilkie's stochastic investment model.

After laying out the necessary notation, definitions and pension plan mathematics, the author studies the variability of cost ratio and funding level using different sets of assumptions. Economic data are provided by assumptions for the pre-1924 period, by the *Report on Canadian Economic Statistics* for 1924 to 1992 and by simulations using Wilkie's model for years after 1992. Unlike all papers referenced, this one contains no analytical results. Hopefully, the assumptions made are somewhat more realistic and compensate for the loss of tractability.

Before concluding with a wrap-up of the results achieved as well as the research that could be undertaken on the subject, comments are made concerning the potential of the software developed by the author for this project, both as it is now and as it could be amended to accommodate some other interesting assumptions.

1. Literature Review

Originally, insurance and annuity contracts were valued in a deterministic setting, with respect to all variables. Then it was acknowledged that mortality and other decrements could be treated as random variables. The probabilistic view of actuarial science was being born. Pollard and Pollard (1969) analyzed the distribution, expectation and variance of whole-life insurance and life annuities. Their results now form part of the classical theory, taught to every actuarial student. Later, Dhaene (1990) extended these results to the distribution of general insurance contracts, thus accommodating deferral and varying benefits.

Still, the interest rate was assumed constant, another assumption that would eventually be relaxed. Various authors have calculated moments of financial and actuarial functions under a stochastic model of the interest rates. Pollard (1971) did so with a second-order autoregressive model. Independent lognormal returns were assumed in Boyle's (1976) and Waters' (1978) papers. Bellhouse and Panjer (1980 and 1981) generalized those results to normal as well as first- and second-order autoregressive processes, in both continuous and discrete time. Later, Beekman and Fuelling (1990) studied continuous annuities, assuming total return to follow a Ornstein-Uhlenbeck process. Adding to the time series already considered, Frees (1990) used a first-order moving-average process to model the yearly interest rate to value life insurance and annuities.

In addition to the individual case, some researchers have looked at the variability of the total claims. As several remarked, unlike the mortality risk which can be diversified away and disappears in the limit, the interest rate risk remains since all policies contained in a single portfolio are subject to the same investment strategy.

Considering both mortality and interest random, Parker (1994) examined the variability of the average cost per life insurance policy for varying portfolio sizes and various models for the rate and force of interest. In the pensions field, both Dufresne (1988) and Haberman (1992 and 1993) investigated the variability of contributions and fund levels with only the yield as a random variable. Their papers provide results for independent and first-order autoregressive returns, and for varying delays with respect to the contribution adjustment. Using historical data, Maynard (1992) calculated the balance sheets of a simplified pension plan for various periods and under different formulae for total liability and net contribution.

The scope of this paper gets its inspiration from Dufresne (1988) for the variables of interest and from Maynard (1992) for the modelling of the actual pension plan. It thus can be viewed as an extension of the two.

2. Notation

Before reviewing pension plan mathematics, notation first has to be specified. That notation which is used in survival analysis is assumed known; i.e. the meaning of ${}_kp_x$, μ_x and the like is clear. Without subscript, those symbols shall mean that death is the only decrement. Otherwise, (d) refers to death; (w), to withdrawal; and (τ), to both decrements. As for the symbols used by Wilkie in his model, they will be presented in conjunction with the model itself.

Then, let us set out the notation for the economic data. To study the evolution of the fund, let y^t be the nominal yield and inf^t be the inflation rate, both in year t. And to calculate the salary earned by a life aged x in year t, Sal_x^t, let w^t be the wage inflation in that year and s_x be the salary scale for that age.

For purposes of pension plan valuation, let i_a and i_r be the interest rate before and after retirement, respectively. When the subscript is omitted, it is then assumed that i is the valuation interest rate common to both periods. Also, let h be the total yearly increase in wages, due to both experience and inflation.

Let α be the age at entry and ρ be the single age at retirement. Let n_x^t be the number of lives aged x in year t; if without subscript, $n_x^t=n_x$, $\forall t$. Let g be the yearly percentage of population increase.

When the plan is over- or underfunded, a special refund or payment may be made. Let f be the funding level (assets, A, as a percentage of liability, L). Then, whenever $f<l$, a payment is in order, while when $f>u$, a refund is necessary. l and u are the lower and upper bounds, respectively, of the interval in which no adjustment to the normal cost, NC, is made for the unfunded liability, UL. The special payment shall be determined using the 'spreading' method: when needed, it will equal $k\cdot(l\cdot L-A)$, where $k=1/\ddot{a}_{\overline{n}|}$, calculated at the appropriate interest rate and with n being the number of years over which the repayment of the unfunded liability is spread. If required, the special refund is calculated in a similar fashion.

Finally, let P_a^t denote the payments made to the active lives and P_r^t, those to the retired lives. Let C^t be the net contribution for year t, i.e. the normal cost plus any special payment (a refund being negative). Let S^t be the surplus at the end of year t. Also, let b_x^t and B_x^t be the benefits accrued to a life aged x, during year t and at the beginning of year t, respectively.

Note that for the years during which indexation is offered, the benefit accrued to the beginning of a year is the sum of the benefit accrued to the beginning of the previous year, multiplied by the indexation factor for that year, and the benefit accrued during that year. The following then holds:

$$B_{x+1}^{t+1} = B_x^t\cdot(1+inf^t)+b_x^t.$$

Also, r denotes the ratio of C to NC. For all symbols needed in the pension plan valuation, the subscript t refers to the year during which or at the beginning or end of which the value is determined.

Note that, in general, the subscript represents the age and the superscript, the year.

3. Types of Benefits

Recall that only defined-benefit pension plans are considered. In that family, there are three major subdivisions: flat-benefit, career-average and final-average. Furthermore, the nominal benefit can be indexed with the Consumer Price Index (CPI) prior to and/or after retirement, so as to keep up with the cost of living. The formulae for B in all of these cases will help clarify the matter.

Throughout the paper, a key assumption is made: the participant's birthday is January 1 and he also starts working on that day. Furthermore, B is calculated at the beginning of the year and the subscript x will refer to the exact age at the time of calculation. Of course, after retirement, B is the same regardless of the time of the year when calculated, since no more benefits accrue.

As the reader will note, indexation with the CPI is delayed by one year, to take into account the time required for Statistics Canada to calculate and publish the figure as well as that needed for the sponsor to readjust all benefits.

Under a flat-benefit pension plan, the employee earns a flat pension amount per year worked. It does not take into account differences in wages among workers. Assuming a nominal benefit of one per year,

$$B_x^t = \begin{cases} \min\{x-\alpha, \rho-\alpha\}, & \textit{when there is no indexation,} \\ \displaystyle\sum_{i=t-(x-\alpha)}^{t-1-\max(0,x-\rho)} \prod_{j=i+1}^{t-1} (1+k^j), & \textit{when there is indexation,} \end{cases}$$

$$\textit{where } k^j = \begin{cases} inf^j, & \textit{in the years in which indexation is offered,} \\ 0, & \textit{in the years without indexation.} \end{cases}$$

For example, if indexation is only offered during active years and $\alpha=25$, $\rho=65$, $x=69$, $t=1992$,

$$B_{69}^{1992} = \sum_{i=1948}^{1987} \prod_{j=i+1}^{1987} (1+inf^j).$$

Whenever wage differences are substantial, the latter type of pension plan is less appropriate. A better one might be a career-average plan, which accrues in each year a defined percentage p of the salary earned in that same year. As a result, the implicit 1 is replaced by $pSal^{i}_{x-(t-i)}$ to produce the following formula:

$$B_x^t = \sum_{i=t-(x-\alpha)}^{t-1-\max\{0,x-\rho\}} pSal^{i}_{x-(t-i)} \cdot \prod_{j=i+1}^{t-1} (1+k^j),$$

where k^j is defined as before.

Again, a specific example may provide some clarification. With $\alpha=20$, $\rho=70$ and indexation both before and after retirement, the accrued benefit for lives aged 40 and 80 in year 1990 are, respectively,

$$B_{40}^{1990} = \sum_{i=1970}^{1989} pSal^{i}_{40-(1990-i)} \cdot \prod_{j=i+1}^{1989} (1+inf^j),$$

$$and\ B_{80}^{1990} = \sum_{i=1930}^{1979} pSal_{80-(1990-i)} \cdot \prod_{j=i+1}^{1989} (1+inf^j).$$

Unless some indexation prior to retirement is included, the adequacy of the benefit accrued at age ρ may be questionable, especially so when the person's early salaries are small relative to the last ones. That is quite common and arises from high wage inflation or progress through the career path.

As a remedy, the sponsor may wish to consider a final-average plan. The accrued benefit is expressed as a percentage p of the average of the m last salaries, multiplied by the number of years of service (active life). In this case, indexation prior to retirement would have little meaning as m is typically relatively small (up to 5) with respect to ρ-α. Of course, indexation after retirement retains its full purpose. The formula for B_x^t follows:

$$B_x^t = p \cdot 1/m \cdot \left(\sum_{i=b-m+1}^{b} Sal^{i}_{x-(t-i)} \right) \cdot (\min\{x,\rho\} - \alpha) \cdot \prod_{j=t-(x-\rho)}^{t-1} (1+k^j),$$

where $b=t-1-\max\{0,x-\rho\}$ and k^j is defined as usual. When $x-\alpha \le m$, the average is calculated over the sum of all $x-\alpha$ annual wages.

Once more, examples are given here to illustrate the situation. Assuming $\alpha=30$, $\rho=60$, m=3 and indexation, values are given for x=32 and x=73 when t=1992.

$$B_{32}^{1992} = p \cdot 1/2 \cdot (Sal_{30}^{1990} + Sal_{31}^{1991}) \cdot 2,$$

$$\text{and } B_{73}^{1992} = p \cdot 1/3 \cdot (Sal_{57}^{1976} + Sal_{58}^{1977} + Sal_{59}^{1978}) \cdot 30 \cdot \prod_{j=1979}^{1991} (1 + inf^{j})$$

Note that whenever the lower bound is greater than the upper bound for the product term, it is assumed to be equal to 1, in all previous formulae. Moreover, the types of plan have been presented in increasing order of relative generosity, assuming the benefit accrued in the first year to be the same for all of them. It is worth mentioning however that the best plan for any individual depends on his own salary history as well as on the flat amount or percentage offered by each of the plans.

4. Pension Plan Mathematics

Before looking at the actual pension plans studied and the results generated from them, the reader may appreciate a brief revision of the mathematics that are particular to pension plans. First of all, simplifying assumptions are given. They then are used to derive the formulae needed to generate the results to be analyzed.

4.1 Assumptions

Some key assumptions have been made in order to simplify the task. In addition to those already mentioned (birth and hiring on January 1), following are the ones common to all results to be presented.

First of all, before the benefits are vested, nothing is paid back to the participant in the event of withdrawal or death. After this period (two years of service), the entire reserve is paid out of the fund.

Although the vesting time is realistic in view of the current regulations affecting most new Canadian pension plan participants, the return of the reserve stands out as a simplification. As a matter of fact, in the case of withdrawal, this makes sense if it is further assumed that all contributions are made by the employer. This eliminates the possibility of reimbursement of contributions, be it due to incomplete vesting or the 50% rule. As for death, some benefit is usually offered from the first day of participation until retirement but may be expressed in a variety of ways (in terms of the contributions or reserve, or as a fixed amount).

Justification for both parts of this first assumption may be given. The time requirement for vesting eliminates part of the cost associated with administering small pensions, given that the employee chooses a deferred pension instead of a transfer. It can represent important savings in industries with a high turnover. Returning the reserve would be highly desirable in order not to penalize people who regularly change jobs, especially in the case of final-average plans.

Second, all deaths and withdrawals occur at the end of the year, with payment of the benefit at the same time. Contributions are made in the middle of the year, while the pensioners are paid at the beginning of the year. As a result, if the retirement age is 65, the participant retires on the December 31 preceding his/her 65^{th} birthday and, the next day, receives his/her first cheque from the pension fund.

Third, taxation issues have been ignored, mainly with regards to the maximum contributions and benefits. Likewise, no harmonization with the Canada/Quebec Pension Plan was implemented.

Fourth, there is no delay in the adjustment to the contribution. Hence, any surplus or deficit is acted upon, if necessary, at the same time (end of year) that it is calculated. In practice, data collection and actuarial computations would cause some delay.

Fifth, only the economic data (inflation, yield and wage increase) are treated as random variables. All other assumptions to be made are assumed to be borne out by experience. In the case of mortality and withdrawal, this can be justified by the law of large numbers. Hence, the risk associated with the decrements is ignored to focus on that which stems from changes in the economy.

Sixth and last, market value is used for the assets. Reasons for doing so may be found in Maynard (1992). Above all, they are simpler to deal with than actuarial or smoothing techniques. Unfortunately, there is a cost attached: the reported rate of return on the assets is inevitably more volatile and this factor will affect the results. But it is also more realistic in that, as of the valuation, a funding level equal to or greater than 100% really means that the plan could face all existing liabilities.

4.2 Formulae

Given termination and mortality tables, an initial number of entrants in a certain year y as well as a growth rate g, it is possible to calculate n_x^t, $\forall(x,t)$, $x \geq \alpha$, as follows:

$$n_x^t = \begin{cases} n_\alpha^y \cdot (1+g)^{t-y} \cdot {}_{x-\alpha}p_\alpha^{(\tau)}, & x<\rho; \\ n_\alpha^y \cdot (1+g)^{t-y} \cdot {}_{\rho-\alpha}p_\alpha^{(\tau)} \cdot {}_{x-\rho}p_\rho, & x \geq \rho. \end{cases}$$

To calculate the annual salary of any life in any year, a salary scale $\{s_x, \alpha \leq x < \rho\}$, a wage index and a reference salary Sal_α^y constitute the requirements. In the results to be derived, the salary can be indexed with the CPI or the wage index, $\{inf^t\}$ or $\{w^t\}$. With respect to the former, there is a delay of one year for the same reasons as given before. When using the latter, there is no delay so as to keep up with what goes on in the rest of the industry. The formula is then:

$$Sal_x^t = \begin{cases} Sal_\alpha^y \cdot (s_x \div s_\alpha) \cdot \prod_{j=y}^{t-1} (1+inf^j), & \alpha \leq x<\rho,\ t>y; \\ Sal_\alpha^y \cdot (s_x \div s_\alpha), & \alpha \leq x<\rho,\ t=y; \\ Sal_\alpha^y \cdot (s_x \div s_\alpha) \div \prod_{j=t}^{y-1} (1+inf^j), & \alpha \leq x<\rho,\ t<y; \end{cases}$$

if indexation mimics the price index. Otherwise, inf is replaced by w, and both upper and lower endpoints for the product are increased by 1.

Considering the liability side of the balance sheet, formulae for the normal cost, payments and actuarial liability are needed. Several methods of funding exist. For this paper, traditional unit credit is used for flat-benefit and career-average pension plans while projected unit credit is preferred in the case of final-average plans. Both methods aim at allocating the pension cost to the year in which it is incurred, with the distinction that the second one makes an assumption with respect to annual wage increases.

Under traditional unit credit, the normal cost, payments and actuarial liability for an individual life can be expressed in these terms:

$$NC_x^t = \begin{cases} b_x^t \cdot {}_{\alpha+v-x}p_x^{(\tau)} \cdot (1+i_a)^{-(\rho-x-1/2)} \cdot \ddot{a}_\rho, & \alpha \le x < \alpha+v, \\ b_x^t \cdot (1+i_a)^{-(\rho-x-1/2)} \cdot \ddot{a}_\rho, & \alpha+v \le x < \rho, \end{cases}$$

where v is the number of years required for vesting;

$$P_x^t = \begin{cases} 0, & \alpha \le x < \alpha+v, \\ AL_x^t, & \alpha+v \le x < \rho, \\ B_x^t, & \rho \le x; \end{cases}$$

$$and\ AL_x^t = \begin{cases} [B_x^t \cdot (1+inf^t) + b_x^t] \cdot {}_{\alpha+v-x-1}p_{x+1}^{(\tau)} \cdot (1+i_a)^{-(\rho-x-1)} \cdot \ddot{a}_\rho, & \alpha \le x+1 < \alpha+v, \\ [B_x^t \cdot (1+inf^t) + b_x^t] \cdot (1+i_a)^{-(\rho-x-1)} \cdot \ddot{a}_\rho, & \alpha+v \le x+1 < \rho, \\ [B_x^t \cdot (1+inf^t) + b_x^t] \cdot \ddot{a}_{x+1}, & \rho \le x. \end{cases}$$

Of course, to find the total normal cost, payments and liability for any year, it suffices to add over all participants. Note, in the following formulae, that the total liability at the end of the year is calculated after payment to those who have died or terminated. So,

$$NC^t = \sum_{x=\alpha}^{\rho-1} n_x^t \cdot NC_x^t,$$

$$P_a^t = \sum_{x=\alpha}^{\rho-1} n_x^t \cdot (q_x^{(w)} + q_x^{(d)}) \cdot P_x^t,$$

$$P_r^t = \sum_{x=\rho}^{\omega-1} n_x^t \cdot P_x^t,$$

$$and\ L^t = \sum_{x=\alpha}^{\omega-1} n_{x+1}^{t+1} \cdot AL_x^t,$$

where ω represents the end of the mortality table ($q_{\omega-1} = 1$).

Adding the prime symbol to distinguish the values obtained using projected unit credit from those using traditional unit credit, the former can be expressed in terms of the latter, as follows:

$$NC'^{t}_{x} = [\, pSal^{t}_{x} \cdot (1+h)^{\rho-x-m} \cdot s_{\overline{m}|h} + m \,] + b^{t}_{x} \cdot NC^{t}_{x},$$

$$P'^{t}_{x} = P^{t}_{x},$$

$$AL'^{t}_{x} = \begin{cases} [\, pSal^{t}_{x} \cdot (1+h)^{\rho-x-m} \cdot s_{\overline{m}|h} + m \cdot (x+1-\alpha) \,] \\ \qquad \cdot AL^{t}_{x} + [\, B^{t}_{x} \cdot (1+inf^{t}) + b^{t}_{x} \,], & x<\rho-1, \\ [\, B^{t}_{x} \cdot (1+inf^{t}) + b^{t}_{x} \,] \cdot \ddot{a}_{x+1}, & \rho-1\leq x. \end{cases}$$

with all summation formulae identical in form, with the obvious exception that the terms being summed over are the ones obtained using projected unit credit. For example, in the formula for the total liability, AL'^{t}_{x} is substituted for AL^{t}_{x}.

Now, the assets need to be considered. Denoting the assets before the special payment is made by IA, the value of the assets from year to year evolves in the following fashion:

$$IA_{t} = (A^{t-1} - P^{t}_{r}) \cdot (1+y^{t}) + NC^{t} \cdot (1+y^{t})^{1/2} - P^{t}_{a};$$

$$\text{and } A_{t} = \begin{cases} IA^{t} - k(IA^{t} - u \cdot L^{t}), & f>u, \\ IA^{t}, & f\in[l,u], \\ IA^{t} + k(l \cdot L^{t} - IA^{t}), & f<l. \end{cases}$$

5. Description of the Software Used

Since no analytical results could be derived, at least not by the author, some programming had to be done in order to generate the desired results. All was done using Microsoft Excel, version 4.0. A worksheet along with a few macros were all that was required.

While a spreadsheet beats a programming language for the 'view' it offers - it shows all results on the screen -, it fails in terms of speed of execution. Because of this limitation, the number of simulations was limited to one hundred. Even in the simpler cases, completing them took three hours on a computer with a 80386 processor.

To provide the reader with an idea of the software implemented, a description of the parameters that can be chosen by the user will be given, followed by a brief explanation of the routine followed to generate the results to be presented in the next section.

5.1 Assumptions to Be Made

Many assumptions have to be made by the user before results are generated. Most have already been mentioned and they all can be classified in one of these six categories: decrements, wages, plan benefits, plan valuation, payment/refund and simulations.

With respect to decrements, one of each of mortality tables and termination tables must be chosen. While the latter are unisex, the former vary by gender. Since the worksheet cannot accommodate two separate sets of $q_x^{(d)}$'s, a mixing ratio male/female must be determined so as to produce a unisex table. Choices available to the user for mortality and termination are reproduced in Appendices 1 and 2, respectively.

For wages, choice must be made between ten salary scales (found in Appendix 3). This will determine how remuneration varies by age, within the same year. Alternatively, the user may specify a constant rate of wage increase, say 4%, in which case $s_x \propto (1.04)^x$. In order to calculate the annual salary for all ages and years, an index which dictates how wages are adjusted from year to year must be selected. Economic statistics offer two possible indices: the Consumer Price Index and the Wage Index. In real life, the first one usually serves as a benchmark, often considered a minimum in order to keep up with the rising cost of living. The second one shows how the average wage evolved over time. As was mentioned before, a one-year lag applies to the first one while no delay is assumed for the second one.

Three types of benefits can be chosen: flat-benefit, career-average and final-average. In the first case, the flat amount earned by year of service must be chosen. The two other ones require p to express the benefit as a percentage of the salary, and the last one requires m, the number of years in the average, as well. The number of years required for vesting, v, and whether indexation is offered or not prior to and after retirement constitute other assumptions that define the plan benefits. Also, the normal (and only possible) age at retirement, ρ, must be determined.

In order to calculate the liabilities at the end of each year, valuation interest rates, i_a and i_r, and rate of salary increase, h, should be chosen so as to reflect their expected values. A downward bias may be desirable for the purpose of conservatism.

Concerning payments and refunds, the user needs to define several parameters, among which there are l, u and n, which have already been defined. Furthermore, the interest rate used to calculate k should be given a value close to the rate of return that the fund may be expected to earn over the next n years. The worksheet offers one more level of flexibility with respect to l and u. As a matter of fact, l^* and u^* may be chosen so that the formula for A^t is modified as follows:

$$A_t = \begin{cases} IA^t - k(IA^t - u^* \cdot L^t), & f > u, \\ IA^t, & f \in [l, u], \\ IA^t + k(l^* \cdot L^t - IA^t), & f < l. \end{cases}$$

Note that, to be appropriate, $l^* \geq l$ and $u^* \leq u$. For example, the total unfunded liability ($l^* = u^* = 100\%$) could be eliminated when f moves outside the interval $[l, u]$.

Finally, the worksheet requires simulation parameters, the most obvious of which is the number of simulations the user desires. Logically enough, it should be set to one whenever the period under study does not extend beyond 1992 since all of the economic data are provided by history. Also, to calculate the first balance sheet, it is necessary to specify the initial funding level f, so that assets can be determined as a percentage of the liabilities. Of course, the first and last years of the period over which the evolution of the pension plan is investigated must be selected, along with a reference year y, which must be smaller than or equal to the greater of 1992 and the last year of calculations.

Up to ten entry ages may be specified by choosing the smallest and largest ones (α_1 and α_q, $q \leq 10$), and their common difference ($\Delta\alpha$). Given these, the balance sheet will be generated using q cohorts having entry ages α_1, $\alpha_1+\Delta\alpha$, $\alpha_1+2\Delta\alpha$, ..., and α_q. Further details about how the worksheet handles them will be provided in the next subsection.

Some data must be provided for the reference year y so that all salaries and numbers of participants may then be extrapolated. Hence, the number of entrants in year y at ages α_1 and α_q must be given and are linearly interpolated to give those at age α_j, $1<j<q$, in that year. Similarly, starting salaries and rates of population growth are needed for the same ages and year, and are linearly interpolated to provide those at intermediate entry ages. For each of the q cohorts, the number of entrants and salary are then determined, for any age and year, using the formulae already developed. As for the growth rate, it is unique to each cohort and remains constant through time.

5.2 What It Does

After the user revised all required assumptions, a macro which does all simulations and generates the results, can be run. Following is a brief survey of how it works.

It is important to understand that there is a nested branching simulation-year-entry age. As a result, each simulation necessitates calculations at all years under study, and, for each year, separate calculations need to be made for each of the q cohorts. It is like having three "do" loops, nested within one another, with the one covering all entry ages being inside the one going over all years, which is itself nested within the one that goes from the first simulation to the last.

For each simulation, the macro first checks if economic data need to be produced (i.e. if the period extends beyond 1992). If so, it generates those required using Wilkie's investment model, to be presented later in this text. It then performs, year by year, the calculations described in the next paragraph. Once completed, it summarizes them using the following four statistics: average and standard deviation of both the cost ratio r and the funding level f. Note that r and f are the two variables of interest.

For each year, it first has to do, cohort by cohort, the routine briefly given in the next paragraph. After that, if it is the first year, it takes the sum of the q cohorts' liabilities as the total actuarial liability to write in the balance sheet for the year. It then multiplies this figure by the initial funding level to value the assets. For all other years, it sums the cohort-specific values to get the total normal cost, payments to active lives, payments to retired lives and actuarial liability. Then, IA^t, A^t, S^t, r^t and f can be calculated and included in the annual balance sheet, using the formulae provided.

For each cohort, the total number of people, normal cost, payments to active lives and to retired lives, and actuarial liability are determined. Note that these values are cohort-specific and are summed to produce the grand totals for each year. These intermediate totals come from calculations done for every age x, $\alpha_j \leq x \leq \omega-1$, included in the j^{th} cohort. They include n_x^t, $\{Sal_i^j, \alpha \leq i \leq \min\{x,\rho-1\}, j=t-(x-i)\}$, NC_x^t, P_x^t and AL_x^t.

In summary, all cohorts' results are summed and are treated as a single cohort for the purposes of the balance sheet, which is produced yearly. All that remains after each simulation are the four statistics generated, which will be analyzed in the next section. The purpose of the cohorts is to allow for different entry ages; the worksheet distinguishes between participants according to their age at entry as well as their current age.

6. Study of the Variables of Interest

Now that notation, definitions, mathematics and software have been taken care of, it is possible to embark on the main subject of this paper, namely the study of the variability of the cost ratio and funding level. Where the economic data comes from will be explained first. Then, all sets of assumptions used for this paper will be given, along with some justification. To complete this section, the actual results will be presented and tentatively commented upon.

6.1 Economic Data

While only the price inflation, wage inflation and yield (inf, w, y) are explicitly needed for each year's computations, y actually is a combination of the yields on different classes of assets (stocks, mortgages, bills and bonds) in years for which the median pension yield is not known. Hence, data about all of those must be available or simulated for the period of interest.

6.1.1 Assumptions prior to 1924

For years before 1924, no historical data were readily available. For that reason, the first year to be studied must be 1924 or after. Unfortunately, this does not solve the problem completely since historical inflation figures may be required when considering career-average or indexed pension plans.

Instead of simulating backwards, the author preferred to make some simplifying assumptions: $inf^t=0.01$ and $w^t=0.02$, $t<1924$. Of course, these arbitrary values are highly debatable. However, in the cases to be considered, they are called upon only for the 1982-1992 and 1995-2005 periods. Hence, they are likely to have limited impact on the final results.

6.1.2 Historical Data for 1924-1992

The Canadian Institute of Actuaries annually publishes its *Report on Canadian Economic Statistics*. Among other information, the one used by the author (published in 1993) contains inf^t and w^t for all years from 1924 to 1992. It also gives the median yields on pension funds, y^t, from 1960 to 1992. For the years before 1960, the historical returns on stocks, mortgages, bonds and bills are used to generate the yields in the same way as Maynard (1992). Table 1 shows the weights that were used in that case to compute their average, used as y^t.

All the data provided by the *Report* and used by the worksheet, along with the hypothetical yields, are reproduced in Appendix 4. Note that while the returns on stock, bonds and bills for the period 1960-1992 are not required to calculate y^t, they are necessary for simulations requiring the use of Wilkie's model.

Period	Weights			
	Common stock	Long bonds	T-bills	Mortgages
1924-1945	37%	63%		
1946-1951	37%	58%	5%	
1952-1959	37%	38%	10%	15%

Table 1. Weights given to each class of assets

6.1.3 Models after 1992

While it is interesting and appropriate to study the variability of r and f over the past, it would be arguably more useful to obtain results pertaining to the future. Of course, no economic data for those years, except 1993, are available yet. Hence, a model must be stipulated.

Wilkie (1993) and Sharp (1993b) together provide all the models necessary to extend the series of inflations and returns beyond 1992, with the exception of mortgages. The former covers price inflation, share dividends, share dividend yields, long- and short-term bond yields. To complete the picture, the latter deals with wage inflation. (Sharp also treats price inflation and obtains results similar to Wilkie's.)

Note that throughout the models that follow, ϵ represents a standard normal random variable, independent of all other ones. Moreover, symbols for the parameters were kept, with their values given in the appendix, since they also constitute assumptions and, as such, could be modified to reflect new information or personal beliefs.

The model developed by Wilkie follows a 'cascade' approach, which will be followed here. It starts with price inflation, with I(t) denoting its force from t-1 to t. Then,

$$\begin{aligned} I(t) &= QMU + QN(t), \\ QN(t) &= QA \cdot QN(t-1) + QE(t), \\ \textit{and } QE(t) &= QSD \cdot \epsilon_1(t). \end{aligned}$$

Its long-term mean is QMU and $\inf' = e^{I(t)}-1$. Estimated values for these parameters and all other ones can be found in Appendix 5; they are all based on Canadian figures. In the case of price inflation, Wilkie obtained them using annual inflation data from 1923 to 1993.

Then, the yield on common stock is obtained as a formula involving share dividend yields and dividends, which are the object of the following two models. Let Y(t) be the dividend yield in June of year t. With I(t), already defined, as an independent variable,

$$
\begin{aligned}
\ln Y(t) &= YW \cdot I(t) + \ln YMU + YN(t), \\
YN(t) &= YA \cdot YN(t-1) + YE(t), \\
\textit{and} \quad YE(t) &= YSD \cdot \epsilon_2(t).
\end{aligned}
$$

$YMU \cdot e^{YW \cdot QMU}$ gives its long-term mean. However, it provides little information unless analyzed in conjunction with the dividends themselves, which are given by D(t) for year t. Their evolution depends on the inflation series and last year's share dividend yield in this way:

$$
\begin{aligned}
\ln D(t) - \ln D(t-1) &= DW \cdot DM(t) + (1-DW) \cdot I(t) + DMU \\
&\quad + DY \cdot YE(t-1) + DB \cdot DE(t-1) + DE(t), \\
DM(t) &= DD \cdot I(t) + (1-DD) \cdot DM(t-1), \\
\textit{and} \quad DE(t) &= DSD \cdot \epsilon_3(t).
\end{aligned}
$$

In order to calculate the return on common stock, the share price itself is needed and is the quotient of D(t) over Y(t). The total return on common stock can then be expressed as a combination of the annual percentage increase in share price and dividend yield, in terms of either a sum or a product. To estimate the parameters, dividend yields in June and annual dividends, spanning 1936 to 1993, were used.

In spite of the relative simplicity of these last two models, their actual implementation turned out to be quite a challenge. In Canada, the data most easily found happen not to be the dividends and dividend yields. Rather, the Stock Price and Total Return Indices are given more coverage and published in the *TSE Review*, among other sources.

Because of that state of matters and because no simple and precise relation seemed to hold between the Stock Price Index (SPI), Total Return Index (TRI), D(t) and Y(t), an alternative had to be found.

Let SPI(t) and TRI(t) denote values at the end of year t. Assuming the dividends to be uniformly distributed over the year and rate of increase in the share price to be constant throughout the year, it follows that:

$$
\begin{aligned}
SPI(t) &= D(t+1) \div Y(t+1), \\
1+\iota(t) &= SPI(t) \div SPI(t-1) \\
\textit{and} \quad TRI(t) &= TRI(t-1)\cdot[1+\iota(t)] + D(t)\cdot s_{\overline{1}|\iota(t)}^{(365)}
\end{aligned}
$$

Of course, both assumptions are questionable. In fact, as was documented by Harvey and Whaley (1992), the payment of dividends is highly seasonal, at least for the S&P 100. In addition, one needs only look at monthly values of the Stock Price Index to realize that their increase is anything but smooth over the year!

With these formulae for SPI(t) and TRI(t) in hand, D(t) and Y(t) are calculated for the period 1957-1992. These values are then used along with Wilkie's models to simulate the dividends and dividend yields for the subsequent years. These same formulae are then used to calculate the simulated stock price and total return index values. The return on common stock for year t is conveniently expressed as the ratio of TRI(t) to TRI(t-1).

So much for common stock! As part of Wilkie's investment model, bonds and bills (long- and short-term bonds) still remain to be treated. Denote by C(t) the yield on bonds in year t. As it turns out, C(t) depends on the inflation history and dividend yield for that year, as seen here:

$$
\begin{aligned}
C(t) &= CW\cdot CM(t) + CMU\cdot e^{CN(t)}, \\
CM(t) &= CD\cdot I(t) + (1-CD)\cdot CM(t-1), \\
CN(t) &= CA\cdot CN(t-1) + CY\cdot YE(t) + CE(t), \\
\textit{and} \quad CE(t) &= CSD\cdot \epsilon_4(t).
\end{aligned}
$$

In this case, the mean is CMU+100·QMU. It is interesting to note that parameters for the Canadian model were estimated using long-term government bonds, which all can be redeemed, whereas the British version modelled the yield on the Consols, which are practically irredeemable. In this case, data for the 1936-1993 period were used.

Also, denote by B(t) the yield on bills in year t. As one might suspect, the model for B(t) involves C(t):

$$\ln B(t) = \ln C(t) + BMU + BN(t),$$
$$BN(t) = BA \cdot BN(t-1) + BC \cdot CE(t) + BE(t),$$
$$\textit{and} \quad BE(t) = BSD \cdot \epsilon_5(t).$$

The product of the mean of C(t) and the exponential of BMU (e^{BMU}) provide the mean of B(t). Although data on three-month Treasury bills are available for as early as 1924, Wilkie only used data for 1956 to 1993 so as to avoid the extremely low yields in the years preceding 1956.

The method followed by Wilkie to develop those models has been termed the 'cascade' approach since variables are introduced in order of dependency. Indeed, I(t) stands alone while Y(t) depends on I(t); D(t), on Y(t) and I(t); and so on.

Of course, these five models do not give values for y^t. Again, a weighted average of the returns on stocks, bonds and bills will be used. The actual weights are adapted from Sharp (1993a). As given in that paper, the percentage of assets in each of four main categories, for a Canadian pension plan, would typically look like this:

Bills and other cash assets	3%
Bonds and mortgages	47%
Canadian equities	41%
American equities	9%

Since neither mortgages nor American equities are modelled, the actual proportions used are as follows: 3% in bills, 47% in bonds and 50% in stock.

Still, one more model remains to be specified: the one for wage inflation. W(t) represents the wage indexation in year t. It was found to depend on price inflation:

$$W(t) = WMU + WW \cdot (e^{I(t)} - 1) + WN(t),$$
$$WN(t) = WA \cdot WN(t-1) + WE(t),$$
$$\textit{and} \quad WE(t) = WSD \cdot \epsilon_6(t).$$

The estimated parameters given in Sharp (1993b), based on the period 1924-1991, were truncated to three decimals. As those for all previous models, they have been reproduced in Appendix 5.

6.2 Sets of Assumptions[2]

Before results may be interpreted, it is necessary to know which assumptions underlie them. This section presents all sets used while actual results are the subject of the next two subsections.

In order to simplify the analysis, many assumptions are common to all sets. For decrements, the 1971 Group Annuity Mortality (GAM71) table and termination table TTW2 were picked, with a ratio male/female of 60/40 to make mortality unisex.

SO3 serves as the common salary scale, while wage indexation varies between certain sets. Since the purpose is to study pension plans, it would make little sense to concentrate on only one type of benefit. However, two years are always required for vesting. Besides, the flat amount is set at two hundred and fifty dollars and the percentage of salary replaced (p), at 2.00%, both per year of service. These two could have taken on any value without affecting the outcome: they act mainly as an arbitrary basis. On the other hand, the number of years m used in final-average plans would make a difference and, for this paper, equals 3. As for the indexation of the benefits, it varies from set to set.

Period	Without indexation	With indexation	Salary increase
1924-1992	8.00%	4.00%	6.35%
1992-2050	7.50%	3.75%	6.50%
1982-1992	13.00%	8.00%	8.00%
1995-2005	7.50%	3.75%	7.85%

Table 2. Rates of interest and salary increase used for plan valuation

The rates of interest and salary increase used for plan valuation depend on the period studied. They also depend on the indexation provided. Hence, if benefits are indexed while active but not while retired, i_a will be a real rate whereas i_r will be a nominal rate. Likewise, if wages increase with the CPI, h will be smaller than it would be if the actual wage index were used.

---

2 All basic sets of assumptions presented in 6.2.1,.2 and .3 can be found in Appendix 6. All others are variations thereof.

The CPI is used for the 1924-1992 and 1992-2050 periods only. For the pre-1993 periods, i_a, i_r and h are equated with their average over the period studied, with a small downward bias to instill some conservatism. For the post-1992 ones, they are equal to the expectations (long-term means). Table 2 shows the actual values assumed.

The nominal rate of interest used for plan valuation is also the rate at which k, used to determine special payments and refunds, is calculated. As was mentioned earlier, at onset, the plan is fully funded (f=100%), and the number of simulations is 1 unless economic data need to be generated, in which case it is 100. Finally, all participants retire at age 65.

6.2.1 Simplified Sets

The main advantage of the so-called simplified sets is that a simple relationship links all liability-related expressions from year to year. As a result, the normal cost, payments and actuarial liability need only be determined for the first year by way of complete calculations. Afterwards, their values for year t are merely multiples of those for the preceding year.

For these sets, a single cohort, whose members enter the plan at age 25, see their wages increased with the Consumer Price Index. The number of entrants is 100 for the first year and the starting salary was two thousand dollars per year in 1924. Note that these last two assumptions have no influence on the final results.

A flat-benefit pension plan without any indexation is, undeniably, the simplest, but also the most unrealistic: it offers the same amount per year of service regardless of age and year. As a result, NC^t, P_a^t, P_r^t and L^t grow at the same rate as the population. For example, $NC^t = (1+g) \cdot NC^{t-1}$.

In real life, the annual amount provided by flat-benefit pension plans is usually renegotiated on a regular basis so that all pension benefits accrued to active lives (sometimes retired lives as well) are increased accordingly. For instance, say a plan pays fifty dollars per year of service, in 1992 and, in 1993, the union negotiates an increase to fifty-five. Then, the pensioner who received five hundred dollars in 1992 for his ten years of service will see his benefit revised to five hundred and fifty dollars in 1993.

To account for that, a flat-benefit pension plan, offering no indexation but subject to annual negotiation, is considered. It is assumed that the annual revision just compensates for the increase in the cost of living. In this case, NC^t, P_a^t, P_r^t and L^t are all linked in this way: $L^t = (1+inf^{t-1}) \cdot (1+g) \cdot L^{t-1}$. Of course, the underlying assumption for that to hold is that the valuation does not account for potential increases in the flat amount. This implies that deficits and special payments will be commonplace.

Only a third type of pension plan seems to lend itself to major simplifications: a fully-indexed career-average plan. Since wages increase with the CPI and accrued benefits increase in the same fashion, so will all liability-related terms. Indeed, this relationship will apply to all four: $P_a^t = (1+inf^{t-1}) \cdot (1+g) \cdot P_a^{t-1}$.

As a matter of fact, the recursion is identical for the last two plans presented so far. The main difference lies in their valuation. While the second one uses nominal rates and effectively ignores potential inflation, the third one values liabilities at real rates.

For the three plans, the risks they face vary. The non-indexed flat-benefit plan is only subject to the variability in the nominal yield (y^t) while the two other ones are also subject to that in inflation. In addition, the fully-indexed flat-benefit plan features an additional risk linked to the inappropriateness of its valuation basis.

Being simplified, these plans can lend themselves, better than others, to sensitivity studies. With nil population growth, partial payment of any unfunded liability (whether positive or negative) and a 'spreading' period of 5 years as a starting point, each of these three assumptions is varied over a set of values while keeping the other two constant. This is accomplished for all plans described above.

Their simplicity also allows for study over longer periods, which explains the choice of 1924-1992 and 1992-2050 in this case, as opposed to 1982-1992 and 1995-2005 for all other sets.

Note that, for all other sets of assumptions to be defined, only the 'starting point' given above is used, with the exception of the realistic sets which also include a shifting population. Unlike the simplified sets, these sets assume wages to increase with the actual wage index.

6.2.2 Sets for All Types of Benefits

As one might argue, the assumptions contained in the simplified sets lack realism. In order to see whether the type of benefit actually influences the variability of the cost ratio and of the funding level, all types of benefits described in Section 3 are investigated.

That is, flat-benefit, career-average and final-average are all considered, along with all possible combinations of indexation before and after retirement.

Again, one hundred new participants enter the plan each year at age 25 and the reference annual salary is $Sal_{25}^{1924}=2{,}000$ although, as pointed out earlier, these assumptions have no material impact.

Typically, one would think that, the more generous the plan is, the greater the risks it faces. However, it might actually be less risky though more costly to have a fully-indexed plan if it so happens that real rates vary less than nominal ones.

In spite of that, risks particular to a certain pension plan may be identified. Thus, all non-indexed plans are exposed to the variability in yields. In addition, the normal cost for career- and final-average plans depends on wage indexation. The final-average ones are even more sensitive to wage indexation since it constitutes one of the assumptions made in the plan valuation.

A similar characterization may be made for partially- as well as fully-indexed pension plans. Of course, they then are also subject to the variability in price inflation to the extent of the indexation provided.

6.2.3 More Realistic Sets

Still, assuming a single entry age bears little resemblance to reality. Hence, in order to add yet another level of realism, three cohorts were assumed (with entry ages of 25, 35 and 45) for a three-year final-average pension plan. The whole population may be either stationary or shifting. In the last case, the distribution of entrants at each age is reversed over ten years. That is, if $n_{25}=38$ and $n_{45}=28$ in the initial year, then, $n_{25}=28$ and $n_{45}=38$ ten years later, while n_{35} remains constant.

These sets, the most realistic built for this paper, may shed some light over the variability of the two variables of interest in an aging population, compared with a stationary one. All risks included in the analysis are faced by the indexed realistic plans: with respect to yield as well as price and wage inflation.

6.3 Historical Results

To begin the analysis, historical data will be used for the economic figures. This will allow an 'all other things being equal' type of study since all calculations share a common economic scenario. In this sense, the results to be given are exact; that is, they are not subject to estimation error.

6.3.1 Period 1924-1992

Results under the basic set of assumptions ($g=0\%$, $l=u=100\%$, $n=5$) will first be studied for all three simplified sets. After that, for each set, n and $[l,u]$ will be varied independently and any impact these variables may have will be examined.

The reader is reminded that the first set refers to the non-indexed flat-benefit pension plan; the second, to the fully-indexed one; and the third, to the fully-indexed career-average plan.

Since the second one does not account for future indexation in its valuation, it is expected that the cost ratio, r=C/NC, will be, on average, appreciably greater than one. For the same reason, the funding level will most likely be lower than 100% as unfunded liabilities are spread over five years and likely to occur. As for the two other ones, given an appropriate valuation basis, the average cost ratio and funding level should both be close to one. As far as the standard deviations of these variables are concerned, their actual magnitude seems harder to guess at.

Set	average r	std. dev. r	average f	std. dev. f
1	1.107382	1.878838	99.33%	11.73%
2	2.923538	1.968080	87.99%	12.29%
3	0.840523	1.131452	101.79%	12.70%

Table 3. Basic results for all sets, n=5, $l=u=100\%$, g=0%

As suspected for the second set and shown in Table 3, the average cost ratio turns out to be well above one and the average funding level, below 100%. However, while the average funding level appears satisfactorily close to 100% for the two other sets, the same cannot be said about the average cost ratio. The inadequacy of the valuation basis could be the cause. Despite that, if one considers that the normal cost is relatively small compared to the liability and that it is on assets of the size of the latter that deficits or surpluses accrue, the basis may well be adequate and the cost ratio, satisfactorily close to one.

One point worth noting is that, whenever the average cost ratio is below one, the average surplus is above 100%, and vice versa. Of course, if the plan tends to have favorable experience, the sponsor will be able to use the surplus to reduce the size of the contributions and thus drive the cost ratio below one.

Now, standard deviations can be looked at. The difference between those for the first and second sets of assumptions is strikingly small. How can that be? Perhaps, the risk arising from ignoring inflation in valuing liabilities acts as a destabilizing factor only in terms of expectations.

While the variabilities of the funding level are not all that different, that is not the case for the cost ratio. The third plan features a lower standard deviation of the cost ratio, implying that fully-indexed plans are less risky in that sense. Further investigation needs to be carried out concerning that matter.

Tables 4, 5 and 6 will permit the study of the dependency of the variables of interest on the number of years n over which any surplus or deficit is spread. Dufresne (1988) has found that, in the long run, σ_f increases as n increases. Indeed, if n equals 1, no variability ensues as f is annually brought back to 100%. As n becomes greater, the interval in which f can potentially lie widens and, as a result, allows for it to assume more variability.

Since Dufresne (1988) does not use the same statistic to study the contributions, his findings cannot be used as a starting point. Nevertheless, it is possible to postulate some hypotheses. As n becomes infinitely large, the cost ratio should tend to one in expectation and to zero in variance. So, as n increases, it is expected that the standard deviation of the cost ratio will decrease.

As far as averages are concerned, given a proper valuation basis, both that of r and f should revolve around one. Of course, since the second set implies the use of inadequate valuation assumptions, the averages are not expected to lie anywhere close to unity. As a matter of fact, as the plan tends to be unfunded, the average cost ratio should increase as n increases — a larger n allows more time for deficits to build up — and the average funding level should decrease as a result.

Assume, for example, that a deficit results in the first year and that, from the next year onward, all economic outcomes are as expected. Then, considering the deficit apart from the rest, it will grow at a rate i, the expected rate of return. That is so since the rest of the assets are always equal, at the end of the year, to the liabilities. Indeed, if $y^t = i$, $\forall t$, the following should hold:

$$L^t = A^t = (A^{t-1} - P_r^t) \cdot (1+i) + C^t \cdot (1+i)^{1/2} - P_a^t,$$

assuming the initial funding level to be 100%. Note that, if such is the case, $C^t = NC^t$, $\forall t$.

Of course, as special payments are made, the deficit decreases and eventually disappears. In the event that the contributions increase at a rate faster than the yield, delaying repayment will result in a lower average ratio. However, for the three sets of assumptions considered here (with $g=0\%$ and $l=u=100\%$), the contributions remain constant unless indexation is provided, in which case it grows with inflation, which is, most of the time, smaller than the yield.

As a result, if the plan is, on average, underfunded, it is expected that the average ratio will increase with n. Similarly, r should decrease as n increases if the opposite holds, that is, if surpluses outweigh deficits.

n	average r	std. dev. r	average f	std. dev. f
1	0.992113	4.238976	100.00%	0.00%
2	0.977119	2.747446	100.04%	4.80%
3	1.002560	2.268823	99.99%	7.63%
4	1.048195	2.034506	99.77%	9.88%
5	1.107382	1.878838	99.33%	11.73%
6	1.172832	1.753611	98.70%	13.20%
7	1.239423	1.644188	97.91%	14.33%
8	1.304219	1.545820	97.01%	15.17%
9	1.365719	1.456712	96.04%	15.78%
10	1.423244	1.375986	95.02%	16.20%
11	1.476571	1.303056	93.97%	16.48%
12	1.525723	1.237423	92.93%	16.65%
13	1.570858	1.178617	91.89%	16.74%
14	1.612202	1.126164	90.88%	16.78%
15	1.650013	1.079590	89.90%	16.78%
16	1.684560	1.038417	88.95%	16.75%
17	1.716107	1.002172	88.05%	16.72%
18	1.744910	0.970395	87.19%	16.69%
19	1.771207	0.942637	86.38%	16.65%
20	1.795223	0.918477	85.60%	16.63%

Table 4. First set, varying n, $l=u=100\%$, $g=0\%$

All three tables seem to confirm the doubts expressed in the previous paragraphs about the standard deviations, with a minor exception in the case of the non-indexed flat-benefit pension plan insofar as the funding level is concerned. In that particular case, s_f seems to reach a plateau at 16.78% when n equals 14 and 15 and decreases thereafter. However, the reader should note that the economic scenario that actually prevailed in the period studied is a special one in the sense that plenty of others *could* equally likely have taken place and may have yielded slightly different outcomes.

Taking the results in Table 3 as a reference point, it would seem that the valuation interest rate used for the first set (i=8.00%) is somewhat too high since the fund is, on average, insufficient to meet current liabilities. The opposite conclusion seems to hold for the third set, which values liabilities at an interest rate of 4.00%, as the fund usually has a surplus.

n	average r	std. dev. r	average f	std. dev. f
1	2.764433	4.764203	100.00%	0.00%
2	2.775761	3.091090	96.90%	5.40%
3	2.831156	2.455284	93.84%	8.25%
4	2.882064	2.142199	90.86%	10.41%
5	2.923538	1.968080	87.99%	12.29%
6	2.955915	1.864186	85.28%	14.03%
7	2.980587	1.799672	82.74%	15.68%
8	2.999047	1.758657	80.38%	17.26%
9	3.012610	1.732203	78.20%	18.76%
10	3.022350	1.714998	76.19%	20.19%
11	3.029126	1.703767	74.33%	21.55%
12	3.033611	1.696442	72.63%	22.83%
13	3.036335	1.691693	71.07%	24.03%
14	3.037711	1.688651	69.64%	25.16%
15	3.038064	1.686743	68.33%	26.21%
16	3.037646	1.685586	67.12%	27.20%
17	3.036657	1.684924	66.02%	28.11%
18	3.035252	1.684586	65.01%	28.97%
19	3.033553	1.684455	64.08%	29.76%
20	3.031653	1.684453	63.22%	30.49%

Table 5. Second set, varying n, $l=u=100\%$, g=0%

Under the same reasoning as that presented for the second set, it is expected that the average cost ratio will increase and the average funding level will decrease as n increases, if i is too high, and vice versa. Again, that makes sense because the greater n, the more time the surplus or deficit has to accumulate and affect both statistics studied.

These last hypotheses are confirmed, at least partially, by a careful look at Tables 4 and 6. Only the results for n=1,2 under the first set raise some suspicion as to their validity. They seem to indicate that the valuation basis might, in fact, have been too conservative. The author checked the balance sheets for that period and could not find any explanation. Again, the two sets of averages which raise concerns are arguably very close to their ideal values of 1 and 100% and thus may mean little after all.

Of all hypotheses considered so far, the ones concerning the averages in Table 5 remain to be verified. Whereas the funding level follows the expected pattern, the cost ratio seems to peak for n equal to 15 and then decline. The relative difference between the average cost ratio at larger values of n is around 0.2%, a value that some may consider negligible. However, the straight decline in its value for n larger than 15 can hardly go unnoticed. Again, unfortunately, the author could not find any reason to support that behavior.

n	average r	std. dev. r	average *f*	std. dev. *f*
1	0.849551	2.718102	100.00%	0.00%
2	0.825251	1.797778	100.55%	5.64%
3	0.830117	1.437191	101.03%	8.69%
4	0.836286	1.247962	101.43%	10.90%
5	0.840523	1.131452	101.79%	12.70%
6	0.842410	1.051760	102.13%	14.23%
7	0.842193	0.993349	102.47%	15.57%
8	0.840310	0.948413	102.82%	16.74%
9	0.837201	0.912569	103.17%	17.78%
10	0.833247	0.883162	103.53%	18.70%
11	0.828754	0.858490	103.89%	19.53%
12	0.823954	0.837420	104.26%	20.26%
13	0.819024	0.819165	104.62%	20.93%
14	0.814089	0.803169	104.98%	21.52%
15	0.809239	0.789023	105.33%	22.05%
16	0.804537	0.776420	105.67%	22.53%
17	0.800024	0.765125	106.00%	22.96%
18	0.795724	0.754952	106.32%	23.34%
19	0.791652	0.745755	106.62%	23.69%
20	0.787812	0.737412	106.91%	24.00%

Table 6. Third set, varying n, $l=u=100\%$, g=0%

Now that the effect of varying the number of years over which repayment is spread has been investigated, it might be interesting to examine the effect of changing the boundaries of the 'reaction' interval $[l,u]$. To do so, the 'spreading' period is held constant at five years. Setting n equal to 1 might have made the analysis easier but would have found little justification with respect to current legislation. Tables 7, 8 and 9 contain the results supporting this analysis. $[l,u]=[0\%,1000\%]$ was only used as an extreme case.

As was done earlier, it is worthwhile first to try to postulate some hypotheses, which can then be compared with the actual results. This makes the exercise more interesting and valuable. For the average funding level, it appears likely that, the larger $\frac{1}{2}\cdot(l+u)$, the greater that statistic should be. To be more precise, on the one hand, keeping l constant and increasing u should lead to a greater value for the average *f* since the interval then allows for larger surpluses to survive. On the other hand, decreasing *l* while keeping *u* at the same level should produce a smaller average *f* as greater deficits may accumulate before special payments are called for.

Guesses concerning the average cost ratio seem harder to make. Nonetheless, it appears reasonable to think that, if a larger surplus can build up (increasing *u* while freezing *l*), the average *r* is likely to drop. This may be hard to understand if one considers the fact that positive experience may not be taken advantage of or only later, while negative experience is addressed quickly; this could lead one to believe that the average r should increase as a result. But if *u* increases, the surplus has the possibility of getting larger with investment income and thus limit the probability of unfunded liabilities. Indeed, the surplus which is not used right away to lower contributions accumulates interest and eventually plays a more effective role. A similar reasoning applies to the opposite case with, of course, the opposite conclusions. Hence, if the 'reaction' interval is widening from the left (*l* decreases), then the average cost ratio should increase as deficits are allowed to take on larger proportions and eventually call for greater adjustments to the level of contributions.

In terms of standard deviations, having *u-l* increase, while modifying only one of the two boundaries, should also increase the variability of the funding level, since it can then take on a greater range of values before contributions are adjusted. It should also make the standard deviation of the cost ratio smaller for the same reason.

l	*u*	average r	std. dev. r	average *f*	std. dev. *f*
0%	1000%	3.570034	2.903984	20.57%	58.85%
80%	100%	1.925059	1.303592	79.03%	15.45%
80%	110%	1.936021	1.137406	81.13%	17.40%
80%	120%	1.943043	1.001683	82.48%	18.29%
90%	100%	1.493609	1.584472	89.71%	13.83%
90%	110%	1.496982	1.435085	92.37%	16.61%
90%	120%	1.522932	1.286683	94.34%	18.90%
100%	100%	1.107382	1.878838	99.33%	11.73%
100%	110%	1.069557	1.804447	102.85%	15.10%
100%	120%	1.070226	1.714716	105.49%	18.07%

Table 7. First set, varying *l* and *u*, n=5, g=0%

All three tables reveal that the average funding level exhibits the suspected behavior. Note that it may be outside the interval [l,u] since f is not constrained to stay within these boundaries. Because the 'spreading' period spans more than one year, the adjustment made to the contributions whenever f lies outside the interval does not force it back in that range.

l	u	average r	std. dev. r	average f	std. dev. f
0%	1000%	3.024169	2.255934	33.20%	65.54%
80%	100%	3.108008	1.855341	70.11%	17.99%
80%	110%	3.045091	1.812209	72.57%	21.09%
80%	120%	2.996263	1.762315	74.78%	23.69%
90%	100%	3.019372	1.879908	78.94%	14.85%
90%	110%	2.969004	1.801781	81.07%	17.57%
90%	120%	2.935956	1.717062	82.83%	19.72%
100%	100%	2.923538	1.968080	87.99%	12.29%
100%	110%	2.880387	1.863819	89.82%	14.62%
100%	120%	2.849899	1.754575	91.51%	16.70%

Table 8. Second set, varying l and u, n=5, g=0%

Unfortunately, the average cost ratios obtained do not allow for such a clear-cut conclusion. While results for the second set agree with the initial hypothesis and those for the first set are not so different either, those for the third set seem to contradict it.

l	u	average r	std. dev. r	average f	std. dev. f
0%	1000%	1.000000	0.000000	286.62%	102.73%
80%	100%	0.910109	1.026420	88.40%	18.49%
80%	110%	0.826361	0.995666	96.05%	20.98%
80%	120%	0.740148	0.909122	107.07%	22.33%
90%	100%	0.903736	1.034427	95.61%	15.07%
90%	110%	0.851140	1.013790	101.29%	17.78%
90%	120%	0.773656	0.947008	110.12%	19.34%
100%	100%	0.840523	1.131452	101.79%	12.70%
100%	110%	0.822401	1.050035	107.37%	15.16%
100%	120%	0.791415	0.985205	114.08%	17.17%

Table 9. Third set, varying l and u, n=5, g=0%

Remember that it is the only set for which the plan is, on average, over-funded. This explains the fact that, unlike the two other ones, under the extreme case ([l,u]=[0%,1000%]), the contributions do not vary and a huge surplus is formed. Then, it would seem that if l is decreased when u=120%, subsequent surpluses can effectively take care of transient deficits and reduce the need for special payments.

Results generated for the standard deviations are in almost perfect agreement with the hypotheses laid out. As a consequence, widening the interval increases the variability of the funding level while doing so decreases that of the cost ratio.

This completes the analysis for the simplified sets using historical data. The effect of varying the growth rate could also have been investigated in that setting, but the author preferred to do so only using simulated data.

6.3.2 Period 1982-1992

For this shorter period, the remaining sets of assumptions will be used. Being more complicated and thus necessitating more computation time, those sets require the study to be restricted to ten years. It would otherwise be, computer-wise, too demanding, especially when simulations are necessary.

First of all, the effect of varying the type of benefit is investigated. In this case, it is hard to develop a feel for what the impact should be. A look at Table 10 will support the analysis. Intuitively, if the risk associated with nominal yields is greater than that arising from real yields, non-indexed plans should show more variability, both in cost ratio and funding level. Of course, a final-average plan is likely to be riskier than a career-average one since salary increases require an additional assumption for valuation. As usual, given a proper valuation basis, the averages should be close to one.

Type of benefit	Indexation prior to retirement	Indexation after retirement	average r	std. dev. r	average f	std. dev. f
Flat-benefit	No	No	1.088613	1.656007	99.68%	5.93%
	No	Yes	1.986014	2.259376	97.46%	5.57%
	Yes	No	1.927649	2.957750	98.39%	5.60%
	Yes	Yes	2.957917	3.723777	97.21%	5.38%
Career-average	No	No	1.066824	0.396811	99.38%	5.67%
	No	Yes	1.149737	0.456300	98.40%	5.27%
	Yes	No	1.190287	0.554276	98.38%	5.33%
	Yes	Yes	1.274233	0.623560	97.86%	5.12%
Final-average	No	No	1.030437	0.802848	99.92%	5.56%
	No	Yes	1.158464	0.904602	99.12%	5.33%

Table 10. Sets for all types of benefits

What surprises the author, and perhaps the reader as well, is the little impact the level of inadequacy of the valuation basis seems to have on the standard deviation of the funding level. It is true that $l=u=100\%$ but, as was pointed out earlier, this does not imply $f\equiv 100\%$. All bases used to value liabilities turn out to be too optimistic: while h (8.00%) and non-indexed i (13.00%) seem only slightly overestimated, the indexed i (8.00%) clearly is too large. This is revealed by the magnitude of the average cost ratios and funding levels.

Providing indexed benefits seems to introduce more risk since the related standard deviations in the cost ratio are greater in that case. However, if there is correlation between its average and standard deviation, the improper valuation basis may explain it all. Figuring which one of the two impacts it most would require a two-way analysis, which has not been done for this project.

Finally, using historical data, only realistic sets remain to be studied. Because of the multiple entry ages, those sets are, undeniably, the most demanding in computer resources. Again, that justifies limiting the period to ten years. As in the previous case, hypotheses are not formulated.

Table 11 provides the results for four different final-average pension plans. Once more, the standard deviation of the funding level undergoes remarkably limited variation from one set to the other, as it was seen when varying the type of benefit, although slightly lower when indexation is offered.

Type of population	Indexation after retirement	average r	std. dev. r	average f	std. dev. f
Stable	No	1.032342	0.602375	99.85%	5.55%
	Yes	1.141917	0.689113	98.97%	5.28%
Shifting	No	1.020728	0.754611	99.89%	5.59%
	Yes	1.194129	0.888437	98.79%	5.31%

Table 11. Realistic sets

Because the average cost ratio is simply a little above 1 and the funding level, negligibly below 100%, the valuation basis is apparently only slightly too optimistic.

Moreover, changing the type of population from stationary to shifting appreciably modifies neither the averages nor the standard deviation of the funding level. However, a shifting population seems to induce a greater variability of the cost ratio, at least historically. This perhaps can be partly justified by the fact that it is easier, for a plan sponsor, to estimate the cost under an unchanging population. However, note that population growth does not constitute a required assumption when using unit credit as the funding method. As a result, this hardly can be considered as the reason. Further investigation shall be conducted using simulated results.

6.4 Simulated Results

For this part of the project, since the periods under study are post-1992, economic scenarios have to be simulated. Consequently, all results contained in this section really are estimates and are thus subject to error. Note that the approximate standard error associated with the average is one tenth ($1/\sqrt{100}$) of the estimated standard deviation. As for the standard deviation itself, its approximate standard error cannot be calculated with the values provided: an estimate of the fourth moment would be required, and of the third moment as well if a normal distribution is not willingly assumed.

Note that, in this case, the variance will be the sum of that between simulations and that within simulations, for both the cost ratio and the funding level.

Since all sets to be considered are the same as those looked at for Section 6.3, the conclusions should also be similar to those found in that section. If not, then it should be noted that the historical results constitute a particular scenario which may yield locally different results.

The same valuation basis is used for all simulated futures, regardless of the actual period studied. The valuation interest rate is 3.75% when indexation is provided, 7.50% when it is not. The rate of salary increase is only required for final-average plans and is assumed to be 7.85%. These are based on the long-term means, obtained analytically for inflation figures (inf and w) and through 1000 simulations for the yield components.

6.4.1 Period 1992-2050

To begin with, all three simplified sets are compared under the basic set of assumptions. These sets refer to the non-indexed flat-benefit, fully-indexed flat-benefit and fully-indexed career-average pension plans. Again, the sets of assumptions may be found in Appendix 6.

Set	average r	std. dev. r	average *f*	std. dev. *f*
1	1.575743	1.874516	96.16%	12.49%
2	3.210511	1.982666	85.26%	13.22%
3	1.165111	1.235650	98.07%	14.41%

Table 12. Basic results for all sets, n=5, $l=u=100\%$, g=0%

Simulated results resemble those previously obtained using past economic data, in order of magnitude. In addition, the average cost ratios imply that the valuation nominal and real interest rates (7.50% and 3.75%, respectively) are a little too high. As for other conclusions that could be reached, they match those already given earlier in the text.

n	average r	std. dev. r	average *f*	std. dev. *f*
1	1.517836	5.353685	100.00%	0.00%
2	1.511597	3.147597	99.05%	5.83%
3	1.578565	2.459109	97.93%	8.78%
4	1.506855	2.151362	97.38%	11.12%
5	1.575743	1.874516	96.16%	12.49%
6	1.580102	1.785189	95.32%	14.39%
7	1.692769	1.689753	93.54%	15.75%
8	1.507903	1.955805	94.65%	20.59%
9	1.630939	1.614078	92.65%	18.79%
10	1.599153	1.781321	92.42%	22.52%
11	1.673765	1.438516	90.78%	19.68%
12	1.659727	1.500533	90.37%	21.90%
13	1.603859	2.586726	90.66%	40.01%
14	1.559598	1.595661	90.99%	25.70%
15	1.708340	1.449837	87.98%	24.60%
16	1.690329	1.472168	87.90%	25.80%
17	1.676182	1.313146	87.75%	23.79%
18	1.646787	1.736493	87.88%	32.55%
19	1.498973	1.610795	90.31%	31.27%
20	1.548053	1.562096	89.10%	31.06%

Table 13. First set, varying n, $l=u=100\%$, g=0%

Then, the effect of varying the length of the 'spreading' period can be revisited. Again, increasing n should decrease the variability of the cost ratio while increasing that of the funding level. As for their average values, that of the funding level should decrease as a result of increasing n since the fund tends to be underfunded while that of the cost ratio should increase as interest accumulates on the unfunded liability.

Pertinent results are found in Tables 13, 14 and 15. On the whole, standard deviations behave as predicted. However, if graphed as a function of n, they look rather bumpy. Again, these are simulated results: all are subject to estimation error and deviations from their true values are bound to take place.

n	average r	std. dev. r	average f	std. dev. f
1	3.253076	5.555492	100.00%	0.00%
2	3.097743	3.360929	96.12%	6.22%
3	3.209744	2.494431	92.10%	8.91%
4	3.164173	2.214352	88.80%	11.46%
5	3.210511	1.982666	85.26%	13.22%
6	3.230188	1.812458	82.04%	14.59%
7	3.252872	1.560401	78.95%	14.58%
8	3.126839	1.601166	77.58%	16.88%
9	3.227028	1.440645	74.04%	16.79%
10	3.293198	1.421470	70.89%	18.05%
11	3.210638	1.407216	69.80%	19.22%
12	3.154166	1.385736	68.64%	20.18%
13	3.190921	1.373455	66.27%	21.15%
14	3.242811	1.257091	63.73%	20.33%
15	3.265178	1.393115	61.73%	23.54%
16	3.335788	1.166421	58.96%	20.49%
17	3.309617	1.244014	57.98%	22.63%
18	3.242887	1.294778	57.89%	24.31%
19	3.290292	1.269909	55.76%	24.53%
20	3.196593	1.255119	56.46%	24.88%

Table 14. Second set, varying n, $l=u=100\%$, g=0%

Also, the average funding level decreases as n increases, as expected because of the too optimistic valuation basis. Again, it does so with some erratic behavior around an imaginary smoothly declining slope.

However, the average cost ratio defies the initial hypothesis: it actually seems to revolve around a single value. Of course, its standard error may, in itself, be the cause. But it would be rather surprising that the errors would be so that the values would be clustered. That there be outliers as a result appears to be much more plausible.

Then, the conclusions reached in Section 6.3.1 were potentially wrong. Indeed, it might be that the expected cost ratio should always be the same, no matter what the value of n is. This makes sense if the impact of interest on the accumulated deficit is limited or if, given enough time, the deficit or surplus disappears by itself, thus reducing the need for adjustments to contributions.

n	average r	std. dev. r	average *f*	std. dev. *f*
1	1.231723	3.293262	100.00%	0.00%
2	1.215364	2.040863	99.30%	6.61%
3	1.247079	1.625335	98.46%	10.15%
4	1.231870	1.373368	97.90%	12.42%
5	1.165111	1.235650	98.07%	14.41%
6	1.251790	1.102466	96.45%	15.55%
7	1.252026	1.056430	95.89%	17.22%
8	1.093391	1.176929	98.28%	21.68%
9	1.228843	0.954803	95.34%	19.45%
10	1.142228	1.003199	96.85%	22.20%
11	1.177477	1.038286	95.75%	24.86%
12	1.130193	1.009876	96.68%	25.79%
13	1.067561	1.009809	98.17%	27.33%
14	1.238342	0.844425	93.28%	23.79%
15	1.199411	0.851223	94.08%	25.27%
16	1.182079	0.944024	94.42%	28.94%
17	1.205219	0.803076	93.50%	25.45%
18	1.079834	1.080899	97.38%	35.45%
19	1.208182	0.819531	92.93%	27.83%
20	1.208624	0.787965	92.74%	27.41%

Table 15. Third set, varying n, $l=u=100\%$, g=0%

The impact of modifying the 'reaction' interval $[l,u]$ was investigated earlier. Tables 16 to 18 make comparisons with historical results possible.

l	*u*	average r	std. dev. r	average *f*	std. dev. *f*
0%	1000%	1.372945	7.792565	124.46%	237.03%
80%	100%	1.905381	1.571585	78.72%	16.92%
80%	110%	1.971327	1.516775	79.84%	19.13%
80%	120%	1.740100	1.595000	84.93%	23.10%
90%	100%	1.702398	1.659571	88.18%	14.48%
90%	110%	1.806316	1.591591	89.46%	17.02%
90%	120%	1.681838	1.643166	93.30%	20.37%
100%	100%	1.575743	1.874516	96.16%	12.49%
100%	110%	1.497095	1.824330	100.08%	15.79%
100%	120%	1.272721	1.949469	105.28%	20.13%

Table 16. First set, varying l and u, n=5, g=0%

Once again, those regarding the volatilities are reproduced, though with some minor discrepancies. Hence, increasing the upper boundary or decreasing the lower one while keeping the other constant increases the standard deviation of the cost ratio while it decreases that of the funding level.

l	*u*	average r	std. dev. r	average *f*	std. dev. *f*
0%	1000%	3.367592	1.963008	7.08%	48.92%
80%	100%	3.220718	1.732768	67.36%	15.91%
80%	110%	3.324051	1.582155	66.92%	15.85%
80%	120%	3.256711	1.659277	68.28%	18.32%
90%	100%	3.236276	1.679013	76.01%	13.04%
90%	110%	3.167480	1.750758	77.48%	15.53%
90%	120%	3.065704	1.759851	78.99%	17.20%
100%	100%	3.210511	1.982666	85.26%	13.22%
100%	110%	3.085022	1.819673	87.23%	14.26%
100%	120%	3.144650	1.837827	87.44%	15.82%

Table 17. Second set, varying *l* and *u*, n=5, g=0%

Likewise, the average funding level increases under the same circumstances. This confirms previous findings using historical data. By looking at Table 17, one may appreciate the fact that increasing only *u* has relatively little effect as the fund is, most of the time, chronically deficient to meet current liabilities. For the same reason, modifying only *l* seems to have a greater impact in that case than for the two other sets.

Lastly, the average cost ratio only roughly exhibits the predicted behavior. Even in some isolated instances, the reverse seems to prevail. However, graphing the averages along with their standard error would likely reveal that the stated hypothesis cannot be rejected with any reasonable level of confidence.

l	*u*	average r	std. dev. r	average *f*	std. dev. *f*
0%	1000%	1.086960	1.715964	105.25%	145.25%
80%	100%	1.153283	1.052782	85.43%	19.48%
80%	110%	1.171657	1.070196	88.42%	23.12%
80%	120%	1.126875	1.010433	92.82%	25.94%
90%	100%	1.234368	1.043863	90.82%	15.84%
90%	110%	1.203745	1.026684	94.78%	19.28%
90%	120%	1.230349	0.999295	97.25%	22.10%
100%	100%	1.165111	1.235650	98.07%	14.41%
100%	110%	1.197063	1.231731	101.41%	18.07%
100%	120%	1.134106	1.228350	105.68%	21.35%

Table 18. Third set, varying *l* and *u*, n=5, g=0%

Now, another variable will be studied for its potential impact on the statistics of interest: the growth rate. At this point, it is important to specify in what way the number of participants shall increase or decrease. All changes take place through the number of entrants; that is, there is no major hiring or lay-off occurring. As a result, the repartition of participants among age groups remains unaltered.

For example, a growth rate of 3.00% causes the number of entrants to increase at that rate each year. If $n_\alpha^{1978}=100$, then $n_\alpha^{1979}=103$. The overall number of participants also increases by that same percentage through aging and passage of time, since the number of people alive at any age depends on the size of the cohort from which they come.

Certainly, changing the growth rate affects the size of the contributions, payments and liabilities. For a growing population, the increase in these values is more than it would be if caused by benefit or salary indexation alone. Thus, all other things being equal, the same deficit and resulting contribution adjustment will lead to a smaller cost ratio for larger values of g.

Because of that damping effect brought about by a bigger g, a reduction in the variance of the cost ratio is also expected. Increasing g effectively reduces the range of values the cost ratio can assume.

Knowing that the valuation bases used for all three sets of assumptions are too high, an average funding level below 100% is expected. Because the liability pertaining to new entrants is funded in the middle of the year, the deficit or surplus that may arise from a yield departing from its expected value will be relatively smaller than that generated from the funds accumulated for the other participants since the latter were there for the whole year (except for their own contributions).

Since the liability originating in the first year of service will on average be better funded, it would seem logical to believe that larger growth rates should lead to greater average funding levels. Note that the opposite would be expected in the case of a generally over-funded pension plan. For the same reason, increasing g should decrease the resulting variability of the funding level.

g	average r	std. dev. r	average f	std. dev. f
-5.00%	2.873743	4.267373	94.37%	12.82%
-2.50%	1.698758	3.057408	96.84%	13.84%
-1.00%	1.738402	2.278311	95.78%	13.03%
0.00%	1.575743	1.874516	96.16%	12.49%
1.00%	1.410926	1.657743	96.82%	12.83%
2.50%	1.367188	1.296874	96.46%	12.51%
5.00%	1.276691	0.834374	96.22%	11.41%

Table 19. First set, varying g, n=5, $l=u=100\%$

Simulated results permitting the verification of the above hypotheses are contained in Table 19 to 21. Indeed, the average and standard deviation of the cost ratio drop as a result of increasing the growth rate.

g	average r	std. dev. r	average f	std. dev. f
-5.00%	7.415097	4.709251	80.71%	14.16%
-2.50%	4.933964	2.848847	82.19%	12.90%
-1.00%	3.637482	2.266703	84.91%	12.97%
0.00%	3.210511	1.982666	85.26%	13.22%
1.00%	2.871326	1.641558	85.51%	12.71%
2.50%	2.302232	1.349170	87.44%	13.02%
5.00%	1.876887	0.904539	88.00%	12.37%

Table 20. Second set, varying g, n=5, $l=u=100\%$

However, for all practical purposes, the statistics associated with the funding level neither differ by much nor show a well-defined increase or decrease. Nonetheless, the average roughly increases with g while the standard deviation exhibits the opposite reaction. The results obtained for the first set are the least conclusive, while the other two are more convincing and seem to support the hypotheses.

g	average r	std. dev. r	average f	std. dev. f
-5.00%	1.554119	2.840186	96.92%	15.78%
-2.50%	1.350083	1.770543	97.14%	14.46%
-1.00%	1.272461	1.487529	97.24%	15.09%
0.00%	1.165111	1.235650	98.07%	14.41%
1.00%	1.171187	1.063579	97.72%	14.19%
2.50%	1.153943	0.841225	97.50%	13.64%
5.00%	1.047940	0.647248	98.95%	14.17%

Table 21. Third set, varying g, n=5, $l=u=100\%$

This period being longer than the next one to be studied, the results were less likely to be influenced by the current economic figures. Time series with properly chosen parameters have the property of being stationary: the cumulative average eventually tends to the long-term mean. Hence, the erratic behavior seen in the previous tables will probably affect the following ones to an even greater extent.

6.4.2 Period 1995-2005

At present, all sets of assumptions other than the simplified ones remain to be examined for their results under simulated economic scenarios. For reasons given before, the period of study is limited to ten years. The first year is set at 1995 so as to distance it a little from prevailing conditions.

If the conclusions reached in the previous subsection are still valid, indexing the benefits would increase the average and standard deviation of the cost ratio. It would also decrease the expected funded level. As the reader may remember, in that case, none of the valuation bases was conservative enough and little variation in the standard deviation of the funding level was apparent.

Type of benefit	Indexation prior to retirement	Indexation after retirement	average r	std. dev. r	average f	std. dev. f
Flat-benefit	No	No	1.141365	1.879626	99.06%	12.53%
	No	Yes	0.929921	2.253973	100.71%	12.87%
	Yes	No	1.617727	3.343042	97.90%	12.64%
	Yes	Yes	1.747435	4.461346	98.89%	15.41%
Career-average	No	No	1.031902	0.709229	99.57%	12.04%
	No	Yes	0.987483	0.777327	100.14%	12.44%
	Yes	No	1.083709	0.898098	99.12%	12.63%
	Yes	Yes	1.029484	1.062935	99.98%	13.84%
Final-average	No	No	1.036304	0.923367	99.45%	12.04%
	No	Yes	0.950359	1.022756	100.59%	12.62%

Table 22. Sets for all types of benefits

Table 22 repeats the findings related to the cost ratio. Again, it would seem that real yields induce more risk than nominal ones, although, as suggested earlier, the only culprit may be the valuation basis, given that there be a positive correlation between the average and the standard deviation.

As for the funding level, the values assumed by both the average and the standard deviation are remarkably close to one another. Because of the inevitable inaccuracy arising from simulating results, it seems impossible to express sensible comments in that case.

To complete this analysis, results will be given and commented upon for the four presumably more realistic sets of assumptions, using simulated data.

Once more, results will be compared to those obtained with the historical data. Table 23, unlike Table 11, does not allow one to draw conclusions related to the average cost ratio: it would be very hard to reject the hypothesis of equal means. However, both tables agree concerning its standard deviation: indexation leads to a larger one. Now, correlation with the average no longer seems to be at the root of the problem. Instead, it appears that protecting the benefits from erosion due to inflation brings more risk from the sponsor's point of view.

Type of population	Indexation after retirement	average r	std. dev. r	average *f*	std. dev. *f*
Stable	No	1.078026	0.815248	98.75%	12.03%
	Yes	0.977929	0.924098	100.25%	12.46%
Shifting	No	0.939588	1.040825	100.54%	12.06%
	Yes	1.070042	1.185223	99.20%	12.87%

Table 23. Realistic sets

Similarly, for the funding level, their averages are pretty much alike. However, the standard deviation increases as a result of indexing benefits. Hence, under these findings, indexation appears riskier on both accounts.

With respect to the type of population, note that the shifting one really has a net growth rate below 0% since the one subject to growth is the one with the greater entry age. As was found when studying the impact of varying g on the standard deviations, both increase when g decreases, for reasons already presented.

6.5 Summary of the Results

After so many tables and so much interpretation, the reader may welcome a summary. Conclusions supported by empirical evidence are reproduced in Table 24. Only those with respect to the standard deviation are included as they really are the important measures of risk to the sponsor. As a matter of fact, the valuation basis can, in any case, be modified so that the expected cost ratio and funded level be 1 and 100%, respectively.

Factor	Effect on...	
	std. dev. r	std. dev. f
'spreading' period (n)	Negative	Positive
call for special payment (l)	Positive	Negative
call for special refund (u)	Negative	Positive
growth rate (g)	Negative	Positive
indexation	Positive	Undetermined

Table 24. Effects of certain factors

Factors negatively related with the volatility of the cost ratio, among those studied, are the length of the 'spreading' period (n), the funding level at which a special refund is called for (u) and the growth rate (g). On the other hand, increasing the minimum level of funding (l) not requiring a special payment and adding indexation appeared to have a positive effect on this statistic.

As usual, there is a trade-off: in general, when one volatility increases, the other one decreases. As a result, all correlations are reversed for the standard deviation of the funding level. While it increases along with n, u and g, it decreases when l takes on a larger value. In the case of indexation, the results do not provide enough information to give a quantitative description of its effect.

7. Comments on Software Potential

Earlier in the text, the software used for this project was presented. The way it was built would readily allow for certain modifications that might be of interest.

First of all, while the investment policy does not vary in the face of a changing economic environment, it would be possible to implement decision criteria that lead to its modification under well-defined circumstances.

Second, though valuation bases only slowly respond to economic signals, they are not static. While they were throughout this study, the program could be amended so as to vary i_a, i_r and h as trends develop on the asset side.

These are the two changes which the software could most easily undergo. Accounting for the Canada/Quebec Pension Plan or different death and termination benefits would be more arduous. Similarly, transaction costs as well as other fees pertaining to pension plans should realistically be included in the analysis, though not without considerable effort.

Of course, taking full account of existing legislation would definitely require much more powerful software as the extent of government's control in this area has increased over 1970s-1990s.

8. Areas for Future Research

Hopefully, this paper may warrant renewed interest for research in the area of pension plans, especially considering the new challenges plan sponsors face. Among topics which could be of interest and have potential applications in the real world, the author identifies a few, in addition to those mentioned in the previous section.

For instance, multi-factor analysis would greatly improve the quality as well as the relevance of the study started here. Other factors could also be introduced and examined. The valuation basis might be an important one, not only for its undeniable influence on the averages but also for its potential impact on the standard deviations.

Maybe, under sufficiently simple assumptions, Wilkie's and Sharp's models may lend themselves to the derivation of analytical results. These would then allow for interpretation without the shadow of doubt.

Another aspect of real life which has been overlooked for this project is the way in which a company grows. A constant rate of growth is anything but likely. Instead, a model could be built which would dictate the evolution of the workforce. This could vary by industry and should have some relation to technology and national population, with respect to both its size and age pyramid. Note that this would require more than changing the number of entrants.

As usual, before actually adding a level of realism, it is necessary for the researcher to evaluate if the gain realized compensates, if such is the case, for the loss in tractability.

Conclusion

As may be gathered from the literature review, research concerning the variability of pension plan contributions and fund levels has been undertaken by only a few people in the recent past. However, the economic situation may renew the interest and need for it.

The object of this paper was to study the effect of different factors on the average and standard deviation of both the cost ratio and funding level. The cost ratio, the ratio of the net contributions to the normal cost, is arguably a variable of interest to the sponsor, who has to assume it.

The funding level, the percentage of assets to liabilities, is important from a solvency point of view: in the participants' best interest, it should not fall too far below 100%. From the sponsor's perspective, it should not attain too high levels at it represents money which he cannot use for expansion, research or other projects he may want to launch.

Hence, these two variables were selected for their apparent importance and their first two moments were calculated under many different sets of assumptions. That allowed the author to state some conclusions, which mostly confirmed anticipations.

In the author's mind, the advantage of this paper is that it uses a comprehensive model for price inflation, asset returns and wage inflation. Its main handicap lies in its incapacity to derive analytical results. However, as pointed out in the previous section, it might be possible to use advanced mathematical tools to resolve that issue.

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Appendix 1. Mortality Tables

The values contained in the table are mortality rates ($q_x^{(d)}$).

Age	GAM51		GAM71		GAM83	
	Male	Female	Male	Female	Male	Female
20	0.000616	0.000371	0.000503	0.000260	0.000377	0.000189
21	0.000640	0.000393	0.000522	0.000275	0.000392	0.000201
22	0.000666	0.000416	0.000544	0.000292	0.000408	0.000212
23	0.000693	0.000440	0.000566	0.000309	0.000424	0.000225
24	0.000724	0.000467	0.000591	0.000327	0.000444	0.000239
25	0.000758	0.000495	0.000619	0.000347	0.000464	0.000253
26	0.000796	0.000524	0.000650	0.000368	0.000488	0.000268
27	0.000838	0.000556	0.000684	0.000390	0.000513	0.000284
28	0.000885	0.000591	0.000722	0.000414	0.000542	0.000302
29	0.000935	0.000628	0.000763	0.000440	0.000572	0.000320
30	0.000991	0.000669	0.000809	0.000469	0.000607	0.000342
31	0.001054	0.000712	0.000860	0.000499	0.000645	0.000364
32	0.001122	0.000760	0.000916	0.000533	0.000687	0.000388
33	0.001198	0.000812	0.000978	0.000569	0.000734	0.000414
34	0.001281	0.000868	0.001046	0.000608	0.000785	0.000443
35	0.001374	0.000930	0.001122	0.000651	0.000860	0.000476
36	0.001475	0.000997	0.001204	0.000698	0.000907	0.000502
37	0.001587	0.001071	0.001295	0.000750	0.000966	0.000536
38	0.001711	0.001152	0.001397	0.000807	0.001039	0.000573
39	0.001849	0.001240	0.001509	0.000869	0.001128	0.000617
40	0.002000	0.001338	0.001633	0.000938	0.001238	0.000665
41	0.002192	0.001446	0.001789	0.001013	0.001370	0.000716
42	0.002450	0.001563	0.002000	0.001094	0.001527	0.000775
43	0.002769	0.001694	0.002260	0.001186	0.001715	0.000842
44	0.003147	0.001836	0.002569	0.001286	0.001932	0.000919
45	0.003580	0.001994	0.002922	0.001397	0.002183	0.001010
46	0.004065	0.002169	0.003318	0.001519	0.002471	0.001117
47	0.004599	0.002361	0.003754	0.001654	0.002790	0.001237
48	0.005180	0.002573	0.004228	0.001802	0.003138	0.001366
49	0.005807	0.002809	0.004740	0.001967	0.003513	0.001505
50	0.006475	0.003070	0.005285	0.002151	0.003909	0.001647
51	0.007187	0.003319	0.005867	0.002324	0.004324	0.001793
52	0.007938	0.003597	0.006480	0.002520	0.004755	0.001949
53	0.008731	0.003908	0.007127	0.002738	0.005200	0.002120
54	0.009563	0.004257	0.007806	0.002982	0.005660	0.002315
55	0.010436	0.004648	0.008519	0.003256	0.006131	0.002541
56	0.011346	0.005102	0.009262	0.003574	0.006618	0.002803
57	0.012298	0.005637	0.010039	0.003948	0.007139	0.003103
58	0.013302	0.006265	0.010889	0.004388	0.007719	0.003443
59	0.014379	0.006997	0.011924	0.004901	0.008384	0.003821
60	0.015555	0.007837	0.013119	0.005489	0.009158	0.004241
61	0.016866	0.008788	0.014440	0.006156	0.010064	0.004703
62	0.018353	0.009848	0.015863	0.006898	0.011133	0.005210
63	0.020068	0.011010	0.017413	0.007712	0.012391	0.005769
64	0.022067	0.012264	0.019185	0.008608	0.013868	0.006386
65	0.024418	0.013597	0.021260	0.009563	0.015592	0.007064
66	0.027193	0.014991	0.023643	0.010565	0.017579	0.007817
67	0.030112	0.016457	0.026316	0.011621	0.019804	0.008681
68	0.032986	0.018198	0.029188	0.012877	0.022229	0.009702
69	0.035943	0.020354	0.032435	0.014461	0.024817	0.010922
70	0.039303	0.023098	0.036106	0.016477	0.027530	0.012385

Age	GAM51		GAM71		GAM83	
	Male	Female	Male	Female	Male	Female
71	0.043183	0.026527	0.040008	0.019000	0.030354	0.014128
72	0.047476	0.030468	0.043827	0.021911	0.033370	0.016160
73	0.052084	0.034779	0.047489	0.025112	0.036680	0.018481
74	0.057077	0.039413	0.051221	0.028632	0.040388	0.021092
75	0.062427	0.044309	0.055293	0.032385	0.044597	0.023992
76	0.068347	0.049512	0.060068	0.036408	0.049388	0.027185
77	0.075132	0.055108	0.065924	0.040769	0.054758	0.030672
78	0.082687	0.061093	0.072595	0.045472	0.060678	0.034459
79	0.090946	0.067459	0.079692	0.050616	0.067125	0.038549
80	0.099679	0.074146	0.087431	0.056085	0.074070	0.042945
81	0.108706	0.081114	0.095445	0.061853	0.081484	0.047655
82	0.117979	0.088374	0.103691	0.067936	0.089320	0.052691
83	0.127437	0.095943	0.112303	0.074351	0.097525	0.058071
84	0.137073	0.103904	0.121116	0.081501	0.106047	0.063807
85	0.146852	0.112328	0.130102	0.089179	0.114836	0.069918
86	0.156836	0.121295	0.139315	0.097468	0.124170	0.076570
87	0.167120	0.130885	0.148714	0.106452	0.133870	0.083870
88	0.177787	0.141188	0.158486	0.116226	0.144073	0.091935
89	0.188919	0.152300	0.168709	0.126893	0.154859	0.101354
90	0.200594	0.164331	0.179452	0.138577	0.166307	0.111750
91	0.212555	0.177144	0.190489	0.151192	0.178214	0.123076
92	0.225161	0.191099	0.201681	0.165077	0.190460	0.135630
93	0.238524	0.206341	0.212986	0.180401	0.203007	0.149577
94	0.252765	0.223029	0.226535	0.197349	0.217904	0.165103
95	0.268025	0.241336	0.241164	0.216129	0.234086	0.182419
96	0.284455	0.261451	0.256204	0.236970	0.248436	0.201757
97	0.302223	0.283581	0.272480	0.258059	0.263954	0.222044
98	0.321515	0.307953	0.290163	0.280237	0.280803	0.243899
99	0.342526	0.334812	0.309125	0.304679	0.299154	0.268185
100	0.365462	0.364429	0.329825	0.331630	0.319185	0.295187
101	0.390538	0.397100	0.352455	0.361361	0.341086	0.325225
102	0.417979	0.433150	0.377220	0.394167	0.365052	0.358897
103	0.450096	0.472930	0.406205	0.430366	0.393102	0.395843
104	0.489201	0.518156	0.441497	0.471522	0.427255	0.438360
105	0.537605	0.570545	0.485182	0.519196	0.469531	0.487816
106	0.597619	0.631813	0.539343	0.574950	0.521945	0.545886
107	0.671554	0.703676	0.606069	0.640345	0.586518	0.614309
108	0.761722	0.787851	0.687444	0.716944	0.665268	0.694885
109	0.870434	0.886054	0.785555	0.806309	0.760215	0.789474
110	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Sources: All three mortality tables were originally published in the *Transactions of the Society of Actuaries*, volumes 4, 23 and 35, respectively for GAM51, GAM71 and GAM83.

Appendix 2. Termination Tables

The values contained in the table are termination rates ($q_x^{(w)}$).

Age	TTW1	TTW2	TTW3	TTW4	TTW5	TTW6
20	0.0700	0.1121	0.1783	0.1551	0.2123	0.2833
21	0.0700	0.1061	0.1709	0.1377	0.1845	0.2453
22	0.0700	0.1018	0.1632	0.1257	0.1747	0.2163
23	0.0700	0.0975	0.1554	0.1147	0.1720	0.1963
24	0.0700	0.0933	0.1478	0.1056	0.1693	0.1821
25	0.0700	0.0891	0.1403	0.0983	0.1665	0.1706
26	0.0700	0.0851	0.1329	0.0927	0.1635	0.1609
27	0.0700	0.0810	0.1256	0.0884	0.1603	0.1562
28	0.0700	0.0770	0.1183	0.0846	0.1569	0.1523
29	0.0700	0.0731	0.1111	0.0814	0.1533	0.1484
30	0.0692	0.0692	0.1042	0.0790	0.1494	0.1445
31	0.0654	0.0654	0.0977	0.0775	0.1453	0.1407
32	0.0618	0.0618	0.0914	0.0762	0.1409	0.1368
33	0.0588	0.0588	0.0856	0.0752	0.1363	0.1329
34	0.0565	0.0565	0.0805	0.0743	0.1317	0.1290
35	0.0546	0.0546	0.0764	0.0736	0.1271	0.1251
36	0.0531	0.0531	0.0737	0.0730	0.1227	0.1212
37	0.0519	0.0519	0.0725	0.0725	0.1183	0.1173
38	0.0508	0.0508	0.0720	0.0720	0.1139	0.1134
39	0.0498	0.0498	0.0715	0.0715	0.1095	0.1095
40	0.0488	0.0488	0.0709	0.0709	0.1052	0.1057
41	0.0478	0.0478	0.0702	0.0702	0.1010	0.1018
42	0.0468	0.0468	0.0692	0.0692	0.0968	0.0987
43	0.0459	0.0459	0.0682	0.0682	0.0926	0.0940
44	0.0449	0.0449	0.0670	0.0670	0.0886	0.0897
45	0.0439	0.0439	0.0657	0.0657	0.0847	0.0858
46	0.0428	0.0428	0.0643	0.0643	0.0801	0.0820
47	0.0416	0.0416	0.0626	0.0626	0.0738	0.0784
48	0.0403	0.0403	0.0591	0.0591	0.0663	0.0747
49	0.0387	0.0387	0.0524	0.0524	0.0587	0.0711
50	0.0365	0.0365	0.0441	0.0441	0.0510	0.0675
51	0.0328	0.0328	0.0359	0.0359	0.0435	0.0640
52	0.0269	0.0269	0.0283	0.0283	0.0362	0.0604
53	0.0198	0.0198	0.0209	0.0209	0.0288	0.0563
54	0.0120	0.0120	0.0131	0.0131	0.0210	0.0508
55	0.0036	0.0036	0.0047	0.0047	0.0124	0.0433
56	0.0000	0.0000	0.0000	0.0000	0.0034	0.0343
57	0.0000	0.0000	0.0000	0.0000	0.0000	0.0242
58	0.0000	0.0000	0.0000	0.0000	0.0000	0.0133
59	0.0000	0.0000	0.0000	0.0000	0.0000	0.0018
60	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
61	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
62	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
63	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
64	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
65	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
66	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
67	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
68	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
69	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
70	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Source: *Pension Tables for Actuaries.*

Appendix 3. Salary Scales

The following columns are salary scales (s_x) with $s_{65} \equiv 1.00000$.

Age	SA1	SB1	SC1	SD1	SA2	SB2	SO2	SO3	SO4	SO5
20	0.12505	0.17868	0.16722	0.25978	0.61875	0.48112	0.41020	0.26444	0.17120	0.11130
21	0.13032	0.19025	0.17910	0.27929	0.62635	0.49740	0.41840	0.27237	0.17805	0.11686
22	0.13650	0.20177	0.19110	0.29906	0.63355	0.51375	0.42677	0.28054	0.18517	0.12270
23	0.14363	0.21372	0.20330	0.31675	0.64012	0.53023	0.43530	0.28896	0.19257	0.12884
24	0.15158	0.22615	0.21585	0.33107	0.64642	0.54685	0.44401	0.29763	0.20028	0.13528
25	0.16042	0.23890	0.22870	0.35194	0.65235	0.56360	0.45289	0.30656	0.20829	0.14205
26	0.17037	0.25190	0.24177	0.37238	0.65777	0.58057	0.46195	0.31575	0.21662	0.14915
27	0.18162	0.26520	0.25512	0.39282	0.66302	0.59800	0.47119	0.32523	0.22529	0.15661
28	0.19420	0.27892	0.26895	0.41326	0.66827	0.61595	0.48061	0.33498	0.23430	0.16444
29	0.20782	0.29317	0.29332	0.43370	0.67440	0.63445	0.49022	0.34503	0.24367	0.17266
30	0.22217	0.30815	0.29840	0.45414	0.68227	0.65355	0.50003	0.35538	0.25342	0.18129
31	0.23717	0.32410	0.31445	0.47458	0.69100	0.67322	0.51003	0.36604	0.26355	0.19035
32	0.25305	0.34105	0.33152	0.49502	0.69975	0.69340	0.52023	0.37703	0.27409	0.19987
33	0.27002	0.35880	0.34947	0.51546	0.70867	0.71130	0.53063	0.38834	0.28506	0.20987
34	0.28785	0.37730	0.36830	0.53590	0.71782	0.72657	0.54125	0.39999	0.29646	0.22036
35	0.30607	0.39677	0.38845	0.55634	0.72705	0.74345	0.55207	0.41199	0.30832	0.23138
36	0.32452	0.41750	0.41010	0.57677	0.73625	0.76055	0.56311	0.42435	0.32065	0.24295
37	0.34327	0.43957	0.43292	0.59721	0.74562	0.77517	0.57437	0.43708	0.33348	0.25509
38	0.36235	0.46317	0.45730	0.61765	0.75525	0.78835	0.58586	0.45019	0.34682	0.26785
39	0.38157	0.48870	0.48387	0.63809	0.76492	0.80070	0.59758	0.46369	0.36069	0.28124
40	0.40100	0.51660	0.51292	0.65853	0.77465	0.81260	0.60953	0.47761	0.37512	0.29530
41	0.42107	0.54707	0.54452	0.67897	0.78455	0.82440	0.62172	0.49193	0.39012	0.31007
42	0.44232	0.57987	0.57822	0.69941	0.79462	0.83617	0.63416	0.50669	0.40573	0.32557
43	0.46417	0.61465	0.61370	0.71985	0.80482	0.84790	0.64684	0.52189	0.42196	0.34185
44	0.48547	0.65122	0.65095	0.74029	0.81510	0.85960	0.65978	0.53755	0.43883	0.35894
45	0.50630	0.68950	0.68977	0.76030	0.82542	0.87127	0.67297	0.55368	0.45639	0.37689
46	0.52712	0.72915	0.72985	0.77851	0.83585	0.88290	0.68643	0.57029	0.47464	0.39573
47	0.54817	0.76927	0.77027	0.79494	0.84642	0.89450	0.70016	0.58739	0.49363	0.41552
48	0.57017	0.80787	0.80882	0.80985	0.85710	0.90605	0.71416	0.60502	0.51337	0.43630
49	0.59565	0.84302	0.84392	0.82345	0.86782	0.91740	0.72845	0.62317	0.53391	0.45811
50	0.62600	0.87457	0.87587	0.83595	0.87860	0.92840	0.74301	0.64186	0.55526	0.48102
51	0.65812	0.90240	0.90402	0.84810	0.88945	0.93902	0.75788	0.66112	0.57748	0.50507
52	0.68912	0.92535	0.92697	0.86024	0.90037	0.94925	0.77303	0.68095	0.60057	0.53032
53	0.71862	0.94262	0.94412	0.87239	0.91128	0.95895	0.78849	0.70138	0.62460	0.55684
54	0.74667	0.95400	0.95530	0.88454	0.92207	0.96782	0.80426	0.72242	0.64958	0.58468
55	0.77592	0.96085	0.96195	0.89668	0.93277	0.97542	0.82035	0.74409	0.67556	0.61391
56	0.80650	0.96600	0.96692	0.90883	0.94332	0.98177	0.83676	0.76642	0.70259	0.64461
57	0.83347	0.97085	0.97165	0.92098	0.95352	0.98722	0.85349	0.78941	0.73069	0.67684
58	0.85722	0.97542	0.97612	0.93312	0.96315	0.99172	0.87056	0.81309	0.75992	0.71068
59	0.88035	0.97967	0.98027	0.94527	0.97202	0.99510	0.88797	0.83748	0.79031	0.74622
60	0.90242	0.98350	0.98400	0.95742	0.97995	0.99745	0.90573	0.86261	0.82193	0.78353
61	0.92355	0.98685	0.98725	0.97115	0.98685	0.99890	0.92385	0.88849	0.85480	0.82270
62	0.94402	0.98982	0.99012	0.98282	0.99252	0.99955	0.94232	0.91514	0.88900	0.86384
63	0.96370	0.99277	0.99297	0.99124	0.99665	0.99977	0.96117	0.94260	0.92456	0.90703
64	0.98230	0.99612	0.99622	0.99653	0.99905	0.99990	0.98039	0.97087	0.96154	0.95238
65	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
66	1.01695	1.00000	1.00000	1.00000	1.00080	1.00000	1.00000	1.00000	1.00000	1.00000
67	1.03355	1.00000	1.00000	1.00000	1.00130	1.00000	1.00000	1.00000	1.00000	1.00000
68	1.04983	1.00000	1.00000	1.00000	1.00170	1.00000	1.00000	1.00000	1.00000	1.00000
69	1.06817	1.00000	1.00000	1.00000	1.00210	1.00000	1.00000	1.00000	1.00000	1.00000
70	1.07100	1.00000	1.00000	1.00000	1.00230	1.00000	1.00000	1.00000	1.00000	1.00000

Source: *Pension Tables for Actuaries.*

Appendix 4. Economic Data from 1924 to 1992

Year	CPI	Common stock	Bonds	Bills	Mortgage	Wage index	Pension median
1924	-2.13	11.25	7.84			0.11	9.10
1925	2.90	28.74	5.17			-0.22	13.89
1926	-1.41	24.42	5.39			1.41	12.43
1927	-1.43	44.92	10.18			1.72	23.03
1928	0.72	32.92	0.56			1.48	12.53
1929	2.88	-11.60	2.34			1.15	-2.82
1930	-6.29	-30.90	9.26			-1.35	-5.60
1931	-9.70	-32.96	-4.97			-5.26	-15.33
1932	-8.26	-12.92	12.37			-6.12	3.01
1933	-1.80	51.63	7.37			-2.08	23.75
1934	0.92	20.26	19.66	0.64		1.90	19.88
1935	2.73	30.63	0.83	1.17		2.35	11.86
1936	0.88	25.35	11.12	0.90		4.64	16.39
1937	4.39	-15.83	-0.58	0.71		5.21	-6.22
1938	-2.52	9.13	5.63	0.62		1.68	6.93
1939	2.59	0.19	-2.98	0.70		2.13	-1.81
1940	5.04	-19.13	8.69	0.73		6.63	-1.60
1941	6.40	1.93	3.80	0.59		7.12	3.11
1942	3.01	13.99	3.08	0.54		7.49	7.12
1943	1.46	19.67	3.88	0.49		5.49	9.72
1944	-1.44	13.47	3.16	0.39		2.01	6.97
1945	1.46	36.05	5.18	0.37		0.98	16.60
1946	5.76	-1.50	6.02	0.39		6.28	2.96
1947	14.97	0.34	3.17	0.41		11.06	1.98
1948	8.88	12.13	-2.38	0.41		8.95	3.13
1949	1.09	22.61	4.85	0.48		6.08	11.20
1950	5.91	48.43	-0.12	0.54		7.93	17.38
1951	10.66	24.04	-3.13	0.77		9.86	7.12
1952	-1.38	-0.42	1.99	1.05	5.18	7.22	1.48
1953	0.00	2.15	3.64	1.65	2.08	4.17	2.66
1954	0.00	39.05	9.99	1.53	7.48	3.01	19.52
1955	0.47	27.80	-0.34	1.45	6.73	4.47	11.31
1956	3.24	13.22	-3.63	2.90	-2.42	5.48	3.44
1957	1.79	-20.58	6.40	3.86	3.23	4.51	-4.31
1958	2.64	31.25	-5.98	2.16	8.86	3.97	10.84
1959	1.29	4.59	-4.67	4.77	1.75	3.77	0.66
1960	1.27	1.78	7.10	3.53	10.32	3.24	9.50
1961	0.42	32.75	9.78	2.89	7.12	3.11	13.30
1962	1.67	-7.09	3.05	4.04	7.12	1.53	2.00
1963	1.64	15.60	4.60	3.66	7.12	4.08	8.10
1964	2.02	25.43	6.59	3.80	7.12	4.77	11.10
1965	3.16	6.68	0.96	4.03	2.59	6.48	3.50
1966	3.45	-7.07	1.55	5.14	1.58	5.47	-2.30
1967	4.07	18.09	-2.20	4.62	2.21	6.78	7.60
1968	3.91	22.45	-0.52	6.47	2.97	7.29	9.40
1969	4.79	-0.81	-2.31	7.43	-3.15	6.32	-3.20
1970	1.31	-3.57	21.98	6.58	11.87	8.90	1.30
1971	5.16	8.01	11.55	3.80	13.90	10.67	12.50
1972	4.91	27.38	1.11	3.59	8.92	7.74	18.40
1973	9.36	0.27	1.71	5.45	6.87	6.88	-2.10
1974	12.30	-25.93	-1.69	8.22	4.50	13.36	-12.70
1975	9.52	18.48	2.82	7.55	12.20	14.42	13.20
1976	5.87	11.02	19.02	9.43	14.21	11.20	12.40

Year	CPI	Common stock	Bonds	Bills	Mortgage	Wage index	Pension median
1977	9.45	10.71	5.97	7.87	14.62	8.04	8.70
1978	8.44	29.72	1.29	8.93	6.84	6.41	13.50
1979	9.69	44.77	-2.62	12.53	5.66	8.60	15.00
1980	11.20	30.13	2.06	13.73	8.10	11.45	18.00
1981	12.20	-10.25	-3.02	20.37	9.98	11.41	1.60
1982	9.23	5.54	42.98	15.25	29.15	9.54	22.60
1983	4.51	35.49	9.60	9.86	20.46	7.89	20.00
1984	3.77	-2.39	15.09	11.94	12.36	3.07	9.20
1985	4.38	25.07	25.26	9.77	16.72	3.91	23.60
1986	4.19	8.95	17.54	9.48	13.34	2.71	13.40
1987	4.12	5.88	0.45	8.45	10.26	4.56	3.80
1988	3.96	11.08	10.45	9.76	10.12	4.23	10.40
1989	5.17	21.37	16.29	12.91	13.06	5.03	15.90
1990	5.00	-14.80	3.34	13.98	10.63	4.67	-0.80
1991	3.78	12.02	24.43	9.58	21.56	4.25	17.60
1992	2.14	-1.43	13.07	6.50	11.25	3.11	7.00

Source: *Report on Canadian Economic Statistics, 1924-1992.*

Appendix 5. Estimated Parameters

The following table provides estimated values for the parameters included in the models built by Wilkie (1993) and Sharp (1993b).

Price inflation	QMU	0.034
	QA	0.640
	QSD	0.032
Dividend yield	YW	1.170
	YA	0.700
	YMU	0.038
	YSD	0.190
Dividends	DW	0.190
	DD	0.260
	DMU	0.001
	DY	-0.110
	DB	0.580
	DSD	0.070
Bond yield	CW	1.000
	CD	0.040
	CA	0.950
	CMU	0.037
	CY	0.100
	CSD	0.185
Bill yield	BMU	-0.260
	BA	0.380
	BC	0.730
	BSD	0.210
Wage inflation	WW	0.408
	WMU	0.035
	WA	0.703
	WSD	0.017

Appendix 6. Sets of Assumptions

1° Simplified sets

		First set	Second set	Third set
Decrements	Mortality table	GAM71	GAM71	GAM71
	% male	60.00%	60.00%	60.00%
	Termination table	TTW2	TTW2	TTW2
Wages	Salary scale	SO3	SO3	SO3
	Index	CPI	CPI	CPI
Plan benefits	Vesting period (v)	2	2	2
	Type of benefit	*Flat-benefit*	*Flat-benefit*	*Career-average*
	Flat amount	250.00	250.00	n.a.
	% salary (p)	n.a.	n.a.	2.00%
	Number of years in average (m)	n.a.	n.a.	n.a.
	Indexation — active	No	Yes	Yes
	Indexation — retired	No	Yes	Yes
	Age at retirement (ρ)	65	65	65
Plan valuation	Interest rate — active (i_a)	cf. Table 2	cf. Table 2	cf. Table 2
	Interest rate — retired (i_r)	cf. Table 2	cf. Table 2	cf. Table 2
	Rate of salary increase (h)	n.a.	n.a.	n.a.
Payment/refund	Call for special payment (l)	100%	100%	100%
	Call for special refund (u)	100%	100%	100%
	'Spreading' period (n)	5	5	5
	Interest rate	cf. Table 2	cf. Table 2	cf. Table 2
Simulations	Number of simulations	1 or 100	1 or 100	1 or 100
	First year of study	1924 or 1992	1924 or 1992	1924 or 1992
	Last year of study	1992 or 2050	1992 or 2050	1992 or 2050
	Reference year (y)	1924	1924	1924
	Initial funding level (f)	100%	100%	100%
	First age at entry (α_1)	25	25	25
	Last age at entry (α_q)	n.a.	n.a.	n.a.
	Common difference ($\Delta\alpha$)	n.a.	n.a.	n.a.
	Number of entrants at age α_1 (n_m^y)	100	100	100
	Number of entrants at age α_q (n_m^y)	n.a.	n.a.	n.a.
	Starting salary at entry age α_1 (Sal_m^y)	2,000	2,000	2,000
	Starting salary at entry age α_q (Sal_m^y)	n.a.	n.a.	n.a.
	Rate of growth of first cohort (g_1)	0.00%	0.00%	0.00%
	Rate of growth of q^{th} cohort (g_q)	n.a.	n.a.	n.a.

2° Sets for all types of benefits

		Flat-benefit	Career-average	Final-average
Decrements	Mortality table	GAM71	GAM71	GAM71
	% male	60.00%	60.00%	60.00%
	Termination table	TTW2	TTW2	TTW2
Wages	Salary scale	SO3	SO3	SO3
	Index	Wage	Wage	Wage
Plan benefits	Vesting period (v)	2	2	2
	Type of benefit	Flat-benefit	Career-average	Final-average
	Flat amount	250.00	n.a.	n.a.
	% salary (p)	n.a.	2.00%	2.00%
	Number of years in average (m)	n.a.	n.a.	3
	Indexation — active	Yes or No	Yes or No	No
	Indexation — retired	Yes or No	Yes or No	Yes or No
	Age at retirement (ρ)	65	65	65
Plan valuation	Interest rate — active (i_a)	cf. Table 2	cf. Table 2	cf. Table 2
	Interest rate — retired (i_r)	cf. Table 2	cf. Table 2	cf. Table 2
	Rate of salary increase (h)	n.a.	n.a.	cf. Table 2
Payment/refund	Call for special payment (l)	100%	100%	100%
	Call for special refund (u)	100%	100%	100%
	'Spreading' period (n)	5	5	5
	Interest rate	cf. Table 2	cf. Table 2	cf. Table 2
Simulations	Number of simulations	1 or 100	1 or 100	1 or 100
	First year of study	1982 or 1995	1982 or 1995	1982 or 1995
	Last year of study	1992 or 2005	1992 or 2005	1992 or 2005
	Reference year (y)	1924	1924	1924
	Initial funding level (f)	100%	100%	100%
	First age at entry (α_1)	25	25	25
	Last age at entry (α_q)	n.a.	n.a.	n.a.
	Common difference ($\Delta\alpha$)	n.a.	n.a.	n.a.
	Number of entrants at age α_1 (n_m^y)	100	100	100
	Number of entrants at age α_q (n_m^y)	n.a.	n.a.	n.a.
	Starting salary at entry age α_1 (Sal_m^y)	2,000	2,000	2,000
	Starting salary at entry age α_q (Sal_m^y)	n.a.	n.a.	n.a.
	Rate of growth of first cohort (g_1)	0.00%	0.00%	0.00%
	Rate of growth of q^{th} cohort (g_q)	n.a.	n.a.	n.a.

3° More realistic sets

		Stable population	Shifting population
Decrements	Mortality table	GAM71	GAM71
	% male	60.00%	60.00%
	Termination table	TTW2	TTW2
Wages	Salary scale	SO3	SO3
	Index	Wage	Wage
Plan benefits	Vesting period (v)	2	2
	Type of benefit	Final-average	Final-average
	Flat amount	n.a.	n.a.
	% salary (p)	2.00%	2.00%
	Number of years in average (m)	3	3
	Indexation — active	No	No
	Indexation — retired	Yes or No	Yes or No
	Age at retirement (ρ)	65	65
Plan valuation	Interest rate — active (i_a)	cf. Table 2	cf. Table 2
	Interest rate — retired (i_r)	cf. Table 2	cf. Table 2
	Rate of salary increase (h)	cf. Table 2	cf. Table 2
Payment/refund	Call for special payment (l)	100%	100%
	Call for special refund (u)	100%	100%
	'Spreading' period (n)	5	5
	Interest rate	cf. Table 2	cf. Table 2
Simulations	Number of simulations	1 or 100	1 or 100
	First year of study	1982 or 1995	1982 or 1995
	Last year of study	1992 or 2005	1992 or 2005
	Reference year (y)	1982 or 1995	1982 or 1995
	Initial funding level (f)	100%	100%
	First age at entry (α_1)	25	25
	Last age at entry (α_q)	45	45
	Common difference ($\Delta\alpha$)	10	10
	Number of entrants at age α_1 ($n_{\alpha_1}^{y}$)	33	38
	Number of entrants at age α_q ($n_{\alpha_q}^{y}$)	33	28
	Starting salary at entry age α_1 ($Sal_{\alpha_1}^{y}$)	2,000 in 1924	2,000 in 1924
	Starting salary at entry age α_q ($Sal_{\alpha_q}^{y}$)	3,612 in 1924	3,612 in 1924
	Rate of growth of first cohort (g_1)	0.00%	-3.00%
	Rate of growth of q^{th} cohort (g_q)	0.00%	3.00%