Risk: Applying a New Portfolio Risk/Return Measurement Methodology Based on Recent Advances in Quantifying Stable Paretian Fat Tailed Distributions and Investor Loss Aversion Preferences

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Abstract

Based on recent work by Kevin Dowd on investor loss aversion preferences and work by Benoit Mandelbrot on Stable Paretian distributions with Huston McCulloch's parameter estimation procedures, this paper recommends the practical application of new portfolio risk/return measurements to achieved and back tested stock portfolio performance. This new risk measurement process addresses the issue of infinite variances empirically observed in most stock return distributions.

1. Introduction

To statistically justify portfolio diversification, Harry Markowitz wrote the classic book[2] on portfolio selection. Markowitz employed a mean variance framework, but recognized the variance risk measure does not fully recognize investor wishes to avoid losses. Benoit Mandelbrot's[3] empirical analysis strongly suggested actual stock returns follow fat tailed Stable Paretian distributions with infinite variances. J. Huston McCulloch[4] recommended a quantile method for estimating the four parameters of the Stable distribution with significance statistics to measure how distant from Gaussian Normal they lie. Kevin Dowd suggested a utility function to reflect observed investor loss aversion over the *entire* distribution of anticipated returns.[5] To the author's knowledge, these significant empirical realities have not been carried forward into common practice.

If, as Mandelbrot suggests, stocks follow Stable Paretian distributions of returns with infinite variances, then traditional risk measures of variance, standard deviation, CAPM Beta and Sharpe ratios do not exist. Index tracking error statistics also do not exist. Consequently, the profession needs to find new, statistically valid, replacement risk measures for portfolio diversification applications.

As a proposed replacement risk measurement process, this paper applies McCulloch's parameter estimation procedure to achieved back tested portfolio performance results to estimate the four parameters of the Stable Paretian distribution. The results derive from both the universe of returns and a subset of undervalued firms. Applying the Kevin Dowd loss aversion utility function enables the calculation and graphical display of risk adjusted mean portfolio returns for both the universe and the subset of undervalued stocks. The display determines over what regions of investor loss aversion risk preferences each portfolio dominates. This proposed risk adjusted mean portfolio return replaces

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[2] Harry M. Markowitz, *Portfolio Selection*: *Efficient Diversification of Investments*, Cowles Foundation Monograph, 1959.

[3] Benoit Mandelbrot, "The Variation of Certain Speculative Prices," in Paul Cootner, *The Random Character of Stock Prices*, MIT Press, 1964, pp. 307-332.

Benoit Mandelbrot and Richard L. Hudson, *The (Mis)Behavior of Markets: A Fractal View of Risk, Ruin, and Reward*, Basic Books, 2004.

[4] J. Huston McCulloch, "Simple Consistent Estimators of Stable Distribution Parameters," *Commun. Statist.—Simula.,* 15(4), 1986, pp. 1109-1136. Programmed with the help of Paul Kettler and Terry Heiland.

[5] Kevin Dowd, *Measuring Market Risk*, 2nd ed., Wiley, 2005.

traditional risk measures which rely on an indeterminate variance of the distributions.

2. Empirical Work on Intrinsic Valuation and Excess Returns

On March 23, 2006, the author presented empirical work to the Midwest Finance Association on "Advanced DCF Valuation Measurement Methodology: Predictive Capability, Accuracy and Robustness." The slides and link below come from that presentation. The NACVA paper link below describes the intrinsic valuation methodology in more detail.

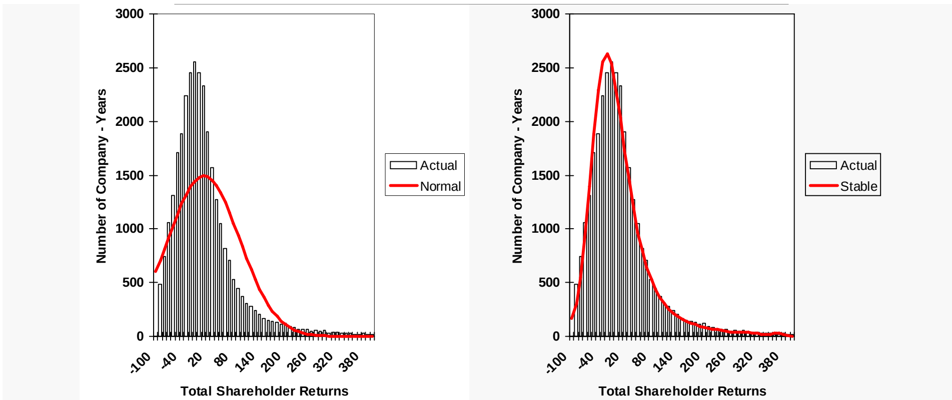
www.lcrt.com/Updates/MidwestFinance3-24-06.pps http://www.LCRT.com/Updates/NACVA PAPER LCRT & DCF.pdf

The Stable distribution contains several important properties. The Gaussian Normal distribution (the "Bell Shaped Curve") is a special case of the Stable Paretian distribution where the alpha peakedness parameter is 2.00. The variance of distributions with alpha peakedness parameters less than 2.00 is infinite. Most all the value-performance data which we analyzed show fat tailed distributions with alpha peakedness significantly less than 2.00 with infinite variances. Therefore, risk measures relying on variance, covariance and standard deviation are indeterminate. This includes CAPM Beta. Consequently, portfolio managers should consider replacement measures of portfolio risk and diversification.

The slide below illustrates that a fat tailed Stable Paretian distribution displays a better visual fit to total shareholder return (TSR) data than does Gaussian Normal.[6] These empirical results suggest potential for the use of non-traditional measures of risk based on fat tailed Stable instead of Gaussian distributions.

For later reference on page 5, note the infrequent, but important large gains in the right tail of the distribution.

[6] Sources: 5,500 industrials 1994–2003, Total Shareholder Return (TSR) Relative to the S&P 500; Hemscott Data; LCRT Platform Calculations. Similar results appear for lognormal rates of return.

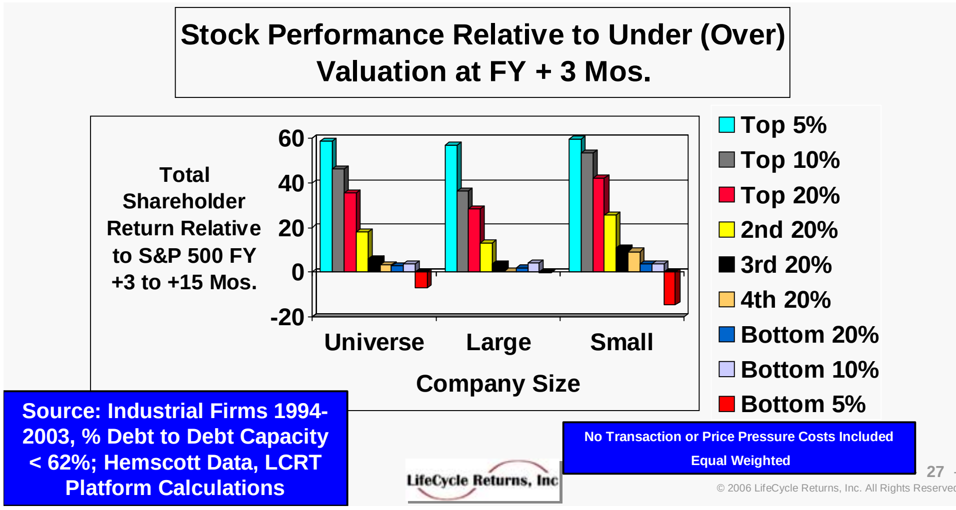


From McCulloch's parameter estimation, the 1.39 alpha peakedness statistical results in the table below confirm the TSR distribution is 41.4 standard errors away from Gaussian Normal (where alpha peakedness is 2.00). This result suggests limitations in the appropriate use of CAPM Beta as a risk measure, since CAPM Beta relies on the existence of the indeterminate covariance statistic.

Results	Value	Std. Error	t-Statistic	
alpha ("peakedness")	1.39	0.01	41.41	Difference from 2.00
beta ("skewness")	0.83	0.03	32.27	Difference from 0.00
c ("dispersion")	33.02	0.01	4,205.23	Difference from 0.00
delta ("location" or "average")	24.12	0.05	449.93	Difference from 0.00

Sources: 5,500 Industrial Firms 1994-2003, Total Shareholder Return (TSR) from FY+3 to +15 Months Relative to S&P 500, Hemscott Data, LCRT Platform Calculations, J. Huston McCulloch, "Simple Consistent Estimators of Stable Distribution Parameters," *Commun. Statist. – Simula*., 15(4), 1986, pp. 1109-1136.

Further research presented at the Midwest Finance Association shows how undervalued stocks perform, depending on the degree of under- and overvaluation, as illustrated in the slide below.



The DCF model on which these back tests rely places all the "risk" in the certainty equivalent cash flows.[7] Consequently, the DCF model employs a single real discount rate each year for all the companies in the super sector of about 5,500 firms and 17,000 company-years. A single discount rate obviously possesses a zero correlation with company CAPM Betas. Consequently, these zero correlation results with traditional risk measures caused the author to rethink proper risk measurement for portfolio diversification applications.

3. Loss Aversion, Gains and Risk Measures

This paper now applies the Kevin Dowd investor loss aversion utility function to these back test results to calculate risk adjusted mean returns.

Most people are loss averse. They hate losses. It takes more gains to offset losses. A 50 percent loss requires a 100 percent gain to break even. Markowitz observed that the problem with variance as a statistic is that it treats both sides of the distribution as equally undesirable,[8] whereas most investors prefer to avoid the left tail of the distribution of losses more strongly than achieving the right tail of gains.

People have suggested various ways to measure risk to address investor wishes to avoid losses (loss aversion). Markowitz suggested standard deviation,

[7] Placing all the "risk" in the certainty equivalent constant dollar net free cash flows requires a series of proprietary option pricing shaped functions on fading cash economic return, constant dollar gross investment size, and percent debt-to-debt capacity leverage. The empirically selected parameters driving these valuations produce results where 50 percent of the firms are undervalued and 50 percent are overvalued, so the model is unbiased across the key drivers.

[8] Harry Markowitz, *Portfolio Selection*, p. 194.

semi-variance, expected value of loss, expected absolute deviation, probability of loss and maximum loss.[9] Others have suggested value at risk (VaR), expected tail losses and other measures. Guided by the economic literature, Kevin Dowd suggests the following utility function to reflect investor loss aversion for each cumulative probability *p* in the *entire* distribution,[10]

$$\phi(p) = \frac{e^{-(1-p)/\gamma}}{\gamma\left(1-e^{-1/\gamma}\right)}$$

The parameter gamma γ reflects the investor's degree of risk aversion, a smaller γ indicating greater risk aversion. Dowd suggests that this utility risk measure possesses several very useful properties:

1. It is non-negative: the function is always greater than or equal to zero.
2. Normalization: the probability-weighted sum of the function weights must be 1.00.
3. Increasing weight: higher losses must have greater weights than lower losses (technically subadditivity).

Dowd's utility function spans the *entire* rate of return spectrum instead of just the loss portion covered by VaR, expected probability of loss and expected tail loss. As illustrated previously on page 3, since a few large gains typically add significantly to the portfolio's return performance, covering the *entire* distribution becomes most important.

On the one hand, the author discovered that equal weighting of the returns in the portfolio of stocks corresponds to a large gamma in the range of 100 to 10,000. These large gammas set the weights to a risk neutral value of 1.000. On the other hand, as gamma approaches zero from values below 0.3, the weighted average of returns of empirical distributions approach zero—a less than useful result. Consequently, the analyses displayed in this paper employ a practical range of gamma of [0.3, 100].[11]

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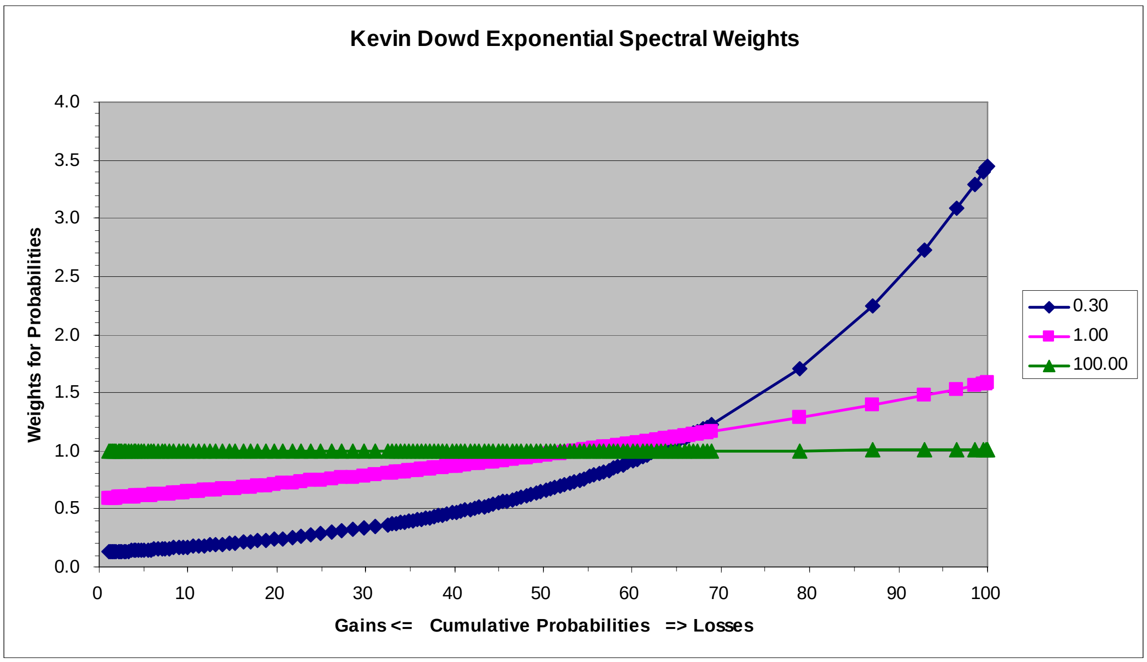
[9] Harry Markowitz, *Portfolio Selection*, p. 287.

[10] Kevin Dowd, *Measuring Market Risk,* p. 40. Also, "Spectral Risk Measures" in *Financial Engineering News* at www.fenews.com/fen42/risk-reward/risk-reward.htm.

[11] The actual points used of [0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 2, 4, 6, 8, 10, 50, 100] cover the full range with good resolution. A natural log ln transform compresses the utility scale for better visualization, making the range [-1.20, -0.92, -0.69, -0.51, -0.36, -0.22, -0.11, 0.00 0.69 1.39 1.79 2.08 2.30 3.91 4.61]. Taking 10 – ln(gamma risk) or [11.20, 10.92, 10.69, 10.51, 10.36, 10.22, 10.11, 10.00 9.31, 8.61, 6.21, 7.92, 7.70, 6.09, 5.39] makes the x-axis reflect increasing loss aversion from a risk neutral position.

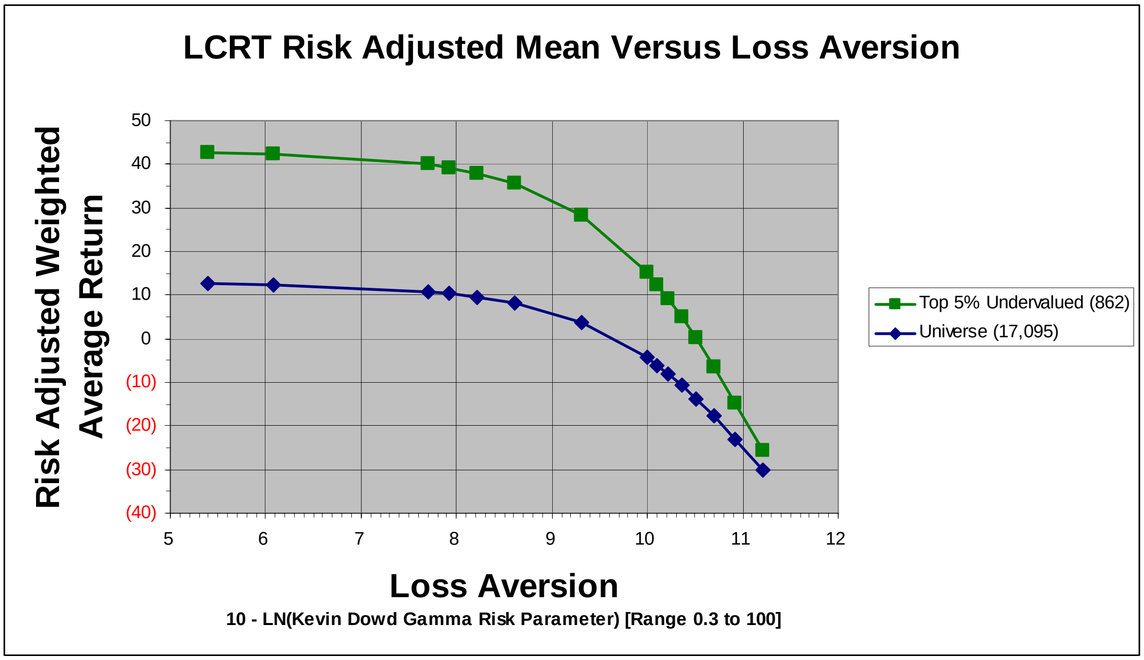
The chart below displays the gamma loss aversion parameters versus a cumulative probability distribution. On the one extreme, a gamma risk of 100 treats all returns as equal with a weight of 1.000, i.e., no loss aversion. On the other extreme, a gamma of 0.30 weights the losses to the right of the distribution substantially more than the gains to the left. A gamma risk of 1.00 falls in the middle of the two extremes.



4. Risk Measurement Applied to Portfolios

The chart below employs the data for both the universe and the top 5 percent of undervalued firms from the back tested portfolio performance on page 4. The x-axis is the measure of loss aversion from Kevin Dowd's gamma risk parameter, charted on the previous page. With zero loss aversion, the calculations use a 100 gamma risk parameter and weigh all returns equally by the 1.00 weighting factor. The result is the mean return for the portfolio: 43 percent for the top 5 percent of undervalued stocks (862 company-years) and 13 percent for the universe (17,095 company-years). As the investor's loss aversion increases, the risk adjusted weighted average returns decline.

As we shall see, sometimes the two lines cross, meaning that the portfolio of choice depends on the degree of loss aversion. However, in this case, the lines never cross. No matter what the investor's degree of loss aversion, the investor should prefer the portfolio of top 5 percent of undervalued stocks over the universe. In statistical terms, the top 5 percent dominate the universe (stochastic dominance).



5. Risk Measurement Applied to the Number of Stocks in the Portfolio

This risk measurement methodology also aids in determining the number of stocks to provide sufficient diversification for the portfolio.

The number of stocks required for a diversified portfolio is a most controversial subject. Meir Statman wrote an excellent summary[12] of the traditional research based on a Gaussian distribution view of the world. Evans and Archer[13] suggested that 10 stocks would do for diversification. Many academics reflect the Evan/Archer view in finance text books:

Francis[14]	10–15 stocks
Stevenson and Jennings[15]	8–16 stocks

[12] Meir Statman, "How Many Stocks Make a Diversified Portfolio?" *Journal of Financial and Quantitative Analysis* (September 1987), reprinted in Keith V. Smith, *Case Problems and Readings: A Supplement for Investments in Portfolio Management*, McGraw-Hill, 1990, pp. 161–171.
[13] J.L. Evans and S.H. Archer, "Diversification and the Reduction of Dispersion: An Empirical Analysis," *Journal of Finance*, 23 (December 1968), pp. 761–767.
[14] J.C. Francis, Investments: Analysis and Management, 4th ed. McGraw-Hill (1986)
[15] R.A. Stevenson and E.H. Jennings, Fundamentals of Investments, 3rd ed., West Publishing (1984).

Gup[16]	9 stocks
Reilly[17]	12–18 stocks

Statman recommends 30 stocks as the minimum for a buy portfolio.

In sharp contrast to these recommendations, Mandelbrot[18] suggests 90–120 stocks based on Fama research[19] with Fat Tailed Stable Paretian distributions.

Just to confound the views, Statman summarizes empirical evidence on how many stocks individuals *actually* own:

> Evidence, however, suggests that the typical investor's stock portfolio contains only a small fraction of the available securities. Blume Crockett and Friend[20] found that in 1971, 34.1 percent of investors in their sample held only one dividend paying stock, 50 per cent held no more than 2 stocks, and only 10.7 per cent held more than 10 stocks. A 1967 Federal Reserve Board Survey of Financial Characteristics of Consumers[21] showed that the average number of stocks in the portfolio was 3.41. A survey of investors who held accounts with a major brokerage company revealed that the average number of stocks in a portfolio ranged from 9.4 to 12.1, depending on the demographic group.[22]

Hopefully, this paper may help to make some sense out of these divergent points of view and empirical evidence on actual individual holdings.

The chart below overlays different sized portfolios from one to 200 stocks from the list of top 5 percent of undervalued stocks against the results from the

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[16] B.E. Gup, The Basics of Investing, Wiley, (1983).

[17] F.K. Reilly, Investment Analysis and Portfolio Management, 2nd, Dryden (1985).

[18] Benoit Mandelbrot, *The (Mis)Behavior of Markets*, p. 266.

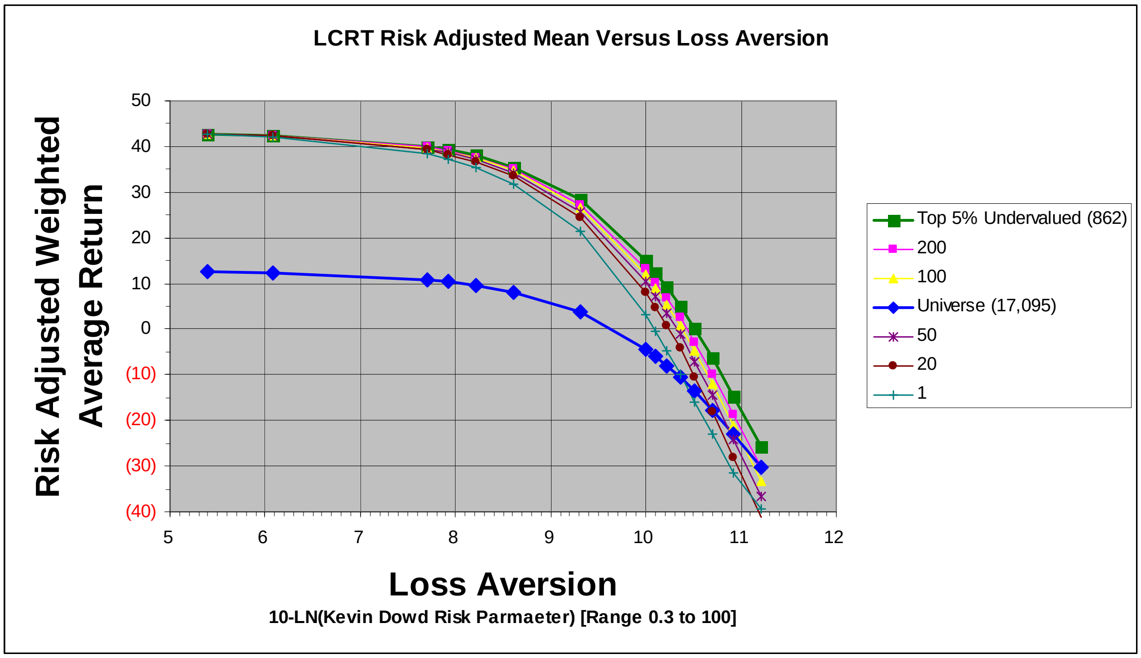
[19] Eugene F. Fama, "Portfolio Analysis in A Stable Paretian Market," *Management Science*, Vol. II, No. 3, January 1965, pp. 404–419, especially p. 415,

[20] M.E. Blume, J. Crockett, and I. Friend, "Stock Ownership in the United States: Characteristics and Trends," *Survey of Current Business*, 54 (Nov. 1974), pp. 16–40.

[21] M.E. Blume and I. Friend, "The Asset Structure of Individual Portfolios and Some Implications for Utility Functions," *Journal of Finance*, 30 (May 1975), pp. 585–603.

[22] R.C. Lease, W. Lewellen, and G. Schlarbaum, "Market Segmentation: Evidence on the Individual Investor," *Financial Analyst Journal*, 32 (September 1976), pp. 53–60.

previous chart on page 7. The one, 20, 50 and 100 stock portfolio lines cross the universe line, but at negative risk adjusted returns. As long as the investor wishes to make a positive return, the investor should prefer the undervalued strategy to the universe. From this risk-return point of view, the number of stocks seems to make less of a material difference, but remains an excellent method for comparing portfolio investing *strategies* to a universe. As the spread between the strategy and the universe narrows, the number of stocks in the portfolio becomes increasingly important.



One major difference between the points of view relates to the distribution assumptions. The Statman summary assumes a Gaussian distribution of returns, while Fama and Mandelbrot assume a Stable Paretian distribution. Fama further quantifies the effect of portfolio size on the dispersion of the distribution:[23]

$$\left(\frac{1}{n}\right)^{\alpha} + |\beta|^{\alpha}$$

where n is the number of stocks in the portfolio, α is the alpha peakedness parameter and β is the beta skewness parameter of the Stable distribution. Instead of the dispersion declining with the square of the number of securities as in a Gaussian world, the Stable dispersion declines with the alpha peakedness power—i.e., at a significantly slower rate.

[23] Fama, “Portfolio Analysis in a Stable Paretian Market,” p. 414.

The table below compares the Fama Stable approach to the Statman Gaussian approach. Statman reproduces the results from the Elton-Gruber study.[24] The first column displays the number of stocks in the portfolio. The second column shows the "c" dispersion from the McCulloch estimation procedure adjusted by the Fama equation above. The third column shows the ratio of the Stable Dispersion from column 2 to the dispersion for a single stock. The fourth column shows the ratio of the standard deviation for multiple stocks to the standard deviation of one stock from the Elton-Gruber study. Note how much faster the ratio declines for the Gaussian ratio, coming close to the minimum at 20-30 stocks. In contrast, to the Gaussian ratio, the Stable ratio continues to decline even at 100 stocks in the portfolio. Stable distributions are simply more risky than Gaussian. These results help to explain Statman's recommendation of 30 stocks to the portfolio and Mandelbrot's recommendation of 90–120 stocks.

# Stocks	Stable Dispersion	Stable Ratio	Gaussian Ratio
1	70.88	1.00	1.00
2	63.74	0.90	0.76
4	58.04	0.82	0.60
10	52.23	0.74	0.49
20	48.85	0.69	0.44
30	47.20	0.67	0.42
40	46.15	0.65	0.42
50	45.40	0.64	0.41
100	43.40	0.61	0.40
200	41.79	0.59	0.39
862	39.40	0.56	0.39

[24] E.J. Elton and M.J. Gruber, "Risk Reduction and Portfolio Size, An Analytical Solution," *Journal of Business*, 50 (October 1977), pp. 415–417. Meir Statman in Keith Smith, *Case Problems*, p. 163.

The table below shows how the number of stocks affects the range of likely average returns. The rows in the table calculate the mean of the distribution +/- 3 or 4 standard errors of return of the delta location parameter of the Stable distribution.[25] Twenty stocks still display a rather large range of possible average returns. That range narrows materially at 100 stocks and even narrows up to 862 stocks in the subset of undervalued firms.

Number of Stocks =>	1	20	50	100	200	Top 5% Undervalued (862)
Mean - 3 Std. Error(delta)	(0.21)	33.42	36.98	38.78	40.05	41.63
Mean + 3 Std. Error(delta)	86.42	52.79	49.23	47.44	46.17	44.58
Mean - 4 Std. Error(delta)	(14.64)	30.20	34.94	37.33	39.03	41.14
Mean + 4 Std. Error(delta)	100.86	56.02	51.28	48.88	47.19	45.08

Yet individuals generally hold fewer than five stocks. Those with brokerage accounts hold fewer than 15. Why?

Please permit the author to offer two hypotheses for this individual behavior of few stocks in their portfolios. These hypotheses deserve more discussion, empirical analysis and testing:

1. Managing a portfolio of 100 stocks requires much work. To manage this many stocks, investors can pay a mutual fund or index fund manager at a lower fee than the opportunity cost of their personal time.
2. Individual investors may invest in companies and industries in which they are intimately familiar with the businesses. Consequently, this information advantage makes them believe that they face substantially lower "risk" than the simple statistics above might indicate.

How many stocks should an investor have in his portfolio? The author cannot answer that question, because it depends on the investor's tolerance for risk of loss and knowledge of the companies/industries. What this analysis does suggest is that in a Stable Paretian world with fat tails, the risks of not diversifying are greater than the traditional 20-30 stock rule of thumb, based on a

[25] McCulloch states the parameter estimates become asymptotically Gaussian Normal.

Gaussian world. The investor should have the confidence in his superior knowledge and devote the time to manage a portfolio of fewer stocks. The investor needs to balance the two admonitions:

1. Don't put all your eggs in one basket and
2. If you have fewer eggs in your basket, watch them very closely.

6. Conclusion

This article applies recent work by Kevin Dowd on investor loss aversion preferences and work by Benoit Mandelbrot on Stable Paretian distributions with Huston McCulloch's parameter estimation procedures to the application of new portfolio risk/return measurements to back tested stock portfolio performance. This practical new risk measurement process addresses the issue of infinite variances empirically observed in most stock return distributions.

This risk-return measurement methodology may apply to the following practical decisions for investors:

1. Risk/return assessment of actual stock portfolio performance.
2. Risk/return assessment of back tested stock portfolio strategies.
3. How many stocks should the portfolio contain to have sufficient diversification?
4. Risk/return assessment of new portfolio products (should, for example, the portfolio purchase 10 percent of out of the money puts in either an index or the individual stocks to buy insurance, lower excess alpha return, but reduce risk?)
5. Time series risk/return measurement of portfolio returns of multiple asset classes—stocks, bonds, commodities, etc.
6. Empirically relate risk adjusted return to the market and fundamental factors affecting likely losses.[26]

[26] For example: market capitalization, constant dollar gross investment, cash economic return, percent debt to capital at market, percent debt to debt capacity, fiscal year high-low price dispersion, stock price level, dividend payout & yield, plant life and the mix of depreciating assets versus non-depreciating assets (asset liquidity).