

Actuarial Approach to Option Pricing

Hans U. Gerber
Ecole des hautes études commerciales
Université de Lausanne
CH-1015 Lausanne, Switzerland
Phone: 21 692 3371
Fax: 21 692 3305
E-mail: hgerber@hec.unil.ch

Elias S.W. Shiu
Department of Statistics and Actuarial Science
The University of Iowa
Iowa City, Iowa 52242-1419, U.S.A.
Phone: 319 335 2580
Fax: 319 335 3017
E-mail: eshiu@stat.uiowa.edu

1. Introduction

Actuaries measure, model and manage risks. Risk associated with the investment function is a major uncertainty faced by many insurance companies. Actuaries should have knowledge of the asset side of the balance sheet of an insurance company and how it relates to the liability side. Such knowledge includes the operation of financial markets, the instruments available to the insurance companies, the options imbedded in these instruments, and the methods of pricing such options and derivative securities.

In this paper we study the pricing of financial options and contingent claims. We show that two time-honored concepts in actuarial science – the Esscher transform and the adjustment coefficient – are efficient tools for pricing many options and derivative securities if the logarithms of the prices of the primitive securities are certain stochastic processes with stationary and independent increments. An Esscher transform of such a security-price process induces an equivalent probability measure on the process. The Esscher parameter or parameter vector is determined so that the discounted price of each primitive security is a martingale under the new probability measure. A derivative

security is valued as the expectation, with respect to this equivalent martingale measure, of the discounted payoffs.

We also study the pricing of American options on two stocks without expiration date and with payoff functions which are homogeneous with respect to the two stock prices. An example of such options is the perpetual Margrabe option, whose payoff is the amount by which one stock outperforms the other. The method is based on the construction of two martingales with respect to the equivalent martingale measure, and applying the optional sampling theorem. The martingale construction is similar to the determination of the adjustment coefficient in collective risk theory. This approach does not involve differential equations and hence is quite different from the traditional approach in financial literature.

This joint work began when Hans Gerber visited to Iowa City in August, 1992, to attend the 27th Annual Actuarial Research Conference. The research on Esscher Transforms was presented in the 28th Annual Actuarial Research Conference in honor of Jim Hickman's retirement and has appeared as [GS94a]. Further results can be found in [GS94b], [GS94c], [GS94d], [GS95] and [GMS95].

2. The Esscher Transform of a Random Variable

Let Y be a given random variable and h a real number for which the expectation

$$E[e^{hY}]$$

exists. The positive random variable

$$\frac{e^{hY}}{E[e^{hY}]} \tag{2.1}$$

can be used (as the Radon-Nikodym derivative) to define a new probability measure, which is *equivalent* to the old measure in the sense that they have the same null sets (sets of measure zero). For a measurable function ψ , the expectation of the random variable $\psi(Y)$ with respect to the new measure is

$$E[\psi(Y); h] = \frac{E[\psi(Y)e^{hY}]}{E[e^{hY}]}. \quad (2.2)$$

We call this new measure the *Esscher measure* of parameter h . The corresponding distribution is usually called the *Esscher transform* in the actuarial literature ([Es32], [Je91]). In some statistical literature, the term *exponential tilting* is used to describe this change of measure.

The method of Esscher transforms was developed to approximate the aggregate claim amount distribution around a point of interest, y_0 , by applying an analytic approximation (the first few terms of the Edgeworth series) to the transformed distribution with the parameter $h = h_0$ chosen such that the new mean is equal to y_0 . Let

$$c(h) = \ln(E[e^{hY}]) \quad (2.3)$$

be the cumulant-generating function. Then

$$c'(h) = \frac{E[Ye^{hY}]}{E[e^{hY}]} = E[Y; h] \quad (2.4)$$

and

$$c''(h) = \frac{E[Y^2e^{hY}]}{E[e^{hY}]} - \left(\frac{E[Ye^{hY}]}{E[e^{hY}]} \right)^2 = \text{Var}[Y; h]. \quad (2.5)$$

Since $\text{Var}[Y; h] > 0$ for a nondegenerate random variable Y , the function $c'(h)$ is strictly increasing; thus the number h_0 for which

$$y_0 = c'(h_0) = E[Y; h_0]$$

is unique. In using the Esscher transform to calculate a stop-loss premium, the parameter h_0 is usually chosen such that the mean of the transformed distribution is the retention limit.

3. Discrete-Time Stock-Price Models

A purpose of this paper is show that the concept of Esscher measures is an effective tool for pricing stock options and other derivative securities. We need to extend

the change of measure for a single random variable to that for a stochastic process. In this section we consider the simpler case of discrete-time stochastic processes.

For $j = 0, 1, 2, \dots$, let $S(j)$ denote the price of a stock at time j . Assume that there is a sequence of independent (but not necessarily identically distributed) random variables $\{Y_k\}$ such that

$$S(j) = S(0) \exp(Y_1 + Y_2 + \dots + Y_j), \quad j = 1, 2, 3, \dots \quad (3.1)$$

We assume that the moment generating function for each Y_i exists, and write

$$M_{Y_i}(h) = E[e^{hY_i}]. \quad (3.2)$$

For a sequence of real numbers $\{h_k\}$, define

$$Z_j = \exp\left(\sum_{k \leq j} h_k Y_k\right) / \prod_{k \leq j} M_{Y_k}(h_k). \quad (3.3)$$

Then $\{Z_j\}$ is a positive martingale which can be used to define a change of measure for the stock-price process. For a positive integer m , let $\psi(m)$ be a random variable that is a function Y_1, \dots, Y_m ,

$$\psi(m) = \psi(Y_1, \dots, Y_m). \quad (3.4)$$

The expected value of $\psi(m)$, with respect to the new measure, is

$$E[\psi(m) Z_m]. \quad (3.5)$$

In (3.5) the random variable Z_m can be replaced by Z_j , $j \geq m$, because of the martingale property.

We assume that the risk-free interest rate is constant through time and the stock pays no dividends. Let δ denote the risk-free force of interest. The *risk-neutral Esscher measure* is the measure, defined by the sequence of numbers $\{h_k^*\}$, with respect to which

$$\{e^{-\delta j} S(j)\} \quad (3.6)$$

is a martingale. This leads to

$$e^{\delta} = M_{Y_k}(1 + h_k^*) / M_{Y_k}(h_k^*), \quad k = 1, 2, 3, \dots \quad (3.7)$$

As we pointed out at the end of the last section, the numbers $\{h_k^*\}$ are unique.

Suppose that each Y_k is a Bernoulli random variable, i.e., it takes on two distinct values, a_k and b_k , only. Then there is only one risk-neutral measure, given by

$$\Pr^*(Y_k = a_k) = \frac{e^\delta - e^{a_k}}{e^{b_k} - e^{a_k}} \quad (3.8)$$

and

$$\Pr^*(Y_k = b_k) = \frac{e^\delta - e^{b_k}}{e^{a_k} - e^{b_k}}. \quad (3.9)$$

(δ is between a_k and b_k for each k .)

If we assume that the random variables $\{Y_k\}$ are identically distributed in addition to being independent, then all h_k^* are the same number. This points to an approach to extend the change of measure to certain continuous-time models, as we shall see in Section 5. On the other hand, the risk-neutral Esscher measure can also be defined for dependent random variables $\{Y_k\}$. In this more general situation, each h_k is a function of Y_1, Y_2, \dots, Y_{k-1} and thus a random variable itself.

4. Fundamental Theorem of Asset Pricing

In this paper we assume that the market is frictionless and trading is continuous. There are no taxes, no transaction costs, and no restriction on borrowing or short sales. All securities are perfectly divisible. It is now understood that, in such a security market model, the absence of arbitrage is “essentially” equivalent to the existence of a *risk-neutral measure* or an *equivalent martingale measure*, with respect to which the *price* of a random payment is the expected discounted value. Dybvig and Ross [DR87] call this result the *Fundamental Theorem of Asset Pricing*. In general, there may be more than one equivalent martingale measure. The merit of the risk-neutral Esscher measure is that it provides a general, transparent and unambiguous solution.

That the condition of no arbitrage is intimately related to the existence of an equivalent martingale measure was first pointed out in Harrison and Kreps [HK79] and Harrison and Pliska [HP81]. Their results are rooted in the idea of risk-neutral valuation

of Cox and Ross [CR76]. In a finite discrete-time model, the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure ([CMW90], [Sc92a]). In a more general setting the characterization is delicate, and we have to replace the term “equivalent to” by “essentially equivalent to”. It is beyond the scope of the present paper to discuss the details. Some recent papers are [Ba91], [BP91], [CH89], [De92], [DS94a], [DS94b], [Mü89], [Sc92b], [Sc94], [Scw92] and [St93].

We note that the idea of changing the probability measure to obtain a consistent positive linear pricing rule had appeared in the actuarial literature in the context of equilibrium reinsurance markets ([Bo60], [Bo90], [Bü80], [Bü84], [CM94], [Ge87], [Li86], [So91]).

5. Continuous-Time Stock-Price Models

In the rest of the paper we consider continuous-time stock-price models. For $t \geq 0$, let $S(t)$ denote the price at time t of a non-dividend-paying stock. We assume that there is a stochastic process $\{X(t)\}$ with independent and stationary increments such that

$$S(t) = S(0) e^{X(t)}, \quad t \geq 0. \quad (5.1)$$

For a theoretical “justification” that stock prices should be modeled with such processes, see Samuelson [Sa65] or Parkinson [Pa77]. (Some authors call $\{X(t)\}$ a *Lévy process*.)

We assume that the moment generating function of $X(t)$,

$$M(h, t) = E[e^{hX(t)}],$$

exists and that

$$M(h, t) = M(h, 1)^t. \quad (5.2)$$

The process

$$\{e^{hX(t)} M(h, 1)^{-t}\} \quad (5.3)$$

is a positive martingale and can be used to define a change of probability measure, i.e., it can be used to define the Radon-Nikodym derivative $\frac{dQ}{dP}$, where P is the original

probability measure and Q is the Esscher measure of parameter h . The *risk-neutral Esscher measure* is the Esscher measure of parameter $h = h^*$ such that the process

$$\{e^{-\delta t} S(t)\} \tag{5.4}$$

is a martingale.

The condition

$$E[e^{-\delta t} S(t); h^*] = e^{-\delta 0} S(0) = S(0)$$

yields

$$\begin{aligned} e^{\delta t} &= E[e^{(1+h^*)X(t)} M(h^*, 1)^{-t}] \\ &= [M(1 + h^*, 1)/M(h^*, 1)]^t, \end{aligned}$$

or

$$e^{\delta} = M(1 + h^*, 1)/M(h^*, 1), \tag{5.5}$$

which is analogous to (3.7) with $\{Y_k\}$ being identically distributed. The parameter h^* is unique. There may be many other equivalent martingale measures.

Because, for $t \geq 0$,

$$e^{hX(t)} M(h, 1)^{-t} = \frac{e^{hX(t)}}{E[e^{hX(t)}]} = \frac{S(t)^h}{E[S(t)^h]}, \tag{5.6}$$

we have the following: Let g be a measurable function and h, k and t be real numbers, $t \geq 0$; then

$$\begin{aligned} E[S(t)^k g(S(t)); h] &= E[S(t)^k g(S(t)) e^{hX(t)} M(h, 1)^{-t}] \\ &= \frac{E[S(t)^{h+k} g(S(t))]}{E[S(t)^h]} \\ &= \frac{E[S(t)^{h+k}]}{E[S(t)^h]} \frac{E[S(t)^{h+k} g(S(t))]}{E[S(t)^{h+k}]} \\ &= E[S(t)^k; h] E[g(S(t)); h + k]. \end{aligned} \tag{5.7}$$

This *factorization formula* simplifies many calculations, and is a main reason why the method of Esscher measures is an efficient device for valuing certain derivative securities. For example, applying (5.7) with $k = 1$, $h = h^*$ and $g(x) = I(x > K)$ [where $I(A)$ denotes the indicator random variable of an event A], we obtain

$$\begin{aligned}
E[S(\tau) I(S(\tau) > K); h^*] &= E[S(\tau); h^*] E[I(S(\tau) > K); h^* + 1] \\
&= E[S(\tau); h^*] \Pr[S(\tau) > K; h^* + 1] \\
&= S(0)e^{\delta\tau} \Pr[S(\tau) > K; h^* + 1]. \tag{5.8}
\end{aligned}$$

The last equality holds because (5.4) is a martingale with respect to the risk-neutral Esscher measure. Thus we have a pricing formula for a European call option on a non-dividend-paying stock,

$$\begin{aligned}
&E[e^{-\delta\tau} (S(\tau) - K)_+; h^*] \\
&= E[e^{-\delta\tau} (S(\tau) - K) I(S(\tau) > K); h^*] \\
&= e^{-\delta\tau} \{E[S(\tau) I(S(\tau) > K); h^*] - KE[I(S(\tau) > K); h^*]\} \\
&= S(0)\Pr[S(\tau) > K; h^* + 1] - Ke^{-\delta\tau}\Pr[S(\tau) > K; h^*]. \tag{5.9}
\end{aligned}$$

For $\{X(t)\}$ being a Wiener process, (5.9) is the celebrated Black-Scholes formula [BS73]; see also (9.20) below.

6. Representative Investor with Power Utility Function

When there is more than one equivalent martingale measure, why should the option price be the expectation, with respect to the risk-neutral Esscher measure, of the discounted payoff? This particular choice may be justified within a utility function framework. Consider a simple economy with only a stock and a risk-free bond and their derivative securities. There is a representative investor who owns m shares of the stock and bases his decisions on a risk-averse utility function $u(x)$. Consider a derivative security that provides a payment of $\pi(\tau)$ at time τ , $\tau > 0$; $\pi(\tau)$ is a function of the stock price process until time τ . What is the investor's *price* for the derivative security, such that it is optimal for him not to buy or sell any fraction or multiple of it? Let $V(0)$ denote this price. Then, mathematically, this is the condition that the function

$$\phi(\eta) = E[u(mS(\tau) + \eta[\pi(\tau) - e^{\delta\tau}V(0)])] \tag{6.1}$$

is maximal for $\eta = 0$. From

$$\phi'(0) = 0$$

we obtain

$$V(0) = e^{-\delta\tau} \frac{E[\pi(\tau)u'(mS(\tau))]}{E[u'(mS(\tau))]} \quad (6.2)$$

(as a necessary and sufficient condition, since $\phi''(\eta) < 0$ if $u''(x) < 0$). In the particular case of a power utility function with parameter $c > 0$,

$$u(x) = \begin{cases} \frac{x^{1-c}}{1-c} & \text{if } c \neq 1, \\ \ln x & \text{if } c = 1 \end{cases}, \quad (6.3)$$

we have $u'(x) = x^{-c}$, and

$$\begin{aligned} V(0) &= e^{-\delta\tau} \frac{E[\pi(\tau)[mS(\tau)]^{-c}]}{E[[mS(\tau)]^{-c}]} \\ &= e^{-\delta\tau} \frac{E[\pi(\tau)S(\tau)^{-c}]}{E[S(\tau)^{-c}]} \end{aligned} \quad (6.4)$$

Formula (6.4) must hold for all derivative securities. For $\pi(\tau) = S(\tau)$ and therefore

$V(0) = S(0)$, (6.4) becomes

$$\begin{aligned} S(0) &= e^{-\delta\tau} \frac{E[S(\tau)^{1-c}]}{E[S(\tau)^{-c}]} \\ &= e^{-\delta\tau} S(0) \frac{M(1-c, \tau)}{M(-c, \tau)}, \end{aligned}$$

or

$$e^{\delta} = \frac{M(1-c, 1)}{M(-c, 1)}. \quad (6.5)$$

On comparing (6.5) with (5.5), we see that the value of the parameter c is $-h^*$. Hence $V(0)$ is indeed the discounted expectation of the payoff $\pi(\tau)$, calculated with respect to the Esscher measure of parameter $h^* = -c$.

By considering different points in time τ , we get a consistency requirement. This is satisfied if the investor has a power utility function. We conjecture that it is violated for any other risk-averse utility function, which implies that the pricing of an option by the risk-neutral Esscher measure is a consequence of the consistency requirement. Some related papers are Rubinstein [Ru76], Bick ([Bi87], [Bi90]), Constantinides [Co89], Naik

and Lee [NL90], Stapleton and Subrahmanyam [SSu90], He and Leland [HL93], Heston [He93] and Wang [Wa93].

7. Logarithm of Stock Price as a Shifted Poisson Process

Here we consider the so-called pure jump model. The assumption is

$$X(t) = kN(t) - ct, \quad (7.1)$$

where $\{N(t)\}$ is a Poisson process with parameter λ , and k and c are constants. The price of the non-dividend-paying stock is modeled as

$$S(t) = S(0)e^{kN(t) - ct}. \quad (7.2)$$

There is only one equivalent martingale measure in this model.

Since

$$E[e^{zN(t)}] = \exp[\lambda t(e^z - 1)],$$

we have

$$\begin{aligned} M(z, t) &= E[e^{zX(t)}] \\ &= E[e^{z(kN(t) - ct)}] \\ &= \exp([\lambda(e^{zk} - 1) - zc]t). \end{aligned} \quad (7.3)$$

Because

$$\begin{aligned} E[e^{zX(t)}; h] &= \frac{M(z + h, t)}{M(h, t)} \\ &= \exp([\lambda e^{hk}(e^{zk} - 1) - zc]t), \end{aligned}$$

we see that, under the Esscher measure of parameter h , the process $\{X(t)\}$ remains a shifted Poisson process, but with modified Poisson parameter λe^{hk} . Formula (5.5) is the condition that

$$\delta = \lambda e^{h^*k}(e^k - 1) - c. \quad (7.4)$$

The equivalent martingale measure is the measure with respect to which $\{N(t)\}$ becomes a Poisson process with parameter

$$\begin{aligned} \lambda^* &= \lambda e^{h^*k} \\ &= (\delta + c)/(e^k - 1). \end{aligned} \quad (7.5)$$

We now show that, by a replicating portfolio argument, the price of a derivative security is indeed the expectation of its discounted payoff, with the expectation taken with respect to the equivalent martingale measure. Let $V(S(t), t)$ be the price of the derivative security at time t . We can form a self-financing portfolio of the stock and risk-free bond replicating the price $V(S(t), t)$ through time. Let

$$\eta = \eta(S(t), t) \tag{7.6}$$

be the amount invested in the stock at time t and therefore the difference $V(S(t), t) - \eta$ is the amount invested in the risk-free bond at time t . The amount η is such that the derivative security price and the portfolio value have equal instantaneous change. By considering whether there will be an instantaneous jump in the stock price or not, we have the following two conditions:

$$V(Se^k, t) - V(S, t) = \eta e^k - \eta, \tag{7.7}$$

and

$$\begin{aligned} V_t(S, t) - cSV_S(S, t) &= -c\eta + \delta[V(S, t) - \eta] \\ &= \delta V(S, t) - (\delta + c)\eta. \end{aligned} \tag{7.8}$$

Formula (7.7) yields

$$\eta = \frac{V(Se^k, t) - V(S, t)}{e^k - 1}. \tag{7.9}$$

Thus (7.8) becomes

$$V_t(S, t) - cSV_S(S, t) = \delta V(S, t) - \lambda^*[V(Se^k, t) - V(S, t)], \tag{7.10}$$

where λ^* is given by (7.5).

Now, let $W(S(t), t)$ denote the value at time t of the expected discounted payoff of the derivative security; the expectation is taken with respect to the probability measure corresponding to the Poisson parameter λ^* . Let s be a very small positive number. By the Poisson process assumption, the probability that a jump in the stock price will occur in the time interval $(t, t + s)$ is $\lambda^*s + o(s)$. Thus, conditioning on whether there are stock-price jumps in the interval $(t, t + s)$, we have

$$W(S, t) = e^{-\delta s}[(1 - \lambda^*s)W(Se^{-cs}, t+s) + \lambda^*sW(Se^{k-cs}, t+s)] + o(s), \tag{7.11}$$

or

$$(1 + \delta s)W(S, t) - W(Se^{-cs}, t+s) = \lambda^*s[W(Se^{k-cs}, t+s) - W(Se^{-cs}, t+s)] + o(s).$$

Dividing the last equation by s and letting s tend to 0 yields

$$\delta W(S, t) + cSW_S(S, t) - W_t(S, t) = \lambda^*[W(Se^k, t) - W(S, t)], \quad (7.12)$$

which is identical to (7.10). Consequently, the price of the derivative security, $V(S(t), t)$, is calculated as the expected discounted payoffs according to the provisions of the contract; the expectation is taken with respect to the measure corresponding to the Poisson process with parameter λ^* .

We note that, in constructing the replicating portfolio, we did not use the assumption that $\{N(t)\}$ is a Poisson process. Thus $N(t)$ in (7.1) and (7.2) may be assumed to come from a counting process; the equivalent martingale measure is the measure with respect to which $\{N(t)\}$ becomes a Poisson process with parameter λ^* given by (7.5). A replicating portfolio can be constructed because at each point of time the stock price has only two possible movements, both with known magnitude.

It is interesting to consider the limiting case where $k \rightarrow 0$ and $c \rightarrow \infty$ such that the variance per unit time of $\{X(t)\}$ in the risk-neutral measure is constant:

$$\lambda^*k^2 = \frac{\delta + c}{e^k - 1} k^2 = \sigma^2. \quad (7.13)$$

This is the classical lognormal model. In the limit (7.9) becomes

$$\eta = SV_S(S, t),$$

showing that the ratio, $\eta(S(t), t)/S(t)$, is given by $V_S(S(t), t)$, which is usually called *delta* in the option literature. Also, by means of the Taylor expansion, we have

$$\begin{aligned} & \lambda^*[V(Se^k, t) - V(S, t)] \\ &= \lambda^*\{(e^k - 1)SV_S(S, t) + [(e^k - 1)S]^2V_{SS}(S, t)/2 + O(k^3)\} \\ &= (\delta + c)SV_S(S, t) + \sigma^2S^2V_{SS}(S, t)/2 + O(k). \end{aligned}$$

Thus in the limit (7.10) becomes

$$V_t(S, t) = \delta V(S, t) - \delta SV_S(S, t) - \frac{\sigma^2}{2}S^2V_{SS}(S, t). \quad (7.14)$$

This partial differential equation was first derived by Black and Scholes [BS73] with a replicating portfolio argument.

8. Extension to Dividend-Paying Stocks

The results in Section 5 can be extended to the case where the stock pays dividends continuously, at a rate proportional to its price. In other words, we assume that there is a nonnegative number φ such that the dividend paid between time t and $t+dt$ is

$$\varphi S(t) dt. \quad (8.1)$$

(The number φ may be called the dividend-yield rate.) If all dividends are reinvested in the stock, each share of the stock at time 0 grows to $e^{\varphi t}$ shares at time t . The risk-neutral Esscher measure is the Esscher measure of parameter $h = h^*$ such that the process

$$\{e^{-(\delta-\varphi)t}S(t)\} \quad (8.2)$$

is a martingale. Condition (5.5) now becomes

$$e^{\delta-\varphi} = M(1 + h^*, 1)/M(h^*, 1). \quad (8.3)$$

Since

$$E[S(\tau); h^*] = S(0) e^{(\delta-\varphi)\tau}, \quad (8.4)$$

the European call option pricing formula (5.9) is generalized as

$$\begin{aligned} & E[e^{-\delta\tau} (S(\tau) - K)_+; h^*] \\ &= S(0)e^{-\varphi\tau} \Pr[S(\tau) > K; h^* + 1] - Ke^{-\delta\tau} \Pr[S(\tau) > K; h^*]. \end{aligned} \quad (8.5)$$

Formula (8.5) may also be used to price currency exchange options, with $S(\tau)$ denoting the spot exchange rate at time τ , δ the domestic force of interest and φ the foreign force of interest. For $\{S(t)\}$ being a geometric Brownian motion, (8.5) is known as the *Garman-Kohlhagen formula*; see also (9.20) below.

We can extend the model to more than one dividend-paying stock. For each j , $j = 1, 2, \dots, n$, let $S_j(t)$ denote the price of stock j at time t , $t \geq 0$, and we assume that there exists a nonnegative constant φ_j such that stock j pays dividends of amount

$$\varphi_j S_j(t) dt$$

between time t and $t+dt$. Write

$$X_j(t) = \ln[S_j(t)/S_j(0)], \quad j = 1, 2, \dots, n, \quad (8.6)$$

and

$$\mathbf{X}(t) = (X_1(t), X_2(t), \dots, X_n(t))'. \quad (8.7)$$

Let \mathbb{R}^n denote the linear space of column vectors with n real entries, and

$$M(\mathbf{z}, t) = E[e^{\mathbf{z}\mathbf{X}(t)}], \quad \mathbf{z} \in \mathbb{R}^n, \quad (8.8)$$

be the moment generating function of $\mathbf{X}(t)$. We assume that $\{\mathbf{X}(t)\}_{t \geq 0}$ is a stochastic process with independent and stationary increments and that

$$M(\mathbf{z}, t) = [M(\mathbf{z}, 1)]^t, \quad t \geq 0. \quad (8.9)$$

Let $\mathbf{h} = (h_1, h_2, \dots, h_n)' \in \mathbb{R}^n$ for which $M(\mathbf{h}, 1)$ exists. The positive martingale

$$\{e^{\mathbf{h}\mathbf{X}(t)} M(\mathbf{h}, 1)^{-t}\}_{t \geq 0} \quad (8.10)$$

can be used to define a new measure, the Esscher measure of parameter vector \mathbf{h} . The risk-neutral Esscher measure is the Esscher measure of parameter vector $\mathbf{h} = \mathbf{h}^*$ such that, for each $j, j = 1, 2, \dots, n$,

$$\{e^{-(\delta - \varphi_j)t} S_j(t)\} \quad (8.11)$$

is a martingale. Condition (8.3) is generalized as n simultaneous conditions:

$$e^{\delta - \varphi_j} = M(\mathbf{1}_j + \mathbf{h}^*, 1)/M(\mathbf{h}^*, 1), \quad j = 1, \dots, n. \quad (8.12)$$

Here

$$\mathbf{1}_j = (0, \dots, 0, 1, 0, \dots, 0)', \quad (8.13)$$

where the 1 in the column vector $\mathbf{1}_j$ is in the j -th position.

For $\mathbf{k} = (k_1, \dots, k_n)'$, write

$$\mathbf{S}(t)^{\mathbf{k}} = S_1(t)^{k_1} \dots S_n(t)^{k_n}. \quad (8.14)$$

Then

$$\begin{aligned}
E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t)); \mathbf{h}] &= \frac{E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t)) e^{\mathbf{h}'\mathbf{X}(t)}]}{E[e^{\mathbf{h}'\mathbf{X}(t)}]} \\
&= \frac{E[\mathbf{S}(t)^{\mathbf{k}} g(\mathbf{S}(t)) \mathbf{S}(t)^{\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}]} \\
&= \frac{E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}] E[g(\mathbf{S}(t)) \mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]}{E[\mathbf{S}(t)^{\mathbf{h}}] E[\mathbf{S}(t)^{\mathbf{k}+\mathbf{h}}]} \\
&= E[\mathbf{S}(t)^{\mathbf{k}}; \mathbf{h}] E[g(\mathbf{S}(t)); \mathbf{k} + \mathbf{h}], \tag{8.15}
\end{aligned}$$

which generalizes the factorization formula (5.7). An immediate consequence of (8.15) and that (8.11) is a martingale under the risk-neutral Esscher measure is

$$\begin{aligned}
E[e^{-\delta t} S_j(t) g(\mathbf{S}(t)); \mathbf{h}^*] &= E[e^{-\delta t} S_j(t); \mathbf{h}^*] E[g(\mathbf{S}(t)); \mathbf{h}^* + \mathbf{1}_j] \\
&= S_j(0) e^{-\phi_1 t} E[g(\mathbf{S}(t)); \mathbf{h}^* + \mathbf{1}_j]. \tag{8.16}
\end{aligned}$$

The *Margrabe option* [Ma78] is the option to exchange one stock for another at the end of a stated period, say time τ , $\tau > 0$. The payoff of this European option is

$$[S_1(\tau) - S_2(\tau)]_+. \tag{8.17}$$

Its value at time 0, calculated with respect to the risk-neutral Esscher measure, is

$$E(e^{-\delta \tau} [S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*). \tag{8.18}$$

Since

$$(s_1 - s_2)_+ = s_1 I(s_1 > s_2) - s_2 I(s_1 > s_2),$$

it follows from (8.16) that

$$\begin{aligned}
&E(e^{-\delta \tau} [S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*) \\
&= S_1(0) e^{-\phi_1 \tau} E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_1) \\
&\quad - S_2(0) e^{-\phi_2 \tau} E(I[S_1(\tau) > S_2(\tau)]; \mathbf{h}^* + \mathbf{1}_2) \\
&= S_1(0) e^{-\phi_1 \tau} \Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_1] \\
&\quad - S_2(0) e^{-\phi_2 \tau} \Pr[S_1(\tau) > S_2(\tau); \mathbf{h}^* + \mathbf{1}_2], \tag{8.19}
\end{aligned}$$

A special case of (8.19) is (8.5).

9. Change of Numéraire, Homogeneous Payoff Function and Wiener Process

Consider a European option or derivative security with exercise date τ and payoff

$$\Pi(S_1(\tau), \dots, S_n(\tau)). \quad (9.1)$$

For example, the Margrabe option has the payoff function

$$\Pi(s_1, s_2) = (s_1 - s_2)_+. \quad (9.2)$$

Let $E_t[\cdot]$ denote the expectation conditional on all information up to time t . For $0 \leq t \leq \tau$, let $V(t)$ denote the price of the security at time t , calculated with respect to the risk-neutral Esscher measure,

$$\begin{aligned} V(t) &= E_t[e^{-\delta(\tau-t)} \Pi(S_1(\tau), \dots, S_n(\tau)); \mathbf{h}^*] \\ &= E_t[e^{-\delta(\tau-t)} S_j(\tau) \Pi(S_1(\tau), \dots, S_n(\tau))/S_j(\tau); \mathbf{h}^*] \\ &= E_t[e^{-\delta(\tau-t)} S_j(\tau); \mathbf{h}^*] E_t[\Pi(S_1(\tau), \dots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j] \\ &= e^{-\varphi_j(\tau-t)} S_j(t) E_t[\Pi(S_1(\tau), \dots, S_n(\tau))/S_j(\tau); \mathbf{h}^* + \mathbf{1}_j]. \end{aligned}$$

Thus

$$\frac{V(t)}{e^{\varphi_j t} S_j(t)} = E_t\left[\frac{1}{e^{\varphi_j \tau} S_j(\tau)} \Pi(S_1(\tau), \dots, S_n(\tau)); \mathbf{h}^* + \mathbf{1}_j\right], \quad (9.3)$$

from which it follows that, with respect to the Esscher measure of parameter vector

$\mathbf{h}^* + \mathbf{1}_j$, the process

$$\left\{ \frac{V(t)}{e^{\varphi_j t} S_j(t)}; 0 \leq t \leq \tau \right\} \quad (9.4)$$

is a martingale. In particular, with respect to the Esscher measure of parameter vector

$\mathbf{h}^* + \mathbf{1}_j$, the processes

$$\left\{ \frac{e^{\delta t}}{e^{\varphi_j t} S_j(t)} \right\} \quad (9.5)$$

and

$$\left\{ \frac{e^{\varphi_k t} S_k(t)}{e^{\varphi_j t} S_j(t)} \right\} \quad (9.6)$$

are martingales. To explain the denominator $e^{\varphi_j t} S_j(t)$, we consider stock j as a standard of value or a *numéraire*. In other words, there is a mutual fund consisting of stock j only and all dividends are reinvested; all other securities are measured in terms of the value of this mutual fund. See also [GER94].

Now, we assume that the payoff function Π is homogeneous of degree one with respect to the j -th variable,

$$\Pi(s_1, \dots, s_n) = s_j \Pi(s_1/s_j, \dots, s_{j-1}/s_j, 1, s_{j+1}/s_j, \dots, s_n/s_j), \quad (9.7)$$

which is a condition satisfied by (9.2) with both $j = 1$ and $j = 2$. Then (9.3) becomes

$$\frac{V(t)}{e^{\varphi_j t} S_j(t)} = E_t\left[\Pi\left(\frac{S_1(\tau)}{e^{\varphi_1 \tau} S_j(\tau)}, \dots, \frac{S_n(\tau)}{e^{\varphi_n \tau} S_j(\tau)}; \mathbf{h}^* + \mathbf{1}_j\right)\right]. \quad (9.8)$$

The right-hand side is a conditional expectation, with respect to the Esscher measure of parameter vector $\mathbf{h}^* + \mathbf{1}_j$, of a function of the $(n - 1)$ -dimensional random vector

$$(X_1(\tau) - X_j(\tau), \dots, X_{j-1}(\tau) - X_j(\tau), X_{j+1}(\tau) - X_j(\tau), \dots, X_n(\tau) - X_j(\tau))'. \quad (9.9)$$

Consider the special case that $\{\mathbf{X}(t)\}$ is an n -dimensional Wiener process, with $\underline{\boldsymbol{\mu}} = (\mu_1, \mu_2, \dots, \mu_n)'$ and $\mathbf{V} = (\sigma_{ij})$ denoting the mean vector and the covariance matrix of $\mathbf{X}(1)$, respectively. It is assumed that \mathbf{V} is nonsingular. Because

$$M(\mathbf{z}, t) = \exp\{t(\mathbf{z}'\underline{\boldsymbol{\mu}} + \frac{1}{2}\mathbf{z}'\mathbf{V}\mathbf{z})\}, \quad \mathbf{z} \in \mathbb{R}^n, \quad (9.10)$$

we have, for $\mathbf{h} \in \mathbb{R}^n$,

$$\begin{aligned} E[e^{\mathbf{z}'\mathbf{X}(t)}; \mathbf{h}] &= M(\mathbf{z} + \mathbf{h}, t)/M(\mathbf{h}, t) \\ &= \exp\{t(\mathbf{z}'(\underline{\boldsymbol{\mu}} + \mathbf{V}\mathbf{h}) + \frac{1}{2}\mathbf{z}'\mathbf{V}\mathbf{z})\}, \quad \mathbf{z} \in \mathbb{R}^n, \end{aligned} \quad (9.11)$$

showing that, under the Esscher measure of parameter vector \mathbf{h} , $\{\mathbf{X}(t)\}$ remains an n -dimensional Wiener process with modified mean vector per unit time

$$\underline{\boldsymbol{\mu}} + \mathbf{V}\mathbf{h}$$

and unchanged covariance matrix per unit time \mathbf{V} . For $k = 1, 2, \dots, n$, it follows from (8.12) that

$$\delta - \varphi_k = \mathbf{1}_k'(\underline{\boldsymbol{\mu}} + \mathbf{V}\mathbf{h}^*) + \frac{1}{2}\mathbf{1}_k'\mathbf{V}\mathbf{1}_k, \quad (9.12)$$

from which we obtain

$$\underline{\boldsymbol{\mu}}^* = E[\mathbf{X}(1); \mathbf{h}^*] \quad (9.13)$$

$$= \underline{\boldsymbol{\mu}} + \mathbf{V}\mathbf{h}^* \quad (9.14)$$

$$= \delta \mathbf{1} - (\varphi_1 + \frac{1}{2}\sigma_{11}, \varphi_2 + \frac{1}{2}\sigma_{22}, \dots, \varphi_n + \frac{1}{2}\sigma_{nn})', \quad (9.15)$$

where

$$\mathbf{1} = (1, 1, 1, \dots, 1)'. \quad (9.16)$$

Also,

$$\begin{aligned} E[\mathbf{X}(1); \mathbf{h}^* + \mathbf{1}_k] &= \underline{\boldsymbol{\mu}} + \mathbf{V}(\mathbf{h}^* + \mathbf{1}_k) = \underline{\boldsymbol{\mu}}^* + \mathbf{V}\mathbf{1}_k \\ &= \delta \mathbf{1} - (\varphi_1 - \sigma_{1k} + 1/2\sigma_{11}, \varphi_2 - \sigma_{2k} + 1/2\sigma_{22}, \dots, \varphi_n - \sigma_{nk} + 1/2\sigma_{nn})'. \end{aligned} \quad (9.17)$$

For $\{\mathbf{X}(t)\}$ being an n -dimensional Wiener process, (9.9) is a normal random vector under the Esscher measure of parameter vector $\mathbf{h}^* + \mathbf{1}_j$, and it follows from (9.17) that its mean does not involve the force of interest δ , and of course its $(n - 1)$ -dimensional covariance matrix, which is the same for all \mathbf{h} , does not depend on δ . Thus $V(t)$, the price of a derivative security with a payoff function which is homogeneous with respect to one of its arguments, does not depend on δ .

For example, consider the European Margrabe option. Here $n = 2$. Let

$$\begin{aligned} v^2 &= \text{Var}[X_1(1) - X_2(1)] \\ &= \sigma_{11} - 2\sigma_{12} + \sigma_{22}, \end{aligned} \quad (9.18)$$

$$\zeta(\tau) = \frac{1}{v\sqrt{\tau}} \ln\left(\frac{e^{-\varphi_1\tau}S_1(0)}{e^{-\varphi_2\tau}S_2(0)}\right), \quad (9.19)$$

and Φ denote the standardized normal distribution function. Then (8.19) becomes

$$\begin{aligned} E(e^{-\delta\tau}[S_1(\tau) - S_2(\tau)]_+; \mathbf{h}^*) \\ = e^{-\varphi_1\tau}S_1(0)\Phi(\zeta(\tau) + 1/2v\sqrt{\tau}) - e^{-\varphi_2\tau}S_2(0)\Phi(\zeta(\tau) - 1/2v\sqrt{\tau}), \end{aligned} \quad (9.20)$$

which does not depend on δ . For non-dividend-paying stocks ($\varphi_1 = \varphi_2 = 0$), formula (9.20) has been given by Margrabe [Ma78]. Fischer [Fi78] has also derived (9.20) with $\varphi_1 = 0$ as a European call option formula; for him, $S_2(\tau)$ is the stochastic exercise price at time τ .

Remarks. In the model of n stocks, the risk-neutral Esscher measure is the Esscher measure corresponding to the n -dimensional vector \mathbf{h}^* such that, for $j = 1, 2, \dots, n$, (8.11) is a martingale. Let us now consider modeling only a subset of the n stocks, say stock 1 to stock k , $k < n$. Then the risk-neutral Esscher measure is the Esscher measure

corresponding to the k -dimensional vector \mathbf{h}^* such that, for $j = 1, 2, \dots, k$, (8.11) is a martingale. To avoid confusion, we write the second \mathbf{h}^* as \mathbf{h}_1^* . One may wonder how \mathbf{h}^* and \mathbf{h}_1^* are related. An explicit answer can be given when $\{\mathbf{X}(t)\}$ is a Wiener process.

Let \mathbf{P} denote the projection matrix from \mathbb{R}^n onto its first k coordinates,

$$\mathbf{P} = (\mathbf{I} \ \mathbf{O}), \tag{9.21}$$

where \mathbf{I} is the k -by- k identity matrix and \mathbf{O} is the k -by- $(n-k)$ zero matrix. (The dimension of \mathbf{P} is k by n .) Then the k -by- k matrix \mathbf{PVP}' is the covariance matrix of the random vector

$$(X_1(1), X_2(1), \dots, X_k(1))'.$$

It now follows from (9.14) that

$$\mathbf{P}\mathbf{V}\mathbf{h}^* = \mathbf{PVP}'\mathbf{h}_1^*, \tag{9.22}$$

or

$$\mathbf{h}_1^* = (\mathbf{PVP}')^{-1}\mathbf{P}\mathbf{V}\mathbf{h}^*. \tag{9.23}$$

There is another way to express the relationship between \mathbf{h}^* and \mathbf{h}_1^* . Let \mathbf{V}_1 denote the covariance matrix \mathbf{PVP}' . Similarly, let \mathbf{V}_2 denote the covariance matrix of the $(n-k)$ -dimensional random vector

$$(X_{k+1}(1), X_{k+2}(1), \dots, X_n(1))'.$$

Consider the model consisting only of stock $k+1$ to stock n ; let \mathbf{h}_2^* denote the $(n-k)$ -dimensional vector determining the risk-neutral measure in the model. Then

$$\mathbf{V}\mathbf{h}^* = \begin{pmatrix} \mathbf{V}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{V}_2 \end{pmatrix} \begin{pmatrix} \mathbf{h}_1^* \\ \mathbf{h}_2^* \end{pmatrix}. \tag{9.24}$$

10. Probability of Ruin

The idea of replacing the original probability measure by an Esscher measure with an appropriately chosen parameter has an elegant application in classical actuarial risk theory. Let $\{U(t)\}$ be the surplus process,

$$U(t) = u + X(t), \tag{10.1}$$

where $u \geq 0$ is the initial surplus, and $X(t)$ the aggregate gains (premiums minus claims) up to time t . We suppose that the process $\{X(t)\}$ has independent and stationary increments, satisfies (5.2), and has a positive drift,

$$E[X(1)] > 0. \quad (10.2)$$

Let

$$T = \inf\{t \mid U(t) < 0\} \quad (10.3)$$

be the time of ruin. The probability of ruin before time m , $m > 0$, is

$$\psi(u, m) = \Pr(T < m) = E[I(T < m)]. \quad (10.4)$$

Let $a \wedge b$ denote the minimum of a and b . By a change of measure,

$$\begin{aligned} \psi(u, m) &= E[I(T < m) e^{-hX(T \wedge m)} M(h, 1)^{T \wedge m}; h] \\ &= E[I(T < m) e^{-hX(T)} M(h, 1)^T; h], \end{aligned} \quad (10.5)$$

which can be simplified if h is chosen as the nontrivial solution of the equation

$$M(h, 1) = 1. \quad (10.6)$$

For simplicity we write

$$M(h) = M(h, 1).$$

It follows from

$$M''(h) = E[X(1)^2 e^{hX(1)}] > 0$$

that $M(h)$ is a convex function. Thus equation (10.6) has at most one other solution besides $h = 0$. Because

$$M'(0) = E[X(1)] > 0,$$

the nontrivial solution for (10.6) is a negative h . Following the usual notation in risk theory, we write this solution of (10.6) as $-R$. (R is called the *adjustment coefficient*.)

With $h = -R$, (10.5) becomes

$$\psi(u, m) = E[I(T < m) e^{RX(T)}; -R]. \quad (10.7)$$

The probability of ruin over an infinite horizon is

$$\begin{aligned} \psi(u) &= \psi(u, \infty) \\ &= E[I(T < \infty) e^{RX(T)}; -R]. \end{aligned} \quad (10.8)$$

Now,

$$\begin{aligned} E[X(1); -R] &= E[X(1) e^{-RX(1)}] \\ &= M'(-R) \\ &< 0, \end{aligned}$$

because M is a convex function. An aggregate gains process with a negative drift means that ruin is certain. Thus, under the Esscher measure of parameter $-R$,

$$I(T < \infty) = 1$$

almost surely, and (10.8) simplifies as

$$\begin{aligned} \psi(u) &= E[e^{RX(T)}; -R] \\ &= E[e^{RU(T)}; -R]e^{-Ru}. \end{aligned} \tag{10.9}$$

This approach to the ruin problem can be found in Chapter XII of Asmussen's book [As87] and he has attributed the idea to von Bahr [vB74] and Siegmund [Si75]. Formula (10.9) should be compared with (12.3.4) on page 352 of *Actuarial Mathematics* [BGHJN86],

$$\psi(u) = \frac{e^{-Ru}}{E[e^{-RU(T)} \mid T < \infty]},$$

where the conditional expectation in the denominator is taken with respect to the original probability measure.

11. Perpetual American Options on Two Stocks

The actuarial concept of the adjustment coefficient turns out to be the right tool for pricing certain *American* options without expiration date. In this section we consider two stocks with positive dividend-yield rates. For $\mathbf{k} = (k_1, k_2)' \in \mathbb{R}^2$, we write

$$S(t)^{\mathbf{k}} = S_1(t)^{k_1} S_2(t)^{k_2}$$

[the same notation as (8.14)]. The condition on \mathbf{k} so that the process

$$\{e^{-\delta t} S(t)^{\mathbf{k}}\}_{t \geq 0} \tag{11.1}$$

becomes a martingale under the risk-neutral Esscher measure is:

$$e^{-\delta t} E[e^{\mathbf{k}'\mathbf{X}(t)}; \mathbf{h}^*] = 1. \quad (11.2)$$

Actually, we are only interested in \mathbf{k} of the form

$$(\theta, 1 - \theta)'. \quad (11.3)$$

With the definition

$$f(\theta) = e^{-\delta} E[\exp[\theta X_1(1) + (1 - \theta)X_2(1)]; \mathbf{h}^*], \quad (11.4)$$

condition (11.2) becomes

$$f(\theta) = 1, \quad (11.5)$$

which is analogous to (10.6). Because

$$f(0) = e^{-\delta} E[e^{X_2(1)}] = e^{-\varphi_2} < 1,$$

$$f(1) = e^{-\delta} E[e^{X_1(1)}] = e^{-\varphi_1} < 1$$

and

$$f''(\theta) = e^{-\delta} E[(X_1(1) - X_2(1))^2 \exp[\theta X_1(1) + (1 - \theta)X_2(1)]; \mathbf{h}^*] > 0,$$

we gather that equation (11.5) has, under fairly mild regularity conditions, exactly two solutions, $\theta_0 < 0$ and $\theta_1 > 1$. [Equation (10.6) also has two solutions, 0 and $-R$.] Thus, for $i = 0, 1$, the process

$$\left\{ e^{-\delta t} S_2(t) \left(\frac{S_1(t)}{S_2(t)} \right)^{\theta_i}; t \geq 0 \right\} \quad (11.6)$$

is a martingale with respect to the risk-neutral Esscher measure.

For the rest of this paper we assume that $\{\mathbf{X}(t)\}$ is a two-dimensional Wiener process, so that there is only one equivalent martingale measure, and $\{S_1(t)\}$ and $\{S_2(t)\}$ have continuous sample paths. With $\underline{\mu}^*$ given by (9.13) ($n = 2$), we have

$$E[e^{\mathbf{k}'\mathbf{X}(1)}; \mathbf{h}^*] = \exp(\mathbf{k}'\underline{\mu}^* + 1/2 \mathbf{k}'\mathbf{V}\mathbf{k}). \quad (11.7)$$

Hence (11.4) becomes

$$f(\theta) = \exp[-1/2 v^2 \theta(1 - \theta) - \varphi_1 \theta - \varphi_2(1 - \theta)],$$

where v^2 is defined by (9.18). Equation (11.5) is now equivalent to the quadratic equation:

$$1/2 v^2 \theta(1 - \theta) + \varphi_1 \theta + \varphi_2(1 - \theta) = 0. \quad (11.8)$$

Again it is clear that one root is less than zero and the other greater than one. The roots are

$$\theta_0 = \omega - \Delta \quad (11.9)$$

and

$$\theta_1 = \omega + \Delta, \quad (11.10)$$

where

$$\omega = \frac{1}{2} + \frac{\varphi_1}{v^2} - \frac{\varphi_2}{v^2} \quad (11.11)$$

and

$$\begin{aligned} \Delta &= \sqrt{\omega^2 + \frac{2\varphi_2}{v^2}} \\ &= \sqrt{\frac{1}{4} + \frac{\varphi_1}{v^2} + \frac{\varphi_2}{v^2} + \left(\frac{\varphi_1 - \varphi_2}{v^2}\right)^2}. \end{aligned} \quad (11.12)$$

Note that Δ is symmetric with respect to the parameters of the two stocks, but ω is not.

The roots θ_0 and θ_1 do not depend on the force of interest δ . Also, $\theta_0 + \theta_1 = 1$ if and only if $\varphi_1 = \varphi_2$.

We are interested in the pricing of a perpetual American option whose payoff is $\Pi(S_1(t), S_2(t))$ if it is exercised at time t . Its price is the supremum, taken over all stopping times T , of

$$E[e^{-\delta T} \Pi(S_1(T), S_2(T)); \mathbf{h}^*].$$

We assume that the payoff function $\Pi(s_1, s_2)$ is nonnegative and homogeneous of degree 1. Thus

$$\Pi(s_1, s_2) = s_2 \pi\left(\frac{s_1}{s_2}\right),$$

where

$$\pi(x) = \Pi(x, 1). \quad (11.13)$$

Examples are

$$\Pi(s_1, s_2) = (s_1 - s_2)_+,$$

which is the payoff function for the Margrabe option,

$$\Pi(s_1, s_2) = \text{Max}(s_1, s_2),$$

the payoff function for the maximum option, and

$$\Pi(s_1, s_2) = |s_1 - s_2|,$$

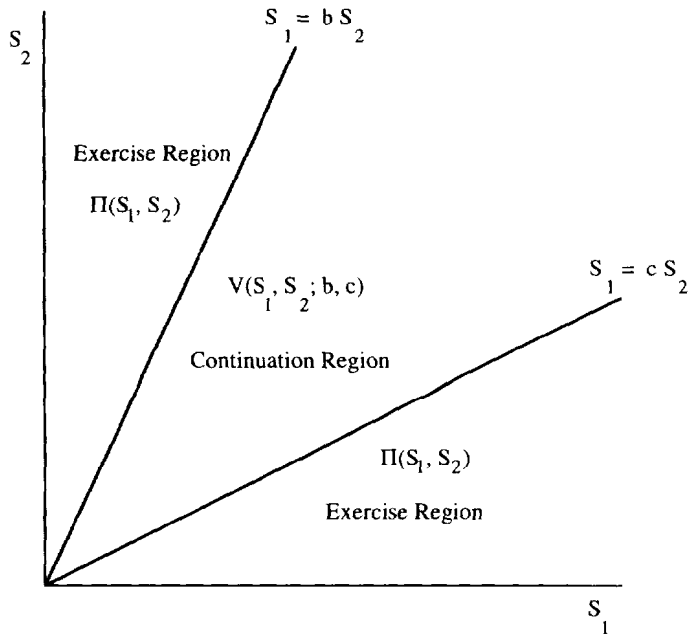
the payoff function for the symmetric Margrabe option.

Because of the homogeneity assumption it suffices to consider stopping strategies where the decision to exercise the option or not at any time t depends only on the ratio of $S_1(t)$ to $S_2(t)$. Then, under some fairly general conditions, we can restrict ourselves to stopping times of the form

$$T_{b,c} = \inf \left\{ t \mid \frac{S_1(t)}{S_2(t)} = b \text{ or } \frac{S_1(t)}{S_2(t)} = c \right\}, \quad (11.14)$$

$0 < b < S_1(0)/S_2(0) < c$. For simplicity we write $S_1 = S_1(0)$ and $S_2 = S_2(0)$. The value of the option-exercise strategy is

$$V(S_1, S_2; b, c) = E[e^{-\delta T_{b,c}} \Pi(S_1(T_{b,c}), S_2(T_{b,c})); \mathbf{h}^*]. \quad (11.15)$$



With the definitions

$$\beta(S_1, S_2; b, c) = E(e^{-\delta T_{b,c}} S_2(T_{b,c}) I[S_1(T_{b,c}) = bS_2(T_{b,c})]; \mathbf{h}^*) \quad (11.16)$$

and

$$\gamma(S_1, S_2; b, c) = E(e^{-\delta T_{b,c}} S_2(T_{b,c}) I[S_1(T_{b,c}) = cS_2(T_{b,c})]; \mathbf{h}^*), \quad (11.17)$$

(11.15) becomes

$$V(S_1, S_2; b, c) = \pi(b)\beta(S_1, S_2; b, c) + \pi(c)\gamma(S_1, S_2; b, c). \quad (11.18)$$

To determine the expectations $\beta = \beta(S_1, S_2; b, c)$ and $\gamma = \gamma(S_1, S_2; b, c)$, we stop the two martingales (11.6) at time $T_{b,c}$ and apply the *optional sampling theorem*. This leads to the equations

$$S_2(S_1/S_2)^{\theta_0} = \beta b^{\theta_0} + \gamma c^{\theta_0}$$

and

$$S_2(S_1/S_2)^{\theta_1} = \beta b^{\theta_1} + \gamma c^{\theta_1}.$$

Their solution is

$$\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} b^{\theta_0} & c^{\theta_0} \\ b^{\theta_1} & c^{\theta_1} \end{pmatrix}^{-1} \begin{pmatrix} S_2(S_1/S_2)^{\theta_0} \\ S_2(S_1/S_2)^{\theta_1} \end{pmatrix}. \quad (11.19)$$

The optimal option-exercise ratios $b = \tilde{b}$ and $c = \tilde{c}$ are obtained from the first order conditions

$$V_b(S_1, S_2; \tilde{b}, \tilde{c}) = 0 \quad (11.20)$$

and

$$V_c(S_1, S_2; \tilde{b}, \tilde{c}) = 0, \quad (11.21)$$

where the subscripts denote partial differentiation. Here we assume that $\tilde{b} > 0$ and $\tilde{c} < \infty$.

We shall see that \tilde{b} and \tilde{c} depend on neither S_1 nor S_2 . Since matrix notation facilitates the further discussion of (11.20) and (11.21), we define the vector-valued functions:

$$\mathbf{f}(S_1, S_2; b, c) = (\beta(S_1, S_2; b, c), \gamma(S_1, S_2; b, c))', \quad (11.22)$$

$$\mathbf{g}(x) = (x^{\theta_0}, x^{\theta_1})' \quad (11.23)$$

and

$$\mathbf{h}(x_1, x_2) = x_2 \mathbf{g}\left(\frac{x_1}{x_2}\right). \quad (11.24)$$

With these definitions, we can rewrite (11.19) as

$$\mathbf{f}(S_1, S_2; \mathbf{b}, \mathbf{c}) = (\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} \mathbf{h}(S_1, S_2), \quad (11.25)$$

and (11.18) as

$$\mathbf{V}(S_1, S_2; \mathbf{b}, \mathbf{c}) = (\pi(\mathbf{b}) \ \pi(\mathbf{c})) \mathbf{f}(S_1, S_2; \mathbf{b}, \mathbf{c}). \quad (11.26)$$

Hence

$$\mathbf{V}(S_1, S_2; \mathbf{b}, \mathbf{c}) = (\pi(\mathbf{b}) \ \pi(\mathbf{c})) (\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} \mathbf{h}(S_1, S_2). \quad (11.27)$$

To find the partial derivatives of \mathbf{V} with respect to \mathbf{b} and \mathbf{c} , we need the partial derivatives of the inverse of the matrix $(\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))$. Let \mathbf{A} be an invertible matrix with elements that are functions of a parameter. If we differentiate the identity

$$\mathbf{A}^{-1} \mathbf{A} = \mathbf{I}$$

with respect to the parameter, we get

$$(\mathbf{A}^{-1})' \mathbf{A} + \mathbf{A}^{-1} \mathbf{A}' = \mathbf{0},$$

or

$$(\mathbf{A}^{-1})' = -\mathbf{A}^{-1} \mathbf{A}' \mathbf{A}^{-1}.$$

Thus, differentiating (11.25) with respect to \mathbf{b} yields

$$\begin{aligned} \mathbf{f}_{\mathbf{b}}(S_1, S_2; \mathbf{b}, \mathbf{c}) &= -(\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} (\mathbf{g}'(\mathbf{b}) \ \mathbf{0}) (\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} \mathbf{h}(S_1, S_2) \\ &= -(\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} (\mathbf{g}'(\mathbf{b}) \ \mathbf{0}) \mathbf{f}(S_1, S_2; \mathbf{b}, \mathbf{c}) \\ &= -\beta(S_1, S_2; \mathbf{b}, \mathbf{c}) (\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} \mathbf{g}'(\mathbf{b}). \end{aligned} \quad (11.28)$$

It now follows from (11.26) that

$$\begin{aligned} \mathbf{V}_{\mathbf{b}}(S_1, S_2; \mathbf{b}, \mathbf{c}) &= (\pi'(\mathbf{b}) \ \mathbf{0}) \mathbf{f} + (\pi(\mathbf{b}) \ \pi(\mathbf{c})) \mathbf{f}_{\mathbf{b}} \\ &= \beta(S_1, S_2; \mathbf{b}, \mathbf{c}) [\pi'(\mathbf{b}) - (\pi(\mathbf{b}) \ \pi(\mathbf{c})) (\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} \mathbf{g}'(\mathbf{b})]. \end{aligned} \quad (11.29)$$

Similarly,

$$\mathbf{V}_{\mathbf{c}}(S_1, S_2; \mathbf{b}, \mathbf{c}) = \gamma(S_1, S_2; \mathbf{b}, \mathbf{c}) [\pi'(\mathbf{c}) - (\pi(\mathbf{b}) \ \pi(\mathbf{c})) (\mathbf{g}(\mathbf{b}) \ \mathbf{g}(\mathbf{c}))^{-1} \mathbf{g}'(\mathbf{c})]. \quad (11.30)$$

Because $\beta > 0$ and $\gamma > 0$, the first order conditions (11.20) and (11.21) are equivalent to the matrix equation

$$(\pi(\tilde{\mathbf{b}}) \ \pi(\tilde{\mathbf{c}})) (\mathbf{g}(\tilde{\mathbf{b}}) \ \mathbf{g}(\tilde{\mathbf{c}}))^{-1} (\mathbf{g}'(\tilde{\mathbf{b}}) \ \mathbf{g}'(\tilde{\mathbf{c}})) = (\pi'(\tilde{\mathbf{b}}) \ \pi'(\tilde{\mathbf{c}})). \quad (11.31)$$

With (11.31) we can determine the optimal option-exercise ratios \tilde{b} and \tilde{c} , which depend on neither S_1 nor S_2 . The price of the perpetual option is

$$\begin{cases} V(S_1, S_2; \tilde{b}, \tilde{c}) & \tilde{b} \leq S_1/S_2 \leq \tilde{c} \\ \Pi(S_1, S_2) & \text{otherwise} \end{cases}, \quad (11.32)$$

where V is given by (11.27).

12. High Contact or Smooth Pasting Condition

The first order conditions (11.20) and (11.21) are closely related to the *high contact condition* in the finance literature [Sa65] and the *smooth pasting condition* in the optimal stopping literature ([Sh78], [SS93]). Also see Dixit [Di93]. In the present context, it means that the gradients of the option-price function $V(\cdot, \cdot; \tilde{b}, \tilde{c})$ and the payoff function $\Pi(\cdot, \cdot)$ coincide on the optimal option-exercise boundaries

$S_1 = \tilde{b}S_2$ and $S_1 = \tilde{c}S_2$, i.e., for $S_1 > 0, S_2 > 0$,

$$V_{S_1}(\tilde{b}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_1}(\tilde{b}S_2, S_2), \quad (12.1)$$

$$V_{S_2}(\tilde{b}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_2}(\tilde{b}S_2, S_2), \quad (12.2)$$

$$V_{S_1}(\tilde{c}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_1}(\tilde{c}S_2, S_2) \quad (12.3)$$

and

$$V_{S_2}(\tilde{c}S_2, S_2; \tilde{b}, \tilde{c}) = \Pi_{S_2}(\tilde{c}S_2, S_2). \quad (12.4)$$

To see this, let \mathbf{v} denote the row vector

$$(\pi(\tilde{b}) \quad \pi(\tilde{c}))(\mathbf{g}(\tilde{b}) \quad \mathbf{g}(\tilde{c}))^{-1}, \quad (12.5)$$

which depends on neither S_1 nor S_2 . Then

$$V(S_1, S_2; \tilde{b}, \tilde{c}) = \mathbf{v}\mathbf{h}(S_1, S_2), \quad (12.6)$$

and (11.31) becomes

$$\mathbf{v}(\mathbf{g}'(\tilde{b}) \quad \mathbf{g}'(\tilde{c})) = (\pi'(\tilde{b}) \quad \pi'(\tilde{c})). \quad (12.7)$$

Because

$$\Pi(S_1, S_2) = S_2\pi(S_1/S_2)$$

and

$$\mathbf{h}(S_1, S_2) = S_2 \mathbf{g}(S_1/S_2),$$

we have

$$\Pi_{S_1}(S_1, S_2) = \pi'(S_1/S_2)$$

and

$$\mathbf{h}_{S_1}(S_1, S_2) = \mathbf{g}'(S_1/S_2).$$

Now (12.1) and (12.3) follow from (12.6) and (12.7).

Similarly, (12.2) and (12.4) can be obtained from (12.6), (12.7),

$$\Pi_{S_2}(S_1, S_2) = \pi(S_1/S_2) - (S_1/S_2)\pi'(S_1/S_2),$$

$$\mathbf{h}_{S_2}(S_1, S_2) = \mathbf{g}(S_1/S_2) - (S_1/S_2)\mathbf{g}'(S_1/S_2)$$

and

$$\mathbf{v}(\mathbf{g}(\tilde{\mathbf{b}}) \mathbf{g}(\tilde{\mathbf{c}})) = (\pi(\tilde{\mathbf{b}}) \pi(\tilde{\mathbf{c}})).$$

Remark. We note that the common gradient, along the line $S_1 = \tilde{\mathbf{b}}S_2$, is the constant vector

$$(\pi'(\tilde{\mathbf{b}}), \pi(\tilde{\mathbf{b}}) - \tilde{\mathbf{b}}\pi'(\tilde{\mathbf{b}})),$$

and, on the line $S_1 = \tilde{\mathbf{c}}S_2$, is the constant vector

$$(\pi'(\tilde{\mathbf{c}}), \pi(\tilde{\mathbf{c}}) - \tilde{\mathbf{c}}\pi'(\tilde{\mathbf{c}})).$$

13. Perpetual Margrabe Option

An interesting limiting case of (11.32) is the pricing formula for the perpetual American Margrabe option. Here, $\tilde{\mathbf{b}} = 0$ and

$$\tilde{\mathbf{c}} = \frac{\theta_1}{\theta_1 - 1}. \quad (13.1)$$

The current price of the option is

$$\begin{aligned}
W(S_1, S_2) &= \begin{cases} V(S_1, S_2; 0, \bar{c}) & \text{if } S_1/S_2 < \bar{c} \\ S_1 - S_2 & \text{if } S_1/S_2 \geq \bar{c} \end{cases} \\
&= \begin{cases} \left(\frac{S_1}{\theta_1}\right)^{\theta_1} / \left(\frac{S_2}{\theta_1 - 1}\right)^{\theta_1 - 1} & \text{if } \frac{S_1}{\theta_1} < \frac{S_2}{\theta_1 - 1} \\ S_1 - S_2 & \text{if } \frac{S_1}{\theta_1} \geq \frac{S_2}{\theta_1 - 1} \end{cases}. \quad (13.2)
\end{aligned}$$

We remark that special cases of the Margrabe option are the call and put options. Also, $\{X(t)\}$ need not be a Wiener process for the pricing formula (13.2) to be valid: let $\{X(t)\}$ be a process with stationary and independent increments, satisfying (8.9); if $\{X_1(t) - X_2(t)\}$ is a skip-free (jump-free) upward process, then (13.2) is a pricing formula for the perpetual American Margrabe option.

14. Forward and Futures Price of the Perpetual Margrabe Option

We conclude this paper by deriving the forward price and futures price of the perpetual Margrabe option. Because of the constant interest rate assumption, these two prices are the same. The current price of the perpetual Margrabe option is given by (13.2),

$$W(S_1, S_2) = \Theta S_1^{\theta_1} S_2^{1 - \theta_1} I(S_1/S_2 \leq \bar{c}) + (S_1 - S_2) I(\bar{c} < S_1/S_2), \quad (14.1)$$

where S_1 and S_2 are the current stock prices and

$$\Theta = \frac{(\theta_1 - 1)^{\theta_1 - 1}}{\theta_1^{\theta_1}}. \quad (14.2)$$

The m -year forward/futures price of the perpetual option is

$$\begin{aligned}
E[W(S_1(m), S_2(m)); \mathbf{h}^*] &= \Theta E[S_1(m)^{\theta_1} S_2(m)^{1 - \theta_1} I(S_1(m)/S_2(m) \leq \bar{c}); \mathbf{h}^*] \\
&\quad + E[(S_1(m) - S_2(m)) I(\bar{c} < S_1(m)/S_2(m)); \mathbf{h}^*]. \quad (14.3)
\end{aligned}$$

Applying the factorization formula (8.15) and that $\{e^{-\delta t} S_1(t)^{\theta_1} S_2(t)^{1 - \theta_1}\}$ is a martingale under the risk-neutral measure, we have

$$\begin{aligned}
& E[S_1(m)^{\theta_1} S_2(m)^{1-\theta_1} I(S_1(m)/S_2(m) \leq \tilde{c}); \mathbf{h}^*] \\
&= E[S_1(m)^{\theta_1} S_2(m)^{1-\theta_1}; \mathbf{h}^*] E[I(S_1(m)/S_2(m) \leq \tilde{c}); \mathbf{h}^* + (\theta_1, 1 - \theta_1)'] \\
&= e^{\delta m} S_1^{\theta_1} S_2^{1-\theta_1} \Pr[S_1(m)/S_2(m) \leq \tilde{c}; \mathbf{h}^* + (\theta_1, 1 - \theta_1)']. \tag{14.4}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& E[(S_1(m) - S_2(m)) I(\tilde{c} < S_1(m)/S_2(m)); \mathbf{h}^*] \\
&= E[S_1(m); \mathbf{h}^*] E[I(\tilde{c} < S_1(m)/S_2(m)); \mathbf{h}^* + (1, 0)'] \\
&\quad - E[S_2(m); \mathbf{h}^*] E[I(\tilde{c} < S_1(m)/S_2(m)); \mathbf{h}^* + (0, 1)'] \\
&= e^{(\delta - \varphi_1)m} S_1 \Pr[\tilde{c} < S_1(m)/S_2(m); \mathbf{h}^* + (1, 0)'] \\
&\quad - e^{(\delta - \varphi_2)m} S_2 \Pr[\tilde{c} < S_1(m)/S_2(m); \mathbf{h}^* + (0, 1)']. \tag{14.5}
\end{aligned}$$

Let us illustrate (14.3), (14.4) and (14.5) with a perpetual put option on a non-dividend-paying stock with a constant exercise price K . Thus we consider

$$S_1(t) \equiv K, \sigma_1 = 0, \varphi_1 = \delta,$$

$$S_2(t) \equiv S(t), \sigma_2 = \sigma, \varphi_2 = 0.$$

[Recall that $S(t) = S(0)e^{X(t)}$, where $\{X(t)\}$ is a Wiener process with variance per unit time σ^2 .] Equation (11.8) simplifies as

$$1/2\sigma^2\theta(1-\theta) + \delta\theta = 0, \tag{14.6}$$

yielding

$$\theta_1 = 1 + \frac{2\delta}{\sigma^2}. \tag{14.7}$$

By (13.1)

$$\tilde{c} = \frac{\sigma^2 + 2\delta}{2\delta}. \tag{14.8}$$

With the definition

$$\kappa = \ln\left(\frac{K}{S(0)\tilde{c}}\right), \tag{14.9}$$

equation (14.5) becomes

$$\begin{aligned}
& E[(K - S(m)) I(\tilde{c} < K/S(m)); h^*] \\
&= K\Pr[X(m) < \kappa; h^*] - e^{\delta m}S(0)\Pr[X(m) < \kappa; h^* + 1] \\
&= K\Phi\left(\frac{\kappa - (\delta - \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) - e^{\delta m}S(0)\Phi\left(\frac{\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right), \tag{14.10}
\end{aligned}$$

where Φ is the standardized normal distribution.

The probability term in (14.4) is

$$\begin{aligned}
\Pr[X(m) \geq \kappa; h^* + 1 - \theta_1] &= \Pr[X(m) \geq \kappa; h^* - \frac{2\delta}{\sigma^2}] \\
&= 1 - \Phi\left(\frac{\kappa - (-\delta - \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) \\
&= \Phi\left(\frac{-\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right). \tag{14.11}
\end{aligned}$$

Also,

$$\begin{aligned}
\Theta K^{\theta_1} S(0)^{1-\theta_1} &= \frac{K}{\theta_1} \left(\frac{K}{S(0)\tilde{c}}\right)^{\theta_1-1} \\
&= \frac{\sigma^2 K}{\sigma^2 + 2\delta} \exp(2\delta\kappa/\sigma^2). \tag{14.12}
\end{aligned}$$

It follows from (14.3), (14.4), (14.11), (14.12) and (14.10) that the m -year futures/forward price of the perpetual Margrabe option is:

$$\begin{aligned}
& \frac{\sigma^2 K}{\sigma^2 + 2\delta} \exp[\delta(m + \frac{2\kappa}{\sigma^2})] \Phi\left(\frac{-\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) \\
&+ K\Phi\left(\frac{\kappa - (\delta - \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right) - S(0)e^{\delta m} \Phi\left(\frac{\kappa - (\delta + \frac{\sigma^2}{2})m}{\sigma\sqrt{m}}\right). \tag{14.13}
\end{aligned}$$

Another derivation of (14.13) can be found in Gerber and Shiu [GS93].

Acknowledgment

Elias Shiu gratefully acknowledges the support from the Principal Financial Group.

References

- [As87] Asmussen, S. *Applied Probability and Queues*. New York: Wiley, 1987.
- [Ba91] Back, K. "Asset Pricing for General Processes," *Journal of Mathematical Economics* 20 (1991): 371-95.
- [BP91] Back, K., and Pliska, S.R. "On the Fundamental Theorem of Asset Pricing with an Infinite State Space," *Journal of Mathematical Economics* 20 (1991): 1-18.
- [vB74] von Bahr, B. "Ruin Probabilities Expressed in Terms of Ladder Height Distributions," *Scandinavian Actuarial Journal* (1974): 190-204.
- [B-N94] Barndorff-Nielsen, O.E. "Gaussian\Inverse Gaussian Processes and the Modelling of Stock Returns," Working Paper, Aarhus University, 1994.
- [Bi87] Bick, A. "On the Consistency of the Black-Scholes Model with a General Equilibrium Framework," *Journal of Financial and Quantitative Analysis* 22 (1987): 259-75.
- [Bi90] Bick, A. "On Viable Diffusion Price Processes of the Market Portfolio," *Journal of Finance* 45 (1990): 673-89.
- [BC89] Bhattacharya, S., and Constantinides, G. (ed.) *Theory of Valuation: Frontier of Modern Financial Theory*, Volume 1. Totowa, N.J.: Rowman & Littlefield, 1989.
- [BS73] Black, F., and Scholes, M. "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (1973): 637-59.
- [BGHJN86] Bowers, N.L., Jr., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. *Actuarial Mathematics*. Itasca, Illinois: Society of Actuaries, 1986.
- [Bo60] Borch, K. "The Safety Loading of Reinsurance Premiums," *Skandinavisk Aktuarietidskrift* (1960): 163-184. Reprinted in [Bo90], 61-81.
- [Bo90] Borch, K. *Economics of Insurance*. Amsterdam: Elsevier, 1990.
- [Bü80] Bühlmann, H. "An Economic Premium Principle," *ASTIN Bulletin* 11 (1980): 52-60.

- [Bü84] Bühlmann, H. "The General Economic Premium Principle," *ASTIN Bulletin* 14 (1984): 13-21.
- [CM94] Chevallier, E., and Müller, H.H. "Risk Allocation in Capital Markets: Portfolio Insurance, Tactical Asset Allocation and Collar Strategies," *ASTIN Bulletin* 24 (1994): 5-18.
- [Co89] Constantinides, G.M. "Theory of Valuation: Overview and Recent Developments." In [BC89], 1-23.
- [CH89] Cox, J.C., and Huang, C.F. "Option Pricing Theory and Its Applications." In [BC89], 272-88.
- [CR76] Cox, J.C., and Ross, S.A. "The Valuation of Options for Alternative Stochastic Processes," *Journal of Financial Economics* 3 (1976): 145-66.
- [CMW90] Dalang, R.C., Morton, A., and Willinger, W. "Equivalent Martingale Measures and No-Arbitrage in Stochastic Securities Market Models," *Stochastics and Stochastic Reports* 29 (1990): 185-201.
- [De92] Delbaen, F. "Representing Martingale Measures When Asset Prices Are Continuous and Bounded," *Mathematical Finance* 2 (1992): 107-30.
- [DH89] Delbaen, F., and Haezendonck, J. "A Martingale Approach to Premium Calculation Principles in an Arbitrage Free Market," *Insurance: Mathematics and Economics* 8 (1989): 269-77.
- [DS94a] Delbaen, F., and Schachermayer, W. "Arbitrage and Free Lunch with Bounded Risk for Unbounded Continuous Processes," *Mathematical Finance* 4 (1994): 343-48.
- [DS94b] Delbaen, F., and Schachermayer, W. "A General Version of the Fundamental Theorem of Asset Pricing," *Mathematische Annalen*, in press.
- [Di93] Dixit, A.K. (1993): *The Art of Smooth Pasting*. Reading, U.K.: Harwood.
- [DR87] Dybvig, P.H., and Ross, S.A. "Arbitrage." In *The New Palgrave: A Dictionary of Economics*, Vol 1, edited by J. Eatwell, M. Milgate and P. Newman. Macmillan, London (1987): 100-106.

- [Es32] Esscher, F. "On the Probability Function in the Collective Theory of Risk," *Skandinavisk Aktuarietidskrift* 15 (1932): 175-95.
- [Fi78] Fischer, S. (1978): "Call Option Pricing When the Exercise Price Is Uncertain, and the Value of Index Bonds," *Journal of Finance*, 33, 169-176.
- [GER94] Geman, H., El Karoui, N., and Rochet, J.C. "Changes of Numéraire, Changes of Probability Measure and Option Pricing," *Journal of Applied Probability*, in press.
- [Ge87] Gerber, H.U. "Actuarial Applications of Utility Functions." In *Advances in the Statistical Sciences*, Vol VI: *Actuarial Science*, edited by I.B. MacNeill and G.J. Umphrey. Dordrecht, Holland: Reidel, 1987, 53-61.
- [GMS95] Gerber, H.U., Michaud, F., and Shiu, E.S.W. "Pricing Russian Options with the Compound Poisson Process," *Transactions of the 25th International Congress of Actuaries* (1995), in press.
- [GS93] Gerber, H.U., and Shiu, E.S.W. Discussion of J.A. Tilley's "Valuing American Options in a Path Simulation Model," *Transactions of the Society of Actuaries* 45 (1993): 524-35.
- [GS94a] Gerber, H.U., and Shiu, E.S.W. "Option Pricing by Esscher Transforms," *Transactions of the Society of Actuaries* 46 (1994), in press.
- [GS94b] Gerber, H.U., and Shiu, E.S.W. "Martingale Approach to Pricing Perpetual American Options," *ASTIN Bulletin* 24 (1994): 195-220.
- [GS94c] Gerber, H.U., and Shiu, E.S.W. "From Perpetual Strangles to Russian Options," *Insurance: Mathematics and Economics* 15 (1994), in press.
- [GS94d] Gerber, H.U., and Shiu, E.S.W. "Pricing Financial Contracts with Indexed Homogeneous Payoff," *Bulletin of the Swiss Association of Actuaries* 94 (1994): 143-66.
- [GS95] Gerber, H.U., and Shiu, E.S.W. "Martingale Approach to Pricing Perpetual American Options on Two Stocks," submitted for publication.

- [HK79] Harrison, J.M., and Kreps, D.M. "Martingales and Arbitrage in Multiperiod Securities Markets," *Journal of Economic Theory* 20 (1979): 381-408.
- [HP81] Harrison, J.M., and Pliska, S. "Martingales and Stochastic Integrals in the Theory of Continuous Trading," *Stochastic Processes and Their Applications* 11 (1981): 215-60.
- [HL93] He, H., and Leland, H. "On Equilibrium Asset Price Processes," *Review of Financial Studies* 6 (1993): 593-617.
- [He93] Heston, S.L. "Invisible Parameters in Option Prices," *Journal of Finance* 48 (1993): 933-47.
- [Je91] Jensen, J.L. "Saddlepoint Approximations to the Distribution of the Total Claim Amount in Some Recent Risk Models," *Scandinavian Actuarial Journal* (1991): 154-68.
- [Kr81] Kreps, D.M. "Arbitrage and Equilibrium in Economics with Infinitely Many Commodities," *Journal of Mathematical Economics* 8 (1981): 15-35.
- [Li86] Lienhard, M. "Calculation of Price Equilibria for Utility Functions of the HARA Class," *ASTIN Bulletin* 16 (1986): S91-S97.
- [Ma78] Margrabe, W. "The Value of an Option to Exchange One Asset for Another," *Journal of Finance* 33 (1978): 177-86.
- [Mü89] Müller, S.M. "On Complete Securities Markets and the Martingale Property of Securities Prices," *Economic Letters* 31 (1989): 37-41.
- [NL90] Naik, V., and Lee, M. "General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns," *Review of Financial Studies* 3 (1990): 493-521.
- [Pa77] Parkinson, M. "Option Pricing: The American Put," *Journal of Business* 50 (1977): 21-36.
- [Ru76] Rubinstein, M. "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics* 7 (1976): 407-25.

- [Sa65] Samuelson, P.A. "Rational Theory of Warrant Pricing," *Industrial Management Review* 6, issue 2 (1965): 13-32.
- [Sc92a] Schachermayer, W. "A Hilbert Space Proof of the Fundamental Theorem of Asset Pricing in Finite Discrete Time," *Insurance: Mathematics and Economics* 11 (1992): 249-57.
- [Sc92b] Schachermayer, W. "A Counterexample to Several Problems in the Theory of Asset Pricing," *Mathematical Finance* 3 (1992): 217-29.
- [Sc94] Schachermayer, W. "Martingale Measures for Discrete-Time Processes with Infinite Horizon," *Mathematical Finance* 4 (1994): 25-55.
- [Scw92] Schweizer, M. "Martingale Densities for General Asset Prices," *Journal of Mathematical Economics* 21 (1992): 363-78.
- [SS93] Shepp, L., and Shiryaev, A.N. "The Russian Option: Reduced Regret," *Annals of Applied Probability* 3 (1993): 631-640.
- [Sh78] Shirayayev, A.N. (1978): *Optimal Stopping Rules*. New York: Springer-Verlag.
- [Si75] Siegmund, D. "The Time Until Ruin in Collective Risk Theory," *Mitteilungen der Vereinigung schweizerischer Versicherungsmathematiker* 75 (1975): 157-66.
- [So91] Sonderman, D. "Reinsurance in Arbitrage-Free Markets," *Insurance: Mathematics and Economics* 10 (1991): 191-202.
- [SSu90] Stapleton, R.C., and Subrahmanyam, M.G. "Risk Aversion and the Intertemporal Behavior of Asset Prices," *Review of Financial Studies* 3 (1990): 677-93.
- [St93] Stricker, C. "Some Remarks on Hedging of Contingent Claims." In *Stochastic Process and Optimal Control*, edited by Engelbert and I. Karatzas. New York: Gordon & Breach, 1993, 177-81.
- [Wa93] Wang, S. "The Integrability Problem of Asset Prices," *Journal of Economic Theory* 59 (1993): 199-213.