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# Some Remarks on Statistical Independence and Fractional Age Assumptions

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**Abstract**

In this expository paper several comments are made with respect to the statistical independence of the curtate future lifetime and the fractional part of the future lifetime, both of a general status. In particular, the conditions for independence need to be stated carefully. The last-survivor status is cited as an example. More general assumptions than the uniform distribution of deaths assumption are then considered, together with applications to insurances, annuities, and reserves. Similar results to those under the uniform distribution of deaths assumption are obtained. Brief comments are made with respect to contingent probabilities in multiple lives and multiple decrement theory.

## 1. Introduction

Consider a general status $(u)$ and its future lifetime random variable $T$. Let ${}_tp_u = Pr(T > t), t \geq 0$, and ${}_tq_u = 1 -{}_t p_u$, with the convention that the leading subscript $t$ is suppressed when $t = 1$. Let the curtate future lifetime be $K = [T]$, and the fractional portion of $T$ be $S = T - [T]$, i.e. $T = K + S$. Assumptions with respect to the joint distribution of $K$ and $S$ are critical for actuarial and demographic analysis within a life table context. See Bowers et al. (1986, chapters 3, 8) for a detailed discussion of these ideas. See also Shiu (1982) and references therein.

In particular, statistical independence of $K$ and $S$ leads to a much simplified analysis of many problems of interest. In this case one has

$$Pr(K = k, S \leq s) = Pr(K = k)Pr(S \leq s) \tag{1}$$

for $k = 0, 1, 2, \ldots$, and $0 \leq s \leq 1$. Stated another way after division of both sides by $Pr(K = k)$, one has

$$Pr(S \leq s|K = k) = Pr(S \leq s) . \tag{2}$$

Conversely, if for some function $H(s)$ not depending on $k$, the relation

$$Pr(S \leq s|K = k) = H(s) \tag{3}$$

holds for all $k$, then

$$\begin{aligned} Pr(S \leq s) &= \sum_{k=0}^{\infty} Pr(S \leq s|K = k)Pr(K = k) \\ &= \sum_{k=0}^{\infty} H(s)Pr(K = k) \\ &= H(s) \sum_{k=0}^{\infty} Pr(K = k) \\ &= H(s) . \end{aligned}$$

That is,

$$H(s) = Pr(S \leq s), \ \ 0 \leq s \leq 1, \tag{4}$$

and so $H(s)$ in (3) must be a distribution function on $(0, 1)$. Thus, if (3) holds,

$$\begin{aligned} Pr(K = k, S \leq s) &= Pr(S \leq s | K = k) Pr(K = k) \\ &= H(s) Pr(K = k) \\ &= Pr(S \leq s) Pr(K = k) \ , \end{aligned}$$

and so $K$ and $S$ must be statistically independent.

To summarize, $K$ and $S$ are independent if and only if

$$Pr(S \leq s | K = k) = \frac{{}_kp_u - {}_{k+s}p_u}{{}_kp_u - {}_{k+1}p_u} = H(s) \tag{5}$$

for all $k = 0, 1, 2, \ldots$, and $0 \leq s \leq 1$. One needs to be careful in some cases in consideration of this independence. In particular, it is not sufficient that ${}_sq_u / q_u$ be a function of $s$ only without additional conditions (exercise 3.40 of Bowers et al. (1986, p. 81) considers this criterion).

*Example 1 - The last survivor status*

Consider the last survivor status $(\overline{xy})$ which fails on the second death of the two lives aged $(x)$ and $(y)$. If the future lifetimes $T(x)$ and $T(y)$ are independent and satisfy (5) with $H(s) = s$ (i.e. the uniform distribution of deaths assumption (UDD) holds for each life), then ${}_sq_x = s \cdot q_x$, ${}_sq_y = s \cdot q_y$, and so ${}_sq_{\overline{xy}} = s^2 \cdot q_{\overline{xy}}$ for the last survivor status. However, $K = K(\overline{xy})$ and $S = S(\overline{xy})$ fail to be independent since (5) does not hold.

To see this, note that in this case one has from (5)

$$\begin{aligned} \frac{{}_kp_{\overline{xy}} - {}_{k+s}p_{\overline{xy}}}{{}_kp_{\overline{xy}} - {}_{k+1}p_{\overline{xy}}} &= \frac{{}_kp_x + {}_kp_y - {}_kp_{xy} - {}_{k+s}p_x - {}_{k+s}p_y + {}_{k+s}p_{xy}}{{}_kp_x + {}_kp_y - {}_kp_{xy} - {}_{k+1}p_x - {}_{k+1}p_y + {}_{k+1}p_{xy}} \\ &= \frac{{}_kp_x \cdot {}_sq_{x+k} + {}_kp_y \cdot {}_sq_{y+k} - {}_kp_{xy} \cdot {}_sq_{x+k:y+k}}{{}_kp_x \cdot q_{x+k} + {}_kp_y \cdot q_{y+k} - {}_kp_{xy} \cdot q_{x+k:y+k}} \\ &= \frac{s\{{}_kp_x q_{x+k} + {}_kp_y q_{y+k}\} - {}_kp_{xy}\{s(q_{x+k} + q_{y+k}) - s^2 q_{x+k} q_{y+k}\}}{\{{}_kp_x q_{x+k} + {}_kp_y q_{y+k}\} - {}_kp_{xy}\{(q_{x+k} + q_{y+k}) - q_{x+k} q_{y+k}\}} \end{aligned}$$

The difficulty in this example is that ${}_{k+s}p_{\overline{xy}} \neq {}_kp_{\overline{xy}} \cdot {}_sp_{\overline{x+k:y+k}}$. □

We shall say that the fractional independence (FI) assumption holds if (5) is satisfied.

## 2. The fractional independence assumption

Consider the ordinary life table so that the status of interest is the future lifetime $T = T(x)$ of a life aged $x$. Since ${}_{k+s}p_x = {}_k p_x \cdot {}_s p_{x+k}$, the FI assumption (5) holds if

$$ {}_sq_x = H(s)q_x \tag{6} $$

for all $0 \le s \le 1, k \in \{0,1,2,\dots,\}$. As shown in section 1, $H(s) = Pr(S \le s)$ and the uniform distribution of deaths (UDD) assumption is obtained with the special case $H(s) = s$.

It is straightforward to verify that the following relations hold if (6) holds:

$$ \mu_{x+s} = -\frac{d}{dx} ln_s p_x = \frac{H'(s)q_x}{1 - H(s)q_x} , \tag{7} $$

$$ {}_sq_{x+t} = \frac{\{H(s+t) - H(t)\} q_x}{1 - H(t)q_x}, 0 \le s,t,s+t \le 1, x = 0,1,2,\dots, \tag{8} $$

$$ {}_sp_x\mu_{x+s} = H'(s)q_x , \tag{9} $$

$$ \ell_{x+s} = \ell_x - H(s)d_x = \ell_x \{1 - H(s)\} + \ell_{x+1}H(s) . \tag{10} $$

$$ L_x = \int_0^1 \ell_{x+s}ds = \ell_x E(S) + \ell_{x+1}\{1 - E(S)\} , \tag{11} $$

$$ m_x = \frac{\ell_x - \ell_{x+1}}{L_x} = \frac{q_x}{1 - \{1 - E(S)\} q_x} , \tag{12} $$

$$ a(x) = E(T|T < 1) = E(S|K = 0) = E(S) , \tag{13} $$

$$ \overset{0}{e}_x = e_x + E(S) . \tag{14} $$

It is evident from (7) that the possible shapes under the FI assumption are much more flexible over integral age ranges than is possible under the UDD assumption. Similarly,

(10) allows for a much more general distribution of deaths over the year of age than simply uniform, as the following example indicates.

*Example 2 - The beta distribution*

A convenient parametric family of distributions for this purpose is the beta family with density

$$H'(s) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} s^{\alpha-1}(1-s)^{\beta-1} , \; 0 \le s \le 1 , \tag{15}$$

when $\alpha > 0$ and $\beta > 0$. The uniform distribution is the special case $\alpha = \beta = 1$. The mean is $E(S) = \alpha/(\alpha+\beta)$. The distribution function is (for general $\alpha$ and $\beta$)

$$H(s) = B(\alpha, \beta, s) \; , 0 \le s \le 1 \tag{16}$$

where $B(\cdot)$ is the incomplete beta function which may be evaluated numerically using a series expansion (e.g. Hogg and Klugman, 1984, p. 219). Of course, (15) may be integrated directly if $\alpha$ or $\beta$ is a positive integer. For example, if $\beta = 1$, (16) simplifies to $H(s) = s^{\alpha}$. □

There are many other examples currently used where $K$ and $S$ are independent besides the UDD assumption. De Moivre's law, for which $\ell_x = \ell_0(1 - x/w)$, $0 \le x \le w$, is a special case of the UDD assumption. If $T$ has an exponential distribution, then $\mu_x = \mu$ for $x \ge 0$, and the FI assumption holds with $H(s) = (1 - e^{-\mu s})/(1 - e^{-\mu})$, $0 \le s \le 1$. Also, discrete assumptions such as a portion of the deaths occurring at mid-year, the beginning of the year, or the end of the year are all FI assumptions and are often used in multiple decrement theory.

Thus, if a portion $\alpha$ of the deaths occur at time $t_0 \in [0,1]$ and the remainder are spread uniformly over the year, then

$$H(s) = \begin{cases} (1-\alpha)s, & 0 \le s < t_0 \\ \alpha + (1-\alpha)s, & t_0 \le s \le 1 \; . \end{cases} \tag{17}$$

If all deaths occur at midyear, then $\alpha = 1$ and $t_0 = 1/2$, etc.

## 3. Insurances

Consider an insurance which pays a death benefit $b_T$ at time $\tau_T$ if death occurs at time $T$. The present value random variable is thus

$$Z = b_T v^{\tau_T} . \tag{18}$$

Under (6), $K = [T]$ and $S = T - [T]$ are independent, so if (18) may be expressed as

$$Z = b_T v^{\tau_T} = \sum_{i=1}^{r} f_i(K) g_i(S) \tag{19}$$

for some functions $f_i(\cdot)$ and $g_i(\cdot)$, then the net single premium may be obtained easily as

$$E(Z) = \sum_{i=1}^{r} E\{f_i(K)\} E\{g_i(S)\} . \tag{20}$$

This idea is utilized consistently in Bowers et al. (1986) and is useful here as well.

Quite generally, for insurances with death benefit depending only on the policy year of death, $b_T = b_{K+1}$. Moreover, if the benefit is payable at the end of the $m$-th of death, one has $\tau_T = e^{-\delta K - \frac{\delta[mS+1]}{m}}$ where $[\cdot]$ is the greatest integer function. This factors as a function of $K$ times that of $S$. Thus, from (19),

$$A^{(m)} = E(b_{K+1} e^{-\delta K - \frac{\delta[mS+1]}{m}}) = \phi(m) A \tag{21}$$

where

$$A = E(b_{K+1} e^{-\delta(K+1)}) \tag{22}$$

is the net single premium for the corresponding insurance with death benefit payable at the end of the year of death, and

$$\phi(m) = (1+i) E\left\{e^{-\frac{\delta[mS+1]}{m}}\right\} . \tag{23}$$

One has

$$\begin{aligned}
\phi(m) &= (1+i)\int_0^1 e^{-\frac{\delta(mS+1)}{m}} H'(s)ds \\
&= (1+i)\int_0^m e^{-\frac{\delta(t+1)}{m}} H'(t/m)\tfrac{dt}{m} \\
&= (1+i)\sum_{j=0}^{m-1}\int_j^{j+1} e^{-\frac{\delta(t+1)}{m}} H'(t/m)\tfrac{dt}{m} \\
&= (1+i)\sum_{j=0}^{m-1} e^{-\frac{\delta(j+1)}{m}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} H'(s)ds \,.
\end{aligned}$$

In other words, (23) may be expressed as

$$\phi(m) = \sum_{j=0}^{m-1}(1+i)^{1-\frac{j+1}{m}}\left\{H(\frac{j+1}{m}) - H(\frac{j}{m})\right\} \,. \tag{24}$$

It is easy to show that (24) reduces to $i/i^{(m)}$ when $H(s) = s$, the well-known UDD formula. The factorization in (21) thus generalizes the UDD formula. One has, for example,

$$A_x^{(m)} = \phi(m)A_x \,. \tag{25}$$

Other special cases of (21) may be found in Bowers et. al (1986, chapter 4). As $m \to \infty$, one obtains the corresponding results for insurance payable at the moment of death. From (23),

$$\phi(\infty) = (1+i)E(e^{-\delta S}) \,, \tag{26}$$

and (21) becomes, in an obvious notation,

$$\overline{A} = \phi(\infty)A \tag{27}$$

under the FI assumption.

*Example 3 - Generalization of the uniform distribution*

Suppose that

$$H(s) = \theta s^2 + (1-\theta)\left\{1-(1-s)^2\right\}, \qquad 0 \le s \le 1$$

where $0 \leq \theta \leq 1$. The uniform distribution is the special case $\theta = 1/2$. Then $H'(s) = 2\theta s + 2(1-\theta)(1-s)$ and from (26)

$$\phi(\infty) = 2(1+i)\left\{\theta(\bar{I}\bar{a})_{\bar{1}|} + (1-\theta)(\bar{D}\bar{a})_{\bar{1}|}\right\} .$$

Thus (26) may be expressed in terms of standard actuarial functions. □

For an ordinary insurance of \$1, i.e. $b_T = 1$ payable at the moment of death of $(x)$, i.e. $\tau_T = T$, results in

$$\bar{A}(x) = E(e^{-\delta T}) = (1+i)E(e^{-\delta S})A_x \tag{28}$$

For a fully continuous and increasing insurance with $b_T = Te^{-\delta T}$,

$$(\bar{I}\,\bar{A})_x = E(Ke^{-\delta(K+1)})E(e^{-\delta(S-1)}) + E(Se^{-\delta(S-1)})E(e^{-\delta(K+1)})$$

$$= (1+i)E(e^{-\delta S})\left\{(IA)_x - A_x\right\} + (1+i)E(Se^{-\delta S})A_x \, . \tag{29}$$

Other examples may be found in Bowers et al. (1986). For $n$-year term insurances, the same relationships hold as for whole life but the functions $f_i(K)$ in (19) are replaced by $f_i(K)1_{\{K\leq n\}}$. This is because $Z$ is unchanged if $K \leq n$ but $Z = 0$ if $K > n$. Thus, for the $n$-year term benefit, (19) yields

$$Z = \sum_{i=1}^{r} \left\{f_i(K)1_{\{K\leq n\}}\right\} g_i(S) \, . \tag{30}$$

For the net single premium for (20), $E\{g_i(S)\}$ is unchanged and $E\{f_i(K)\}$ is the corresponding term expectation. For example, (21) holds for $n$-year term coverages with death benefit depending on the year of death. Similarly, (28) and (29) become, under FI

$$\bar{A}\,^{1}_{x:\overline{n}|} = (1+i)E(e^{-\delta S})A^{1}_{x:\overline{n}|} \, , \tag{31}$$

$$(\bar{I}\,\bar{A})^{1}_{x:\overline{n}|} = (1+i)E(e^{-\delta S})\left\{(IA)^{1}_{x:\overline{n}|} - A^{1}_{x:\overline{n}|}\right\} + (1+i)E(Se^{-\delta S})A^{1}_{x:\overline{n}|} \, . \tag{32}$$

## 4. Annuities

Turning now to annuities, it follows easily from the relation (25) and $A_x^{(m)} = 1 - d^{(m)}A_x$ that

$$\ddot{a}_x^{(m)} = \alpha(m)\ddot{a}_x - \beta(m) \tag{33}$$

where

$$\alpha(m) = \frac{d}{d^{(m)}}\phi(m) = \frac{i}{d^{(m)}}\sum_{j=0}^{m-1} v^{-\frac{j+1}{m}}\left\{H(\frac{j+1}{m}) - H(\frac{j}{m})\right\} . \tag{34}$$

and

$$\beta(m) = \frac{\phi(m)-1}{d^{(m)}} = \frac{1}{m}\sum_{j=0}^{m-1}(1+i)^{1-\frac{j}{m}}H(\frac{j}{m}) . \tag{35}$$

To see the right-hand side of (35), one may use summation by parts on (24). Thus,

$$v\phi(m) + \sum_{j=0}^{m-1} H(\frac{j}{m})\left\{v^{\frac{j+1}{m}} - v^{\frac{j}{m}}\right\} = \sum_{j=0}^{m-1}\left\{v^{\frac{j+1}{m}}H(\frac{j+1}{m}) - v^{\frac{j}{m}}H(\frac{j}{m})\right\} = v$$

which may be rewritten as

$$\phi(m) = 1 + \left\{\frac{1+(1+i)d^{(m)}}{m}\right\}\sum_{j=0}^{m-1} v^{\frac{j}{m}}H(\frac{j}{m}) .$$

It is interesting to note that $\beta(m)$ in (35) is the "average" value of $(1+i)^{1-\frac{j}{m}}H(\frac{j}{m})$. Using the average value $j = (m-1)/2$ and ignoring interest this becomes $H(\frac{m-1}{2m})$ which is often used under UDD, i.e. when $H(x) = x$ ( e.g. Jordan, 1967, p. 46).

Relation (33) is standard under UDD, and holds for the more general FI assumption with the more general definitions (34) and (35).

Also, letting $m \to \infty$, one obtains from (26) and (33)

$$\bar{a}_x = \frac{i}{\delta}E(e^{-\delta S})\ddot{a}_x - \frac{(1+i)E(e^{-\delta S}) - 1}{\delta} . \tag{36}$$

Using integration by parts and $a_x = \ddot{a}_x - 1$, one obtains an alternative to (36), namely

$$\bar{a}_x = \frac{i}{\delta}M_S(-\delta)a_x + \int_0^1 e^{-\delta s}\left\{1 - H(s)\right\} ds . \tag{37}$$

Note that $\lim_{i\to 0} i/\delta = \lim_{i\to 0} i/ln(1+i) = \lim_{i\to 0}(1+i) = 1$ by L'Hôpital's rule, and so (37) reduces to (14) as $i \to 0$. For temporary annuities, one may write, using (33)

$$\begin{aligned}\ddot{a}^{(m)}_{x:\overline{n}|} &= \ddot{a}^{(m)}_{x} - v^n \ {}_np_x \cdot \ddot{a}^{(m)}_{x+n} \\ &= \alpha(m)\ddot{a}_{x:\overline{n}|} - \beta(m)\left\{1 - v^n \ {}_np_x\right\} .\end{aligned}$$

But $1 - v^n \ {}_np_x = d\ddot{a}_{x:\overline{n}|} + A^{\ 1}_{x:\overline{n}|}$, and so under FI one has

$$\ddot{a}^{(m)}_{x:\overline{n}|} = \psi(m)\ddot{a}_{x:\overline{n}|} - \beta(m)A^{\ 1}_{x:\overline{n}|} \tag{38}$$

where

$$\psi(m) = \alpha(m) - d\beta(m) . \tag{39}$$

Again, (38) which is a standard formula under UDD, holds more generally under FI. Also, when $n = \infty$, (38) becomes

$$\ddot{a}^{(m)}_{x} = \psi(m)\ddot{a}_x - \beta(m)A_x .$$

## 5. Reserves

Suppose now that premiums are payable $m$-thly and benefits payable at the end of the $j$-th of the year of death. The $k$-th year terminal reserve is ${}_kV^{(m)}(A^{(j)})$. From (25), one has easily

$${}_kV^{(m)}(A^{(j)}) = \frac{\phi(j)}{\phi(m)} \ {}_kV^{(m)}(A^{(m)}) . \tag{40}$$

The relationship for semi-continuous reserves with $m = 1$ and $j = \infty$ is, from (26) and $\phi(1) = 1$,

$${}_kV(\overline{A}) = (1+i)E(e^{-\delta S})_kV . \tag{41}$$

For ordinary whole-life insurance, one has from

$${}_kV^{(m)}(A^{(m)}_x) = \left\{A^{(m)}_{x+t} - A^{(m)}_x\right\} / \left\{1 - A^{(m)}_x\right\}$$

and (40) that under FI

$$ {}_kV^{(m)}(A_x^{(j)}) = \frac{A_{x+t}^{(j)} - A_x^{(j)}}{1 - A_x^{(m)}} \ . \qquad (42) $$

The analogous relationship to (42) does not hold for endowment insurance since (21) does not hold for endowments.

Next, consider quite generally, two $h$-pay insurance policies issued to $(x)$ both providing the same (arbitrary) benefits and differing only in the payment mode. That is, the first policy has annual premiums ${}_hP$ payable at the beginning of the year while $(x)$ is alive (for $h$ years), and terminal reserves ${}^h_kV$. The second policy has premiums ${}_hP^{(m)}/m$ payable at the beginning of each $m$'th while $(x)$ is alive (for $h$ years), and terminal reserves ${}^h_kV^{(m)}$. Suppose that ${}_hP$ and ${}_hP^{(m)}$ are both calculated according to the equivalence principle. Assume that $k < h$, and consider the difference between the two reserves. Since the benefits under each policy are the same, the difference between the two reserves (viewed prospectively) is simply the difference between the present values of the premiums under each. Thus,

$$ {}^h_kV^{(m)} - {}^h_kV = {}_hP \cdot \ddot{a}_{x+k:\overline{h-k}|} - {}_hP^{(m)} \cdot \ddot{a}^{(m)}_{x+k:\overline{h-k}|} \ . $$

Clearly, since the benefits are the same, one must have
${}_hP/{}_hP^{(m)} = \ddot{a}^{(m)}_{x:\overline{h}|}/\ddot{a}_{x:\overline{h}|}$, and so

$$ {}^h_kV^{(m)} - {}^h_kV = {}_hP^{(m)} \left\{ \frac{\ddot{a}^{(m)}_{x:\overline{h}|}}{\ddot{a}_{x:\overline{h}|}} \ddot{a}_{x+k:\overline{h-k}|} - \ddot{a}^{(m)}_{x+k:\overline{h-k}|} \right\} \ . $$

From (38), one has

$$ \begin{aligned} & \frac{\ddot{a}^{(m)}_{x:\overline{h}|}}{\ddot{a}_{x:\overline{h}|}} \ddot{a}_{x+k:\overline{h-k}|} - \ddot{a}^{(m)}_{x+k:\overline{h-k}|} \\ = & \left\{ \psi(m) - \beta(m) \frac{A^1_{x:\overline{h}|}}{\ddot{a}_{x:\overline{h}|}} \right\} \ddot{a}_{x+k:\overline{h-k}|} - \left\{ \psi(m) \ddot{a}_{x+k:\overline{h-k}|} - \beta(m) A^{\ 1}_{x+k:\overline{h-k}|} \right\} \ . \\ = & \ \beta(m) \left\{ A^{\ 1}_{x+k:\overline{h-k}|} - P^1_{x:\overline{h}|} \cdot \ddot{a}_{x+k:\overline{h-k}|} \right\} \ . \end{aligned} $$

To summarize, we have under FI the simple relationship

$$ {}^{h}_{k}V^{(m)} = {}^{h}_{k}V +_{h} P^{(m)} \cdot \beta(m) \cdot_{k} V^{1}_{x:\overline{h}|} \cdot \tag{43}$$

Equation (43) is given by Shiu (1982, p. 596) under the UDD assumption.

If the policy is of the whole life type, then $h = \infty$ and (42) becomes

$$ {}_{k}V^{(m)} =_{k} V + P^{(m)} \cdot \beta(m) \cdot_{k} V_{x} \, . \tag{44}$$

Continuous reserves may be obtained with $m = \infty$. Also, as discussed in Bowers et al. (1986, p. 206), the terms on the right hand side of (43) and (44) may be viewed as an "unearned premium" reserve. Special cases include, for example,

$$ {}^{h}_{k}V^{(m)}_{x:\overline{n}|} = {}^{h}_{k}V_{x:\overline{n}|} +_{h} P^{(m)}_{x:\overline{n}|} \cdot \beta(m) \cdot {}_{k}V^{1}_{x:\overline{h}|} \, , \tag{45}$$

$$ {}_{k}\overline{V}(\overline{A}^{1}_{x:\overline{n}|}) = {}_{k}V(\overline{A}^{1}_{x:\overline{n}|}) + \overline{P}(\overline{A}^{1}_{x:\overline{n}|}) \cdot \beta(\infty) \cdot {}_{k}V^{1}_{x:\overline{n}|} \, , \tag{46}$$

$$ {}_{k}V^{(m)}_{x} = \left\{1 + \beta(m)P^{(m)}_{x}\right\} {}_{k}V_{x} \, . \tag{47}$$

It is instructive to note that the general relationship (44) relating reserves with a different payment mode was derived under the FI assumption but actually holds more generally. One can see fairly easily that (44) depends only on the linearized annuity form (33). In particular, the classical formula for $\ddot{a}^{(m)}_{x}$ derived via Woolhouse's formula is of the form (33) with $\alpha(m) = 1$ and $\beta(m) = (m-1)/(2m)$. See Jordan (1967, p. 47) for details. Thus, this classical approximation for $\ddot{a}^{(m)}_{x}$ also leads to a reserve formula of the form (44), but with $\beta(m)$ replaced by $(m-1)/(2m)$. Also, the second term on the right hand side of (44) may be viewed quite generally as an "unearned premium" reserve.

## 6. Multiple Lives

As discussed in Bowers et al. (1986, chapter 8), if $T(x)$ and $T(y)$ are independent and the UDD assumption applies to each, then the UDD assumption does not apply to the joint-life random variable $T(xy) = min\,\{T(x), T(y)\}$. It also does not apply to the last survivor random variable $T(\overline{xy}) = max\,\{T(x), T(y)\}$ (see Section 1). The same remains true for the more general FI assumption. Evaluation of quantities of interest under FI for each of the individual times may be handled as in Bowers et al. (1986, chapter 8, 17).

It is instructive to note, however, that certain contingent probabilities and insurances are invariant under the choice of $H(s)$, given that the FI assumption applies. Thus, assume that the future life random variables $T(x_j)$ are independent and satisfy ${}_{s+k}p_{x_j} = {}_k p_{x_j} \cdot {}_s p_{x_j+k}$ and ${}_s q_{x_j} = H(s) q_{x_j}$ for $0 \le s \le 1$. That is, the FI assumption (5) holds.

Then, for example,

$$
\begin{aligned}
{}_m q^1_{x_1 x_2 \ldots x_n} &= \int_0^m \left\{ \prod_{j=1}^n {}_t p_{x_j} \right\} \mu_{x_1+t} dt \\
&= \sum_{k=0}^{m-1} \left\{ \prod_{j=1}^n {}_k p_{x_j} \right\} \int_0^1 \left\{ \prod_{j=1}^n {}_s p_{x_j+k} \right\} \mu_{x_1+k+s} ds \\
&= \sum_{k=0}^{m-1} \left\{ \prod_{j=1}^n {}_k p_{x_j} \right\} \int_0^1 \left\{ \prod_{j=2}^n (1 - H(s) q_{x_j+k}) \right\} H'(s) q_{x_1+k} ds \\
&= \sum_{k=0}^{m-1} \left\{ \prod_{j=1}^n {}_k p_{x_j} \right\} \int_0^1 \left\{ \prod_{j=2}^n (1 - r q_{x_j+k}) \right\} q_{x_1+k} dr
\end{aligned}
$$

where the last line follows from the change of varibles from $s$ to $r = H(s)$. No matter what $H(s)$ is, the value of ${}_m q^1_{x_1 x_2 \ldots x_m}$ is the same as if $H(s) = s$, i.e. under UDD. In fact, the same is true for more general probabilities such as ${}_m q^i_{x_1 x_2 \ldots x_n}$, or even situations where the order of death is restricted. See Bowers et al. (1986, chapter 17) for details. In general, these probabilities involve integrals of the form

$$
\begin{aligned}
&\int_0^1 \left\{ \prod_j {}_s p_{x_j}^{\Delta_j} \right\} \left\{ \prod_j {}_s q_{x_j}^{1-\Delta_j} \right\} \mu_{x_i+s} \, ds \\
= &\int_0^1 \left\{ \prod_j (1 - H(s) q_{x_j})^{\Delta_j} \right\} \left\{ \prod_j (H(s) q_{x_j})^{1-\Delta_j} \right\} \frac{H'(s) q_{x_i}}{1 - H(s) q_{x_i}} ds \\
= &\int_0^1 \left\{ \prod_j (1 - \ s q_{x_j})^{\Delta_j} \right\} \left\{ \prod_j (s \cdot q_{x_j})^{1-\Delta_j} \right\} \frac{q_{x_i}}{1 - \ s \cdot q_{x_i}} ds \ ,
\end{aligned}
$$

which is the same as one obtains by assuming $H(s) = s$.

## 7. Multiple decrements

Consider the probabilities ${}_sq_x^{(j)}$ for $j = 1, 2, \ldots, m$ in the multiple (not the associated single) decrement table (Bowers et al., 1986, chapter 9). If, for $j = 1, 2, \ldots, m$,

$$ {}_sq_x^{(j)} = H(s)q_x^{(j)} \ , \ 0 \le s \le 1 \ , \tag{48} $$

then

$$ {}_sq_x^{(\tau)} = \sum_{j=1}^{m} {}_sq_x^{(j)} = H(s)q_x^{(\tau)} \ . \tag{49} $$

Thus,

$$ \mu_{x+s}^{(j)} = \frac{H'(s)q_x^{(j)}}{1 - H(s)q_x^{(\tau)}} $$

and

$$ \begin{aligned} q_x'^{(j)} &= 1 - \exp\left\{ -\int_0^1 \mu_{x+s}^{(j)} ds \right\} \\ &= 1 - \exp\left\{ -q_x^{(j)} \int_0^1 \frac{H'(s)}{1-H(s)q_x^{(\tau)}} ds \right\} \\ &= 1 - \exp\left\{ \frac{q_x^{(j)}}{q_x^{(\tau)}} \ ln(1 - q_x^{(\tau)}) \right\} \ . \end{aligned} $$

In other words, under the FI assumption one has the well known and very general result (Bowers et al., 1986, pp. 274-5)

$$ q_x'^{(j)} = 1 - \left\{ 1 - q_x^{(\tau)} \right\}^{q_x^{(j)}/q_x^{(\tau)}} \ . \tag{50} $$

## 8. References


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