MULTIPLE CURRENCY OPTION SELECTION USING STOCHASTIC CONSTRAINTS*

Kelly T. Au,[†] Joel R. Barber,[‡] and David C. Thurston[§]

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Abstract

This paper examines the problem of hedging foreign exchange risk across multiple countries using currency options. The profit target requirement is formulated as a stochastic constraint in the context of a binomial lattice. Models are presented for European options. In a numerical example, five currency options are selected, allowing for long or short positions, depending on the target value.

I. INTRODUCTION

In the current business environment, companies have found that they must increasingly buy and sell in international markets. In this environment, companies often find that they have accumulated substantial holdings in foreign currencies, or face future transactions in foreign currencies. Many of these companies wish to limit their exposure to foreign exchange risk.

†DataMost Corporation, Salt Lake City, UT 84165.

‡Department of Finance, Florida International University, Miami, FL 33199.

§School of Business, Henderson State University, Arkadelphia, AR 71999.

Some commonly used hedging instruments for foreign exchange risk are forward and future contracts. However, these contracts lock in fixed exchange rates for future transactions, and consequently these contracts result in market losses if exchange rate movements are unfavorable. Currency option contracts are flexible alternatives that offer the *right*, but not the obligation, to trade foreign currencies at specific exchange rates. Currency options insure against unfavorable exchange rate movements while allowing participation in favorable exchange rate movements. For example, an importer due to pay Japanese yen at a predetermined time can insure against an exchange rate higher than a certain amount (exercise price) by buying calls on yen. At expiration, if the spot exchange rate on Japanese yen is higher than the exercise price, the importer can benefit by exercising the call option. Conversely, if the spot rate is lower than the option's exercise price, the importer can buy yen in the market and let the option expire.

This hedging mechanism is especially useful when a company is not certain whether a future foreign exchange transaction will occur. In this situation a company may be unwilling to lock in a fixed exchange rate through forward or futures contracts because if the transaction does not occur the company could face large losses.

Although it costs nothing to enter into a forward rate agreement, there is an upfront cost to hedging with options. For a given expiration date and currency, a number of options with different exercise values are usually available in the market. The greater the degree of "moneyness" (and hence the greater certainty of value at expiration) of an option, the larger is premium.[1] This risk-return trade-off motivates our development of *cost-effective* hedging models for currency options.

This paper analyzes the effectiveness of hedging foreign exchange risk in multiple

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[1]A call currency call option is said to be *in the money* if the spot exchange rate exceeds the exercise value. Conversely, a currency put option is said to be *in the money* if the exercise value exceeds the spot exchange rate.

countries with currency options when a company faces the problem of selecting from available options with different exercise prices. The selection of options with perfect information (no uncertainty involved) can be modeled as a standard linear program. When only statistical information (such as probability distributions) are available, the optimal selection problem can be formulated as a stochastic program.

Stochastic programs have been used in many engineering applications to solve problems such as aircraft allocation with uncertain demands, Dantzig (1963) and reservoir design with random stream inputs, Prékopa and Szántai (1978). Stochastic programs have also been used for and short term asset management in Kallberg, White, and Ziemba (1982) for uncertain transaction costs. Ermoliev and Wets (1988) provide a comprehensive source of recent applications of stochastic programs.

The definitive treatment of pricing options using binomial lattices is Cox, Ross, and Rubenstein (1979).[2] Lattices have proven to be an extremely successful numerical procedure for the evaluation of a variety of options. They have been used in a variety of contexts, (see e.g. valuing interest continent claims in Ho and Lee (1986)). Binomial lattice pricing approximates the evolution of the underlying asset values in a risk neutral framework using recombining trees.

In this paper, the techniques of a stochastic program will be combined with binomial lattices to guide the selection of a proportion of options so that stochastic constraints (attributed to uncertain exchange rate movements) are satisfied at a minimal cost.

Previous work on the performance of hedging schemes, such as Boyle and Emanuel (1980) and Galai (1983), has required continuous adjustments in response the uncertain changes in foreign exchange rates. In this paper, we establish a "here and now" decision model based on average exchange rate movements. This methodology offers managers a

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[2]William Sharpe used binomial lattices prior to Cox, Ross, and Rubenstein as a simple expositional tool to teach option pricing theory to his MBA students at Stanford.

direct and systematic approach of selecting options for hedging purposes. It is useful as a support system that can be incorporated into the managers' interactive decision making process.

The organization of the remainder of this paper is as follows. Section II presents the preliminaries. Section III describes the basic model for European options. Section IV illustrates the basic model with a numerical example using five currencies. The final section summarizes and concludes the paper.

II. NOTATION AND ASSUMPTIONS

Suppose that a firm wishes to hedge foreign exchange risk of a future transaction by selecting a portfolio of currency options. Unlike hedging with forward or futures contracts, there is a significant upfront cost to using options as a hedging tool. The desired portfolio of currency options is required to satisfy certain criteria at a minimum cost. In this paper, these criteria are formulated as two constraints: 1.) quantity requirement of each option and 2.) performance of expected pay-off of option portfolio.

To simplify our exposition, we assume constant risk-free domestic and foreign interest rates and exchange rate volatility, following Garman and Kohlhagen (1983).[3]

Let $i = 1, \ldots, N$, denote the index of the currency options. The exchange rate S_{it} is assumed to follow a geometric Brownian motion. A binomial model is used to approximate a geometric Brownian motion following Cox, Ross and Rubenstein (1979). The time horizon is divided by a total number of time stages, T. The time increment between stages is denoted by $\triangle t$. In each stage, the exchange rate S_{it} either moves up to $S_{i,t+1} = S_{it}u_i$ with probability p_i or moves down to $S_{i,t+1} = S_{it}d_i$ with probability $1 - p_i$. Let I_{it} denote

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[3]The Garman-Kohlhagen European foreign currency option pricing model is a simple and robust model, often used by practitioners. However, the stochastic analysis can be extended to analyze more realistic and complicated exchange rate models, such as Amin and Jarrow (1991).

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the random pay-off for option i at stage t. Then, I_{it} equals $\text{Max}\{S_{it} - E_i, 0\}$ for a call option, and I_{it} equals $\text{Max}\{E_i - S_{it}, 0\}$ for a put option. The up parameter u_i, down parameter d_i, and probability p_i can be chosen as follows to assure convergence of a binomial model to geometric Brownian motion when the time stage increment, Δt, goes to zero. Following Hull (1993), we choose

$$u_i = e^{\sigma_i \sqrt{\Delta t}} \tag{1}$$

$$d_i = e^{-\sigma_i \sqrt{\Delta t}} \tag{2}$$

$$p_i = \frac{e^{(r - r_{if})\Delta t} - d_i}{u_i - d_i} \tag{3}$$

where

r: U. S. interest rate.

r_{if}: Foreign interest rate for currency i.

σ_i: Volatility of the exchange rate, S_i.

Observe that p_i is the risk neutral probability of the exchange rate going up.

At the terminal stage of a stochastic binomial model, K overall scenarios are generated. For example, if there are T time stages, $T + 1$ possible scenarios will occur. Each possible combination of movements constitute a scenario $k = 1, \ldots, K$. Thus, for each currency i, the scenario probability, p_{ik} is a product of all possible combinations of one-stage up probabilities, p_i's and down probabilities, $(1 - p_i)$. With these risk-neutral scenario probabilities: p_{ik}, $k = 1, \ldots, K$, we apply stochastic programs to currency option selection.

III. THE BASIC MODEL

For ease of exposition we only consider European call options. A company that anticipates trading in foreign currency at a future date may wish to hedge its foreign exchange

exposure by buying or selling currency options maturing at the same date. A collection of such options are available at different premiums c_i, depending on the country's exchange rate. The model developed here will help a financial manager select an portfolio of options at the lowest total cost that satisfies the quantity and performance constraints. The total cost is $\sum_{i=1}^{N} x_i c_i$, where the quantity, x_i, can be positive (short position) or negative (long position). The bounds on x_i's depend on the firm's level of risk aversion.

Here, we require

$$-1 \leq x_i \leq 1, i = 1, \ldots, N$$

The uncertain pay-off at expiration (stage T) is required to be greater than a given target value τ:

$$\sum_{i=1}^{N} x_i I_{iT} \geq \tau.$$

There are numerous ways to handle such stochastic constraints. One common approach is to use the expected value of the random variables (exchange rates). The exchange rates' correlations are not required, since expected values are used. With the scenario probabilities in a binomial lattice, the expected total pay-off must at least exceed a target value τ,[4] Therefore,

$$\sum_{i=1}^{N} x_i \sum_{k=1}^{K} p_{ik} I_{iTk} \geq \tau.$$

A stochastic program can be set up as follows.

$$\begin{aligned}
\text{Min} \quad & \sum_{i=1}^{N} x_i c_i \\
\text{s.t.} \quad & \sum_{i=1}^{N} x_i \sum_{k=1}^{K} p_{ik} I_{iTk} \geq \tau \\
& -1 \leq x_i \leq 1 \quad i = 1, \ldots, N.
\end{aligned} \tag{4}$$

Various modifications and enhancements of our basic model are readily accomplished. For

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[4]Here, the discount factor is included in τ.

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example, variance can be included as an alternative measure of risk.[5]

IV. AN EXAMPLE

Consider a company that wishes to hedge foreign exchange risk using the following European call options: German Marks (GM), Canadian dollars (CD), Japanese Yen (JY), Swiss-Francs (SF) and British Pounds (BP). We use one-month prime lending rates given in *Wall Street Journal* on November 13, 1991; (see also Stoll and Whaley (1993) p. 129.) The U.S. prime rate is 7.50%. The prime rates for the German, Canadian, Japanese, Swiss, and British markets, respectively are, 11.5%, 8.5%, 7.0%, 10.0%, and 10.5%.

Table 1. Call Option Quotes: Philadelphia Exch., Nov. 13, 1991

| Currency (i):	GM	CD	JY	SF	BP
Cost (c_i):	0.0259	0.0035	0.0296	0.0024	0.022
Spot rate (S_{i0}):	0.6103	0.8823	0.7699	0.6885	0.0464
Strike price (E_i):	0.5890	0.8800	0.7400	0.6900	0.5890
Foreign rate (r_{if}):	0.115	0.085	0.07	0.10	0.105
Implied Volatility (σ_i):	0.375	0.0452	0.035	0.629	0.209

For the purpose of our demonstration, the time horizon of one month is divided into four stages. The basic model can select a combination of these options so that the total expected pay-off at maturity will meet the desired target value. It follows from equations (1–3) that the binomial lattice parameters are given as in Table 2.

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[5]This will lead to a quadratic programming problem.

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Table 2. Binomial Lattice Parameters

Parameter:	GM	CD	JY	SF	BP
up-parameter (u_i):	1.0556	1.0065	1.0051	1.0950	1.0307
down-parameter (d_i):	0.9473	0.9935	0.9950	0.9132	0.9702
up-probability (p_i):	0.4733	0.4368	0.4501	0.4712	0.4723

Applying the binomial lattice with four stages, we obtain five scenarios in the terminal stage. For example, for the GM exchange rate, the binomial lattice can be represented as in Figure 1. At each stage the upper value represents the exchange rate and the lower value its corresponding scenario probability. For example, the exchange rate in the final stage, 0.4951, is obtained from $S_{10}d_1^4$ and the scenario probability 0.0502 is from $(1 - p_1)^4$.
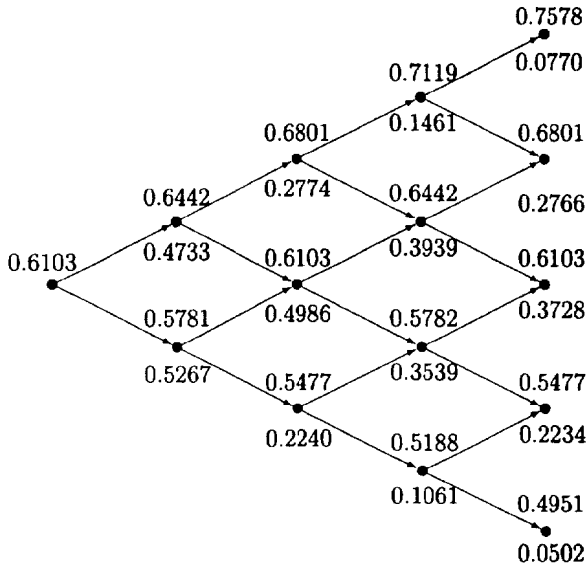
Figure 1: German Mark Exchange Rate Movements

To formulate the stochastic constraint, we require the final stage scenario probabilities and exchange rates for each currency i. Table 3 provides the scenario probabilities for each currency in the final stage, $(T = 4)$.

Table 3. Scenario Probabilities

Probability:	GM	CD	JY	SF	BP
p_{i1}	0.0770	0.1006	0.0914	0.0782	0.0776
p_{i2}	0.2766	0.3121	0.2994	0.2787	0.2776
p_{i3}	0.3728	0.3631	0.3676	0.3725	0.3727
p_{i4}	0.2234	0.1877	0.2006	0.2213	0.2224
p_{i5}	0.0502	0.0365	0.0410	0.0493	0.0497

Table 4 provides the final stage (T=4) scenario exchange rate for each currency. Here, S_{iTk} represents the final stage (T) and scenario (k) exchange rate for currency i.

Table 4. Scenario Exchange Rates

Exchange Rate	GM	CD	JY	SF	BP
S_{iT1}	0.4915	0.8596	0.7545	0.4788	0.5408
S_{iT2}	0.5477	0.8709	0.7622	0.5742	0.5745
S_{iT3}	0.6103	0.8823	0.7699	0.6885	0.6103
S_{iT4}	0.6801	0.8939	0.7777	0.8256	0.6483
S_{iT5}	0.7578	0.9056	0.7856	0.9901	0.6888

Each pay-off is evaluated according to the final exchange rate and its strike price. For example, in the GM exchange rate, the pay-off in the first scenario $I_{1T0} = \text{Max}\{S_{1T0} - X_1, 0\} = \text{Max}\{0.4915 - 0.589, 0\} = 0$. The final stage scenario pay-off for each currency is given in Table 5.

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Table 5. Scenario Pay-Off

Exchange Rate	GM	CD	JY	SF	BP
I_{i1}	0	0	0.0145	0	0
I_{i2}	0	0	0.0222	0	0
I_{i3}	0.0213	0.0023	0.0299	0	0.0213
I_{i4}	0.0911	0.0139	0.3772	0.1356	0.0593
I_{i5}	0.1688	0.0256	0.0456	0.3001	0.0998

With the values from Tables 1 through 5, the basic model in equation (4) can be represented as:

$$\text{Min} \quad 0.0259x_1 + 0.0035x_2 + 0.0296x_3 + 0.0024x4 + 0.022x_5$$

$$\text{s.t.} \quad 0.0368x_1 + 0.0044x_2 + 0.0284x_3 + 0.0448x_4 + 0.0261x_5 \geq \tau$$

$$-1 \leq x_i \leq 1 \quad i = 1, \ldots, N.$$

We use a standard linear programming software package to obtain the optimal proportion of options to select at a minimum total cost. Table 6 provides a comparison of various target values and their corresponding solutions.

Table 6. Optimal Currency Option Holdings: 4 Stages

Target Value	Proportion suggested for each option					Total Cost
τ	GM	CD	JY	SF	BP	$\sum_{i=1}^{N} x_i c_i$
0.01	0.655	−1	−1	1	−1	−0.0356
0.02	0.927	−1	−1	1	−1	−0.0287
0.05	1	1	−1	1	−0.2912	−0.0042
0.07	1	1	−0.426	1	1	0.0416

In this example, a negative proportion means a *long* position is recommended. In the first three target values, the total costs are negative, which represents a gain instead of a cost. The option cost is proportional to the target value of the expected pay-off. One point to emphasize here, is that the target value is achieved stochastically, since we use the binomial probability measure to approximate the average value of the exchange rate. As such, this model is referred to as binomial stochastic optimization. Alternatively, the difference of the total cost and the target value provides another performance measure of this hedging scheme.

V. SUMMARY AND CONCLUSIONS

The techniques of binomial lattices and stochastic programming are combined to develop a model for the selection of available currency options, with the budgeting and pay-off targets satisfied stochastically. This model provides guidelines for companies to efficiently hedge their multicurrency foreign exposure with currency options. In the context of options-based hedging strategies this model investigates hedging efficiency using average exchange rates and offers alternative guidelines to a standard hedging performance measure scheme. Modifications of this model are readily possible in order to meet a company's specific needs. This paper presents a new technique for optimizing hedging policies based upon stochastic programming.

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