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A Note on Annuities Payable at a Different Frequency than Interest Is Compounded

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Abstract

Kellison's *The Theory of Interest* develops annuities with payments more frequent than interest is converted separately from those with payments less frequent than interest is converted. This paper unifies the two subjects. Mathematically there is no difference between the two, so the approach used for one can be used for the other. In particular, the formula representing the present value of an annuity of 1 per year payable m times per year as $1 - v^n$ divided by the interest factor that corresponds to the timing of the payments is seen to hold regardless of the relationship between the payment frequency and the compounding frequency. The recognition of this fact should help students of the subject to understand both approaches and both types of annuities.

I. Introduction.

The Theory of Interest takes one approach to annuities with payments more frequent than interest is converted and another approach to those with payments less frequent. There is no reason why either approach cannot be used for either type of annuity, and that is what this paper does. Three types of formulas are developed that apply to both types of annuity. It is seen that there is no difference conceptually between the two; they can both be viewed more generally as annuities with payments at a *different* frequency than interest is converted.

The development here is in terms of present values of annuities-immediate. The formulas for present values of annuities-due and for the accumulated values of both types of annuities are also presented.

II. First Approach.

The approach that Kellison uses to analyze the annuity with payments more frequent than interest is compounded is as follows [p. 103]:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\left[v^{\frac{1}{m}} + v^{\frac{2}{m}} + \ldots v^{n-\frac{1}{m}} + v^{n}\right]$$

$$= \frac{1}{m}\left[\frac{v^{\frac{1}{m}} - v^{n+\frac{1}{m}}}{1 - v^{\frac{1}{m}}}\right]$$

$$= \frac{1 - v^{n}}{m\left[(1+i)^{\frac{1}{m}} - 1\right]}$$

$$= \frac{1 - v^{n}}{i^{(m)}} .$$

This formula and the analogous formula for the annuity-due,

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^{n}}{d^{(m)}} ,$$

generalize the formulas for the basic case where interest is converted at the same frequency as the payment interval. Students learn in Chapter 3, "Basic Annuities," that the formula representing the present value of an annuity of 1 per year payable m times per year is $(1 - v^{n})$ divided by the interest factor that corresponds to the timing of the payments. That this result holds even when payments are more frequent than interest is converted is a felicitous result of Chapter 4, "More General Annuities."

The following formulas follow easily from the preceding ones:

$$a_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}} a_{\overline{n}|}$$

and

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{i}{d^{(m)}} a_{\overline{n}|} .$$

This approach is equally applicable to the situation with interest convertible more frequently than payments are made. All that is necessary is to define k as the number of interest conversion periods per payment period and m as 1/k so that k=1/m. For example, if interest is convertible annually and we have an annuity that pays every four years, then k=4 and m=1/4.

Kellison states that there is no symbol defined for the annuity with interest convertible more frequently than the payment interval. The natural symbol for this is $a_{\overline{n}|}^{(1/k)}$, where n is the number of years (with n an integral multiple of k), k is the number of years between payments (so that there are n/k payments), and the symbol means the present value of a payment of 1 per year. This means that the payment every k years is k, not 1, so that the present value of 1 payable at times k, 2k, 3k, . . . n-k, n is $\frac{1}{k} a_{\overline{n}|}^{(1/k)}$.

Nothing in the development of the above formulas required m to be an integer, so they should hold with m=1/k. This leads to the following formulas:

$$a_{\overline{n}|}^{(1/k)} = \frac{1 - v^n}{i^{(1/k)}} = \frac{i}{i^{(1/k)}}\, a_{\overline{n}|}$$

and

$$\ddot{a}_{\overline{n}|}^{(1/k)} = \frac{1 - v^n}{d^{(1/k)}} = \frac{i}{d^{(1/k)}}\, a_{\overline{n}|}\,.$$

and the even more felicitous result that the rule given above holds whether payments are more frequent, less frequent, or of equal frequency to the interest conversion frequency. Section IV shows the equivalence of this approach to the traditional approach to annuities with payments less frequent that interest is converted.

III. Second Approach.

What I call the Second Approach is developed by Kellison prior to what I call the Fist Approach, on pages 98-99. The Second Approach replaces the payments of 1 every k periods with payments at the end of every period which are equal in present value to the original payments. This means that the amount of each replacement payment is, in the case of the annuity-immediate, $\frac{1}{s_{\overline{k|}}}$, and in the case of the annuity-due, $\frac{1}{a_{\overline{k|}}}$. This leads to the following formulas (using the notation that I have suggested):

$$\frac{1}{k}\, a_{\overline{n|}}^{(1/k)} = \frac{a_{\overline{n|}}}{s_{\overline{k|}}}$$

and

$$\frac{1}{k}\, \ddot{a}_{\overline{n|}}^{(1/k)} = \frac{a_{\overline{n|}}}{a_{\overline{k|}}} .$$

The 1/k on the left side of each equation is necessary since Kellison's expressions are for a payment of 1 each k periods, whereas the annuity symbol on the left side of each equation means the present value of a payment of k each k years.

Again, there is nothing in the development of these formulas that requires k to be an integer. If we let m be an integer and k=1/m, we obtain the following formulas:

$$m\, a_{\overline{n|}}^{(m)} = \frac{a_{\overline{n|}}}{s_{\overline{1/m|}}}$$

and

$$m\, \ddot{a}_{\overline{n|}}^{(m)} = \frac{a_{\overline{n|}}}{a_{\overline{1/m|}}} .$$

(Recall that the symbol $a_{\overline{1/m|}}$ means the present value of a payment of $\frac{(1+i)^{1/m} - 1}{i}$ at time 1/m, so that $a_{\overline{1/m|}} = \frac{1 - v^{1/m}}{i}$. Likewise,

$s_{\overline{1/m}|} = \frac{(1+i)^{1/m} - 1}{i}$.) The algebraic equivalence of these and the traditional formulas for this situation is demonstrated in the next section.

IV. Algebraic Equivalence.

It is an easy matter to see that these two approaches are algebraically equivalent. We start with the traditional formula for an annuity with payments less frequent than interest is converted.

$$\frac{1}{k} a_{\overline{n}|}^{(1/k)} = \frac{a_{\overline{n}|}}{s_{\overline{k}|}}$$

$$= \frac{a_{\overline{n}|}}{[(1+i)^k - 1]/i}$$

$$= i \frac{a_{\overline{n}|}}{k\, i^{(1/k)}}$$

$$a_{\overline{n}|}^{(1/k)} = k \frac{a_{\overline{n}|}}{s_{\overline{k}|}} = \frac{i}{i^{(1/k)}} a_{\overline{n}|} = \frac{1 - v^n}{i^{(1/k)}} .$$

The above development starts with one of the important formulas for the present value of an annuity-immediate with payment period differing from the interest conversion period and ends with the other two. For completeness, the analogous formulas for the present value of an annuity-due and for the accumulated value of both types of annuities are presented below.

$$\ddot{a}_{\overline{n}|}^{(1/k)} = k\,\frac{a_{\overline{n}|}}{a_{\overline{k}|}} = \frac{i}{d^{(1/k)}}\,a_{\overline{n}|} = \frac{1 - v^{n}}{d^{(1/k)}}$$

$$s_{\overline{n}|}^{(1/k)} = k\,\frac{s_{\overline{n}|}}{s_{\overline{k}|}} = \frac{i}{i^{(1/k)}}\,s_{\overline{n}|} = \frac{(1 + i)^{n} - 1}{i^{(1/k)}}$$

$$\ddot{s}_{\overline{n}|}^{(1/k)} = k\,\frac{s_{\overline{n}|}}{a_{\overline{k}|}} = \frac{i}{d^{(1/k)}}\,s_{\overline{n}|} = \frac{(1 + i)^{n} - 1}{d^{(1/k)}}$$

V. A Further Generalization

Note that nothing in the development of the above formulas required k to be an integer, just as nothing required m to be. Kellison mentions the problem where the ratio of the larger of k and n to the smaller of the two is not an integer [p. 100]. There is no reason why this problem cannot be handled using the above formulas. For example, Problem 12 on page 123 states

> Find and expression for the present value of an annuity on which payments are 1 at the beginning of each 4-month period for 12 years, assuming a rate of interest per 3-month period.

The answer can be immediately given as

$$\frac{1}{4/3}\,\ddot{a}_{\overline{48}|}^{(3/4)} = \frac{3}{4}\left(\frac{1 - v^{48}}{d^{(3/4)}}\right),$$

which is equal to the answer in the back of the book, $\dfrac{1 - v^{48}}{1 - v^{4/3}}$.

VI. Conclusion.

I believe that the unification of these two approaches into three formulas that apply to either specific case is a pedagogical improvement that can help students to better understand the material. In particular, the annuity formula that most students know best, namely that the present value of an annuity is equal to $(1 - v^n)$ divided by an interest factor that relates to the timing of the payments, can be seen to hold in general.

VII. Reference

Kellison, Stephen G., *The Theory of Interest*, 2nd Edition, Homewood, Ill.: Irwin, 1991.