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A Stochastic Model for CCRCs

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Abstract

This paper presents a multi-state stochastic model for analyzing continuing care retirement community (CCRC) populations. The model considers CCRCs with a number of independent living units and a skilled nursing facility. Residents may transfer temporarily or permanently to this facility. It is assumed that units vacated by deaths, withdrawals or permanent transfers are immediately occupied by new residents. Transfers are modeled using a time-homogeneous Markov process. The paper provides some probabilistic results and some numerical results obtained using this model. Some important generalizations of the model are also briefly discussed.

1 Introduction

Actuarial Standard of Practice No. 3 [see Actuarial Standards Board (1994)] defines a continuing care retirement community (CCRC) as "a residential facility for retired people that provides stated housekeeping, social, and health-care services in return for some combination of an advance fee, periodic fees, and additional fees." CCRCs house up to several hundred residents and have facilities to provide one or more levels of long-term care. Residency agreements are typically of long duration and care may be provided with no adjustment of fees. Winklevoss and Powell (1984) discuss the operation of CCRCs in considerable detail.

The uncertainty of future services required by CCRC residents has created a need for actuarial analyses. As outlined in Actuarial Standards Board (1994), actuaries may be asked to:

- design and price residency agreements in order to (1) provide for the economic survival of the community in the short and long run; and (2) fairly represent to the user the economic consequences of entering into a residency agreement;
- project future cash flows;
- project changes in the future population of residents and estimate the future needs for health care beds;
- determine actuarial assets and liabilities, and plan for surplus needs;
- participate in the design of a CCRC's financial management and accounting systems;

- assist in developing financial feasibility studies;
- provide appropriate rates of mortality, morbidity, or life expectancy for the community's use; and
- perform mortality, morbidity, and withdrawal experience studies.

These and other actuarial issues relating to CCRCs are discussed by Brace (1994).

In order to carry out actuarial analyses of CCRCs one requires an appropriate model that describes the CCRC population over time. The model should allow for the movement of residents among the various levels of care and should provide information on the variability of future outcomes as well as expected values. Such a model is necessarily rather complicated.

Cumming and Bluhm (1992) describe a CCRC population and financial model that allows one to perform actuarial valuations and cash flow and population projections. Expected results can be obtained using the decrement rates, and random variation may be estimated by simulation. A limitation of the model is that utilization of the care facility due to temporary stays is reflected on an average basis; the model does not permit transfers from higher to lower care states.

Jones (1994a and b) analyzed a simplified stochastic model for CCRCs. The model considers a community that offers single independent living units and one level of care. Residents may transfer either temporarily or permanently to the care facility. Transfers occur according to a Markov process with constant forces of transition. It is also assumed that the CCRC operates in a high demand environment, so that living units vacated by death, withdrawal or permanent transfer (to the care facility) of a resident are immediately occupied by new residents.

The purpose of this paper is to present some results obtained using the latter model and to point out some generalizations that must be considered in developing a practical stochastic model for analyzing CCRCs. Section 2 describes the model in more detail. Probabilistic results are discussed in Section 3, and some numerical results are provided in Section 4. Section 5 considers a number of generalizations.

2 The Model

Consider a CCRC with m single independent living units (ILU) and a skilled nursing facility (SNF). The SNF is assumed to have an *infinite capacity* in that *no restriction* is placed on the number of residents in the facility at one time. Practically, this represents a situation in which the CCRC's commitment to residents is such that care will be provided even if residents must be moved off site.

Residents transfer from their living unit to the SNF when care is needed. Transfers are deemed to be either temporary or permanent based on an assessment of whether the individual will ever again be capable of living independently. Thus, at any time after entering the community, a resident is in one of the following four states:

1. **ILU**

   Residents in this state live normal active lives and occupy independent living units.

2. **SNF (temporary)**

   Residents in this state have transfered temporarily to the SNF. They are expected to recover and return to their living units.

3. **SNF (permanent)**

   Residents in this state have transferred permanently to the SNF. Upon entry to this state, their living units are made available to new residents.

4. **Dead or Withdrawn**

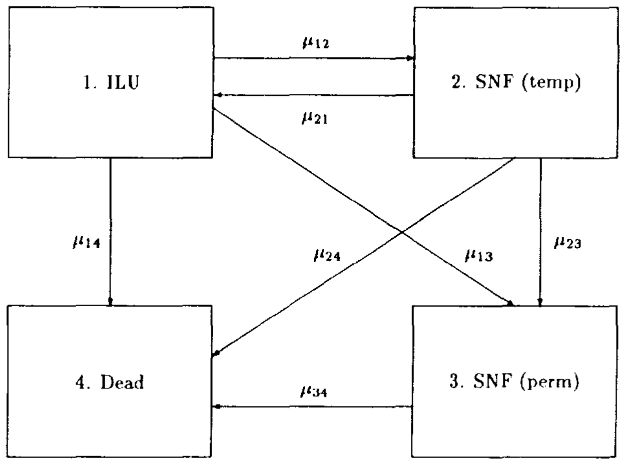
   Individuals in this state are previous residents who have either died or otherwise left the CCRC. Such individuals' living units are made available to new residents upon death or withdrawal, or earlier permanent transfer to the SNF.

Figure 1 illustrates the setup, showing the four states and the possible transitions.

Now assume that the demand for living units in the CCRC is such that those units vacated by permanent transfers, deaths and withdrawals are immediately occupied by new residents. This "high demand" assumption has some interesting and rather nice consequences. The first is that the sum of the number of ILU residents and the number of temporary transfers is m at all times. The second is that the m living units can be assumed to operate independently with respect to the movement of residents. This allows one to analyze the CCRC by first considering just one living unit.

Transitions between the four states in Figure 1 occur according to a Markov process with constant forces of transition. Let μ_{hi} represent the force of transition from state h to state

Figure 1: State Transition Diagram for Individual Residents



i, where $h, i \in \{1, 2, 3, 4\}$. Thus, the probability that a resident in state h at time $t - dt$ moves to state i during the small time interval $(t - dt, t]$ is $\mu_{hi}dt$. Note that this probability does not depend on t. Also, the time an individual spends in state h (for $h = 1, 2, 3$) has an exponential distribution with mean $1/\sum_{i \neq h} \mu_{hi}$.

The assumption of constant forces of transition is questionable in many actuarial applications. One might expect such quantities to depend on the age of the resident and perhaps on the time since entry to the current state. However, allowing for the current state of a resident may remove some of the effect of age and duration. Clearly, though, the assumption must be tested.

Let $J(t)$ represent the number of permanent transfers in the community at time $t \geq 0$.

Also, let $K(t)$ be the number of temporary transfers at time t. Then $m - K(t)$ is the number of ILU residents at time t. Time is measured from some arbitrary date that we let be time 0. We are interested in the bivariate stochastic process $\{(J(t), K(t)), t \geq 0\}$. It will be convenient to assume that $J(0) = K(0) = 0$, though this assumption can be relaxed without difficulty.

Define $J_l(t)$ and $K_l(t)$ to be the number of permanent transfers and temporary transfers associated with living unit l at time t, $l = 1, 2, \ldots, m$. Clearly, $K_l(t)$ must be either 0 or 1 and $J_l(t)$ is a non-negative integer. $J_l(t)$ is the number of permanent transfers at time t who once resided in living unit l. We have $J(t) = \sum_{l=1}^{m} J_l(t)$ and $K(t) = \sum_{l=1}^{m} K_l(t)$. Also, $J_l(0) = K_l(0) = 0$ for all l.

Assume that the processes $\{(J_l(t), K_l(t)), t \geq 0\}$, $l = 1, 2, \ldots, m$ are mutually independent. That is, the state of any living unit at a given time is independent of the states of all other living units at all points in time. This assumption is reasonable in light of the high demand assumption and the fact that the SNF has infinite capacity.

According to this setup, the paths of the m processes are identically distributed. Thus, in seeking information about the distribution of $(J(t), K(t))$, we can first consider the distribution associated with an arbitrary unit, say, $(J_1(t), K_1(t))$.

3 Probabilistic Results

Considerable information about the process $\{(J_1(t), K_1(t)), t \geq 0\}$ is contained in the probabilities $\Pr(J_1(t) = j, K_1(t) = k)$, $j = 0, 1, 2, \ldots; k = 0, 1$; $t > 0$. Though we cannot find explicit expressions for these probabilities, we can for a number of related quantities.

Since $K_1(t)$ is either 0 or 1 for all t, the (marginal) distribution of this quantity is characterized by $\Pr(K_1(t) = 1)$. This probability is actually $\Pr(K_1(t) = 1 | K_1(0) = 0)$ because we have assumed that $K_1(0) = 0$. Now the process $\{K_1(t), t \geq 0\}$ is a two-state Markov process with state space $\{0, 1\}$. The force of transition from state 0 to state 1 is μ_{12}, and the force of transition from state 1 to state 0 is $\mu_{2\cdot} \equiv \mu_{21} + \mu_{23} + \mu_{24}$. Therefore [see Ross (1983, p. 150)],

$$\Pr(K_1(t) = 1) = \frac{\mu_{12} - \mu_{12} e^{-(\mu_{12}+\mu_{2\cdot})t}}{\mu_{12} + \mu_{2\cdot}}. \tag{1}$$

It follows from (1) that

$$\lim_{t \to \infty} \Pr(K_1(t) = 1) = \frac{\mu_{12}}{\mu_{12} + \mu_{2\cdot}}. \tag{2}$$

Equation (2) gives the long-run proportion of time that the SNF has a temporary transfer from a given living unit.

It was shown by Jones (1994a) that if $\mu_{12}, \mu_{23}, \mu_{34} > 0$ and $\mu_{hi} < \infty$ for all h, i, then $\lim_{t \to \infty} \Pr(J_1(t) = j, K_1(t) = k)$ exists and is positive for $j = 0, 1, 2, \ldots; k = 0, 1$. These limits form what is called an equilibrium or stationary distribution for the state of a living unit. Jones (1994a) provides a method for finding the probabilities associated with this distribution.

The marginal distribution of $J_1(t)$ is considerably more complicated than that of $K_1(t)$. Some information can be gained by considering a new process $\{N(t), t \geq 0\}$ where $N(t)$ represents the number of permanent transfer occurrences from a given living unit by time t. Of course not all of those included in this count are still in the community at time t. It is easily seen that $\{N(t), t \geq 0\}$ is a renewal process. Furthermore, as shown by Jones (1994b), the inter-occurrence time distribution function is

$$F(x) = 1 + \beta_1 e^{-\alpha_1 x} - \beta_2 e^{-\alpha_2 x}, \ x \geq 0, \tag{3}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{\mu_{12} + \mu_{13} + \mu_{2\cdot} + \sqrt{(\mu_{12} + \mu_{13} + \mu_{2\cdot})^2 - 4[\mu_{13}\mu_{2\cdot} + \mu_{23}\mu_{12}]}}{2}, \\
\alpha_2 &= \frac{\mu_{12} + \mu_{13} + \mu_{2\cdot} - \sqrt{(\mu_{12} + \mu_{13} + \mu_{2\cdot})^2 - 4[\mu_{13}\mu_{2\cdot} + \mu_{23}\mu_{12}]}}{2}, \\
\beta_1 &= \frac{\mu_{12} + \mu_{2\cdot} - \alpha_1}{\alpha_1 - \alpha_2}, \\
\beta_2 &= \frac{\mu_{12} + \mu_{2\cdot} - \alpha_2}{\alpha_1 - \alpha_2}.
\end{aligned}$$

Let $m(t) = E[N(t)]$ be the renewal function for $\{N(t), t \geq 0\}$. Recognizing that

$$\begin{aligned}
dm(y) &= \Pr\Big(\text{permanent transfer occurs during } (y - dy, y]\Big) \\
&= \Big[\Pr(K_1(y) = 0)\mu_{13} + \Pr(K_1(y) = 1)\mu_{23}\Big]dy \\
&= [\gamma_1 + \gamma_2 e^{-(\mu_{12}+\mu_{2\cdot})y}]dy \qquad (4)
\end{aligned}$$

where

$$\gamma_1 = \frac{\mu_{13}\mu_{2\cdot} + \mu_{12}\mu_{23}}{\mu_{12} + \mu_{2\cdot}}$$

and

$$\gamma_2 = \frac{\mu_{12}(\mu_{13} - \mu_{23})}{\mu_{12} + \mu_{2\cdot}},$$

we find that

$$m(t) = \gamma_1 t + \frac{\gamma_2}{\mu_{12} + \mu_{2\cdot}}\left[1 - e^{-(\mu_{12}+\mu_{2\cdot})t}\right]. \tag{5}$$

Then

$$\lim_{t\to\infty} \frac{m(t)}{t} = \gamma_1. \tag{6}$$

Equation (6) provides the long-run expected number of permanent transfer occurrences per unit of time. Since the expected time that each permanent transfer spends in the SNF is $1/\mu_{34}$, from Little's Result [see Kleinrock (1973, p.17)] the long-run expected number of permanent transfers in the community is

$$\lim_{t\to\infty} E[J_1(t)] = \gamma_1/\mu_{34}. \tag{7}$$

Also,

$$\begin{aligned} E[J_1(t)] &= \int_0^t e^{-\mu_{34}(t-y)}dm(y) \\ &= \varepsilon_1 + \varepsilon_2 e^{-\mu_{34}t} + \varepsilon_3 e^{-(\mu_{12}+\mu_{2\cdot})t}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{\gamma_1}{\mu_{34}} \\ \varepsilon_2 &= -\frac{\gamma_1}{\mu_{34}} - \frac{\gamma_2}{\mu_{34} - \mu_{12} - \mu_{2\cdot}} \\ \varepsilon_3 &= \frac{\gamma_2}{\mu_{34} - \mu_{12} - \mu_{2\cdot}}. \end{aligned}$$

Clearly, (7) is the limit of (8).

It is also possible to obtain $Var[J_1(t)]$ and $Cov[J_1(t), K_1(t)]$ using the following approach. Conditioning on the time of the first permanent transfer we have

$$\begin{aligned}\Pr(J_1(t) = j) &= \int_0^t \Pr(J_1(t-x) = j)[1 - e^{-\mu_{34}(t-x)}]dF(x) \\ &\quad + \int_0^t \Pr(J_1(t-x) = j-1)e^{-\mu_{34}(t-x)}dF(x), \qquad (9) \\ &\qquad j = 1, 2, 3, \ldots .\end{aligned}$$

Multiplying both sides of (9) by j^2 and summing from $j = 1$ to ∞ we obtain

$$\begin{aligned}E[J_1^2(t)] &= \int_0^t \sum_{j=1}^{\infty} j^2 \Pr(J_1(t-x) = j)[1 - e^{-\mu_{34}(t-x)}]dF(x) \\ &\quad + \int_0^t \sum_{j=1}^{\infty}[(j-1)^2 + 2(j-1) + 1]\Pr(J_1(t-x) = j-1)e^{-\mu_{34}(t-x)}dF(x) \\ &= \int_0^t E[J_1^2(t-x)][1 - e^{-\mu_{34}(t-x)}]dF(x) \\ &\quad + \int_0^t \Big\{E[J_1^2(t-x)] + 2E[J_1(t-x)] + 1\Big\}e^{-\mu_{34}(t-x)}dF(x) \\ &= g(t) + \int_0^t E[J_1^2(t-x)]dF(x), \qquad (10)\end{aligned}$$

where

$$g(t) = \int_0^t \Big\{2E[J_1(t-x)] + 1\Big\}e^{-\mu_{34}(t-x)}dF(x). \qquad (11)$$

The interchange of summation and integration is justified by the fact that the integrands are positive. Equation (10) is a renewal equation with well known solution [see Feller (1971)]

$$E[J_1^2(t)] = g(t) + \int_0^t g(t-x)dm(x). \qquad (12)$$

To find a more convenient expression for $E[J_1^2(t)]$ we must first evaluate the integral in (11).

From (3) and (8) we have

$$
\begin{aligned}
g(t) &= \int_0^t \Big\{ 2\Big[\varepsilon_1 + \varepsilon_2 e^{-\mu_{34}(t-x)} + \varepsilon_3 e^{-(\mu_{12}+\mu_{2\cdot})(t-x)}\Big] + 1 \Big\} e^{-\mu_{34}(t-x)} \Big\{ \beta_2\alpha_2 e^{-\alpha_2 x} - \beta_1\alpha_1 e^{-\alpha_1 x} \Big\} dx \\
&= \zeta_1 e^{-\alpha_1 t} + \zeta_2 e^{-\alpha_2 t} + \zeta_3 e^{-\mu_{34} t} + \zeta_4 e^{-2\mu_{34} t} + \zeta_5 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})t}, \qquad (13)
\end{aligned}
$$

where

$$
\begin{aligned}
\zeta_1 &= -\frac{(2\varepsilon_1+1)\beta_1\alpha_1}{\mu_{34}-\alpha_1} - \frac{2\varepsilon_2\beta_1\alpha_1}{2\mu_{34}-\alpha_1} - \frac{2\varepsilon_3\beta_1\alpha_1}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_1} \\
\zeta_2 &= \frac{(2\varepsilon_1+1)\beta_2\alpha_2}{\mu_{34}-\alpha_2} + \frac{2\varepsilon_2\beta_2\alpha_2}{2\mu_{34}-\alpha_2} + \frac{2\varepsilon_3\beta_2\alpha_2}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_2} \\
\zeta_3 &= \frac{(2\varepsilon_1+1)\beta_1\alpha_1}{\mu_{34}-\alpha_1} - \frac{(2\varepsilon_1+1)\beta_2\alpha_2}{\mu_{34}-\alpha_2} \\
\zeta_4 &= \frac{2\varepsilon_2\beta_1\alpha_1}{2\mu_{34}-\alpha_1} - \frac{2\varepsilon_2\beta_2\alpha_2}{2\mu_{34}-\alpha_2} \\
\zeta_5 &= \frac{2\varepsilon_3\beta_1\alpha_1}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_1} - \frac{2\varepsilon_3\beta_2\alpha_2}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_2}
\end{aligned}
$$

Substituting (13) and (4) into (12) we obtain

$$
\begin{aligned}
&E[J_1^2(t)] \\
&= \zeta_1 e^{-\alpha_1 t} + \zeta_2 e^{-\alpha_2 t} + \zeta_3 e^{-\mu_{34} t} + \zeta_4 e^{-2\mu_{34} t} + \zeta_5 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})t} \\
&\quad + \int_0^t \Big\{ \zeta_1 e^{-\alpha_1(t-x)} + \zeta_2 e^{-\alpha_2(t-x)} + \zeta_3 e^{-\mu_{34}(t-x)} + \zeta_4 e^{-2\mu_{34}(t-x)} + \zeta_5 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})(t-x)} \Big\} \\
&\qquad\qquad \times [\gamma_1 + \gamma_2 e^{-(\mu_{12}+\mu_{2\cdot})x}] dx \\
&= \eta_1 + \eta_2 e^{-\alpha_1 t} + \eta_3 e^{-\alpha_2 t} + \eta_4 e^{-\mu_{34} t} + \eta_5 e^{-2\mu_{34} t} + \eta_6 e^{-(\mu_{12}+\mu_{2\cdot})t} + \eta_7 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})t}, \quad (14)
\end{aligned}
$$

where

$$
\eta_1 = \frac{\zeta_1\gamma_1}{\alpha_1} + \frac{\zeta_2\gamma_1}{\alpha_2} + \frac{\zeta_3\gamma_1}{\mu_{34}} + \frac{\zeta_4\gamma_1}{2\mu_{34}} + \frac{\zeta_5\gamma_1}{\mu_{12}+\mu_{2\cdot}+\mu_{34}}
$$

$$
\begin{aligned}
\eta_2 &= \zeta_1 - \frac{\zeta_1\gamma_1}{\alpha_1} - \frac{\zeta_1\gamma_2}{\alpha_1 - \mu_{12} - \mu_{2\cdot}} \\
\eta_3 &= \zeta_2 - \frac{\zeta_2\gamma_1}{\alpha_2} - \frac{\zeta_2\gamma_2}{\alpha_2 - \mu_{12} - \mu_{2\cdot}} \\
\eta_4 &= \zeta_3 - \frac{\zeta_3\gamma_1}{\mu_{34}} - \frac{\zeta_3\gamma_2}{\mu_{34} - \mu_{12} - \mu_{2\cdot}} \\
\eta_5 &= \zeta_4 - \frac{\zeta_4\gamma_1}{2\mu_{34}} - \frac{\zeta_4\gamma_2}{2\mu_{34} - \mu_{12} - \mu_{2\cdot}} \\
\eta_6 &= \frac{\zeta_1\gamma_2}{\alpha_1 - \mu_{12} - \mu_{2\cdot}} + \frac{\zeta_2\gamma_2}{\alpha_2 - \mu_{12} - \mu_{2\cdot}} + \frac{\zeta_3\gamma_2}{\mu_{34} - \mu_{12} - \mu_{2\cdot}} + \frac{\zeta_4\gamma_2}{2\mu_{34} - \mu_{12} - \mu_{2\cdot}} + \frac{\zeta_5\gamma_2}{\mu_{34}} \\
\eta_7 &= \zeta_5 - \frac{\zeta_5\gamma_1}{\mu_{12} + \mu_{2\cdot} + \mu_{34}} - \frac{\zeta_5\gamma_2}{\mu_{34}}.
\end{aligned}
$$

Then, of course, $Var[J_1(t)] = E[J_1^2(t)] - \{E[J_1(t)]\}^2$.

To find $Cov[J_1(t), K_1(t)]$, consider the following equation which is analogous to (9).

$$
\begin{aligned}
&\Pr(J_1(t)=j, K_1(t)=1) \\
&= \int_0^t \Pr(J_1(t-x)=j, K_1(t-x)=1)[1-e^{-\mu_{34}(t-x)}]dF(x) \\
&\quad + \int_0^t \Pr(J_1(t-x)=j-1, K_1(t-x)=1)e^{-\mu_{34}(t-x)}dF(x), \qquad (15) \\
&\qquad j = 1, 2, 3, \ldots.
\end{aligned}
$$

Multiplying both sides of (15) by j and summing from $j=1$ to ∞ we have

$$
\begin{aligned}
E[J_1(t)K_1(t)] &= \int_0^t \sum_{j=1}^{\infty} j \Pr(J_1(t-x)=j, K_1(t-x)=1)[1-e^{-\mu_{34}(t-x)}]dF(x) \\
&\quad + \int_0^t \sum_{j=1}^{\infty} [(j-1)+1] \Pr(J_1(t-x)=j-1, K_1(t-x)=1)e^{-\mu_{34}(t-x)}dF(x) \\
&= \int_0^t E[J_1(t-x)K_1(t-x)][1-e^{-\mu_{34}(t-x)}]dF(x) \\
&\quad + \int_0^t \Big\{E[J_1(t-x)K_1(t-x)] + \Pr(K_1(t-x)=1)\Big\}e^{-\mu_{34}(t-x)}dF(x) \\
&= h(t) + \int_0^t E[J_1(t-x)K_1(t-x)]dF(x), \qquad (16)
\end{aligned}
$$

where

$$h(t) = \int_0^t \Pr(K_1(t-x) = 1)e^{-\mu_{34}(t-x)}dF(x). \tag{17}$$

Equation (16) is a renewal equation with solution

$$E[J_1(t)K_1(t)] = h(t) + \int_0^t h(t-x)dm(x). \tag{18}$$

From (1) and (3), (17) becomes

$$\begin{aligned} h(t) &= \int_0^t \frac{\mu_{12} - \mu_{12}e^{-(\mu_{12}+\mu_{2\cdot})(t-x)}}{\mu_{12}+\mu_{2\cdot}} e^{-\mu_{34}(t-x)}(\beta_2\alpha_2 e^{-\alpha_2 x} - \beta_1\alpha_1 e^{-\alpha_1 x})dx \\ &= \theta_1 e^{-\alpha_1 t} + \theta_2 e^{-\alpha_2 t} + \theta_3 e^{-\mu_{34} t} + \theta_4 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})t}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \theta_1 &= \frac{\mu_{12}}{\mu_{12}+\mu_{2\cdot}}\left[\frac{\beta_1\alpha_1}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_1} - \frac{\beta_1\alpha_1}{\mu_{34}-\alpha_1}\right] \\ \theta_2 &= \frac{\mu_{12}}{\mu_{12}+\mu_{2\cdot}}\left[\frac{\beta_2\alpha_2}{\mu_{34}-\alpha_2} - \frac{\beta_2\alpha_2}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_2}\right] \\ \theta_3 &= \frac{\mu_{12}}{\mu_{12}+\mu_{2\cdot}}\left[\frac{\beta_1\alpha_1}{\mu_{34}-\alpha_1} - \frac{\beta_2\alpha_2}{\mu_{34}-\alpha_2}\right] \\ \theta_4 &= \frac{\mu_{12}}{\mu_{12}+\mu_{2\cdot}}\left[\frac{\beta_2\alpha_2}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_2} - \frac{\beta_1\alpha_1}{\mu_{12}+\mu_{2\cdot}+\mu_{34}-\alpha_1}\right]. \end{aligned}$$

Substituting (19) and (4) into (18) we have

$$\begin{aligned} &E[J_1(t)K_1(t)] \\ &= \theta_1 e^{-\alpha_1 t} + \theta_2 e^{-\alpha_2 t} + \theta_3 e^{-\mu_{34}t}\theta_4 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})t} \\ &\quad + \int_0^t \left\{\theta_1 e^{-\alpha_1(t-x)} + \theta_2 e^{-\alpha_2(t-x)} + \theta_3 e^{-\mu_{34}(t-x)}\theta_4 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})(t-x)}\right\} \\ &\qquad\qquad \times [\gamma_1 + \gamma_2 e^{-(\mu_{12}+\mu_{2\cdot})x}]dx \\ &= \kappa_1 + \kappa_2 e^{-\alpha_1 t} + \kappa_3 e^{-\alpha_2 t} + \kappa_4 e^{-\mu_{34}t} + \kappa_5 e^{-(\mu_{12}+\mu_{2\cdot})t} + \kappa_6 e^{-(\mu_{12}+\mu_{2\cdot}+\mu_{34})t}, \end{aligned} \tag{20}$$

where

$$
\begin{aligned}
\kappa_1 &= \frac{\theta_1\gamma_1}{\alpha_1}+\frac{\theta_2\gamma_1}{\alpha_2}+\frac{\theta_3\gamma_1}{\mu_{34}}+\frac{\theta_4\gamma_1}{\mu_{12}+\mu_{2\cdot}+\mu_{34}}\\
\kappa_2 &= \theta_1-\frac{\theta_1\gamma_1}{\alpha_1}-\frac{\theta_1\gamma_2}{\alpha_1-\mu_{12}-\mu_{2\cdot}}\\
\kappa_3 &= \theta_2-\frac{\theta_2\gamma_1}{\alpha_2}-\frac{\theta_2\gamma_2}{\alpha_2-\mu_{12}-\mu_{2\cdot}}\\
\kappa_4 &= \theta_3-\frac{\theta_3\gamma_1}{\mu_{34}}-\frac{\theta_3\gamma_2}{\mu_{34}-\mu_{12}-\mu_{2\cdot}}\\
\kappa_5 &= \frac{\theta_1\gamma_2}{\alpha_1-\mu_{12}-\mu_{2\cdot}}+\frac{\theta_2\gamma_2}{\alpha_2-\mu_{12}-\mu_{2\cdot}}+\frac{\theta_3\gamma_2}{\mu_{34}-\mu_{12}-\mu_{2\cdot}}+\frac{\theta_4\gamma_2}{\mu_{34}}\\
\kappa_6 &= \theta_4-\frac{\theta_4\gamma_1}{\mu_{12}+\mu_{2\cdot}+\mu_{34}}-\frac{\theta_4\gamma_2}{\mu_{34}}.
\end{aligned}
$$

Then $Cov[J_1(t), K_1(t)] = E[J_1(t)K_1(t)] - E[J_1(t)]E[K_1(t)]$ can be obtained using (20), (8) and (1).

The fact that we can find the first two moments of $J_1(t)$ and $K_1(t)$ is quite significant. Since CCRCs typically have up to several hundred living units, it will usually be reasonable to approximate the distributions of $J(t)$ and $K(t)$ by normal distributions. We therefore require only the first two moments. They are

$$E[J(t)] = \sum_{l=1}^{m} E[J_l(t)] = mE[J_1(t)], \tag{21}$$

$$E[K(t)] = \sum_{l=1}^{m} E[K_l(t)] = mE[K_1(t)], \tag{22}$$

$$Var[J(t)] = \sum_{l=1}^{m} Var[J_l(t)] = mVar[J_1(t)] \tag{23}$$

and

$$Var[K(t)] = \sum_{l=1}^{m} Var[K_l(t)] = mVar[K_1(t)]. \tag{24}$$

We may also be interested in the total number of residents, $S(t) \equiv J(t) + K(t)$, in the SNF at a given time. We have

$$E[S(t)] = E[J(t)] + E[K(t)] \tag{25}$$

and

$$Var[S(t)] = Var[J(t)] + Var[K(t)] + 2Cov[J(t), K(t)], \tag{26}$$

where

$$Cov[J(t), K(t)] = \sum_{l=1}^{m} Cov[J_l(t), K_l(t)] = mCov[J_1(t), K_1(t)]. \tag{27}$$

4 Numerical Results

This section illustrates the results of the previous section with a numerical example. The following (arbitrary) parameter values will be used.

$$\mu_{12} = 0.12 \quad \mu_{13} = 0.05$$

$$\mu_{21} = 0.05 \quad \mu_{23} = 0.07 \quad \mu_{24} = 0.12$$

$$\mu_{34} = 0.20$$

Consider a CCRC with 100 living units. Using equations (21) through (27) we have the following moments for various time points.

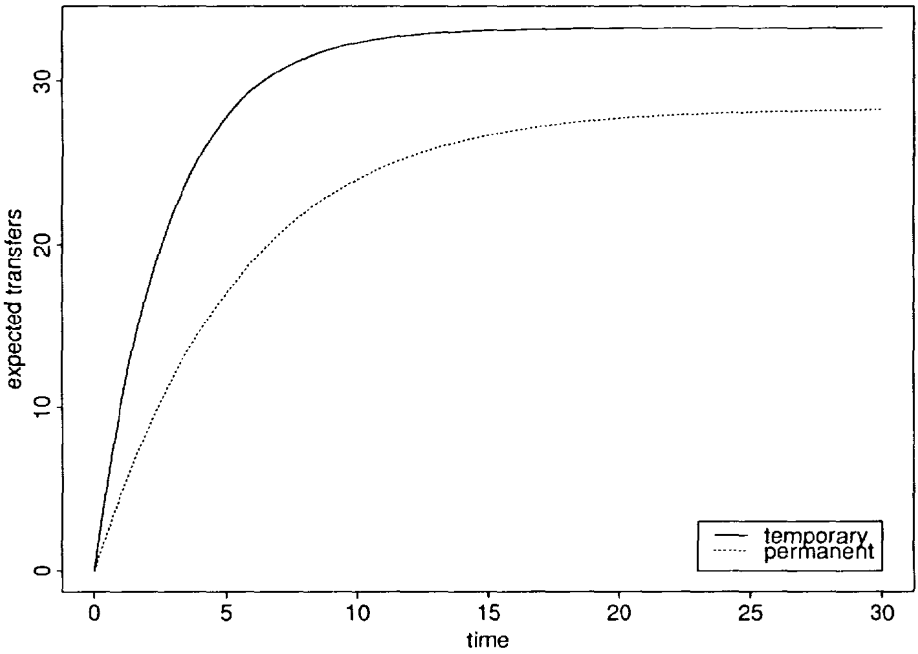
	t				
	1	5	10	20	∞
$E[J(t)]$	4.6316	17.0660	24.0488	27.7412	28.3333
$E[K(t)]$	10.0775	27.8234	32.4225	33.3084	33.3333
$E[S(t)]$	14.7090	44.8894	56.4713	61.0496	61.6667
$Var[J(t)]$	4.6286	16.9497	23.7845	27.4077	27.9960
$Var[K(t)]$	9.0619	20.0820	21.9103	22.2139	22.2222
$Var[S(t)]$	13.2332	32.8087	39.4643	42.8911	43.4722
$Cov[J(t), K(t)]$	-0.2287	-2.2225	-3.1153	-3.3653	-3.3730

Figures 2 through 5 display the results graphically. In Figure 2 the expected numbers of temporary and permanent transfers are shown for times from 0 to 30 years. We see that the expected number of temporary transfers is larger than the expected number of permanent transfers at all points in time. Also, the expected number of temporary transfers appears to converge more quickly to its limit. Figures 3 through 5 show the expected numbers of permanent, temporary and total transfers along with 95 percent (pointwise) confidence intervals. The confidence limits were calculated as the expected value plus or minus 1.96 times the standard deviation. Clearly the number of transfers to the SNF is subject to considerable variation.

5 Generalizations

The model discussed in this paper makes a number of assumptions for simplicity and mathematical convenience. They allow one to obtain explicit expressions for various quantities of interest. However, in order to make the model more realistic, certain generalizations must be considered. These generalizations pertain to the CCRC structure, the forces of decrement

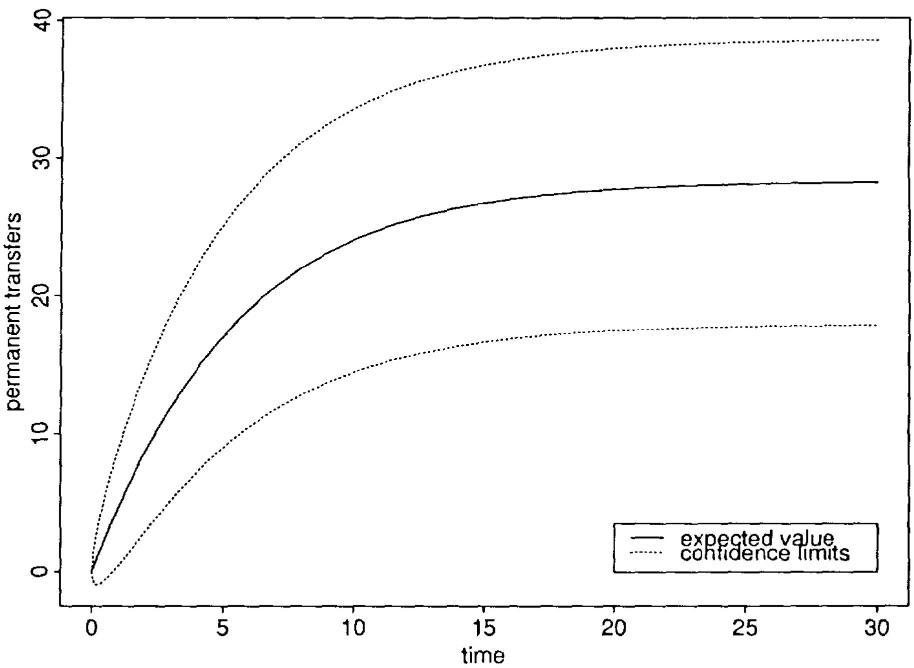
Figure 2: Expected Numbers of Permanent Transfers



and the demand for living units.

The model considers CCRCs that offer only single living units. In fact, most CCRCs provide double units for couples. The state of a double unit at time t cannot be described by $J_1(t)$ and $K_1(t)$ as is the case for single units. We require a third quantity, $L_1(t)$, which represents the number of ILU residents associated with a living unit at time t. $L_1(t)$ may be 0, 1 or 2. $K_1(t)$ may also be 0, 1 or 2. However, $K_1(t) + L_1(t)$ must be 1 or 2. As in the single unit case, $J_1(t)$ is a non-negative integer. Thus, the possible outcomes for $(J_1(t), K_1(t), L_1(t))$

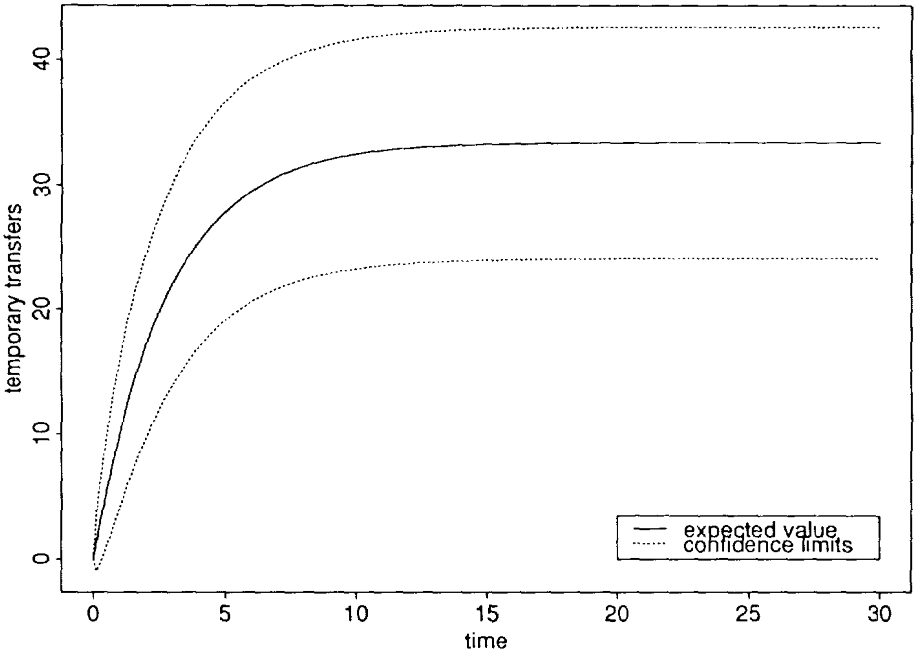
Figure 3: 95% Confidence Interval for Permanent Transfers



are $(j, 0, 2)$, $(j, 0, 1)$, $(j, 1, 0)$, $(j, 2, 0)$ and $(j, 1, 1)$ for $j = 0, 1, 2, \ldots$. The joint distribution of $K_1(t)$ and $L_1(t)$ can be analyzed by recognizing that $\{(K_1(t), L_1(t)), t \geq 0\}$ is a five-state Markov process. Results for $J_1(t)$ are more difficult to obtain since the permanent transfers no longer occur according to a renewal process. We can still find the long-run expected number of permanent transfers using Little's result.

CCRCs usually provide more than one level of care, often a personal care facility and a skilled nursing facility. This also increases the number of states and possible transitions.

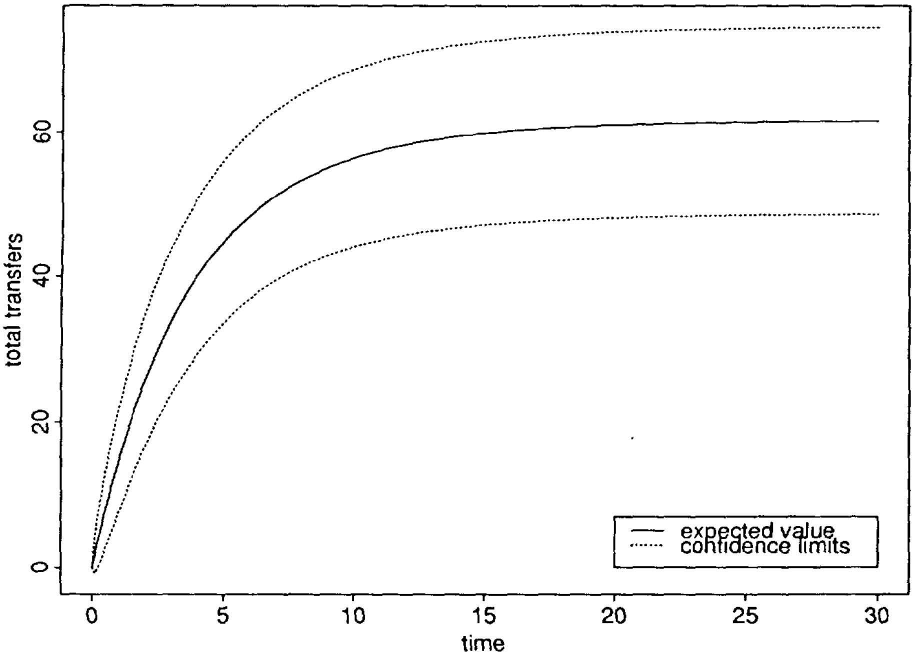
Figure 4: 95% Confidence Interval for Temporary Transfers



However, the analysis of the state of a CCRC unit can be approached in much the same way as described above.

As stated in Section 2, one must test whether or not it is reasonable to assume that forces of transition are constant. It may be that one must assume that forces vary by age of resident and/or time since entry to the various model states. If this is the case, the model becomes much less tractable mathematically. Numerical results can, however, be found using simulation techniques.

Figure 5: 95% Confidence Interval for Total Transfers



Finally, in the analysis of CCRCs with lower demand for living units, one cannot assume that the units are independent. The time until a vacant unit becomes occupied depends on the number of vacant units. Therefore, one must consider the state of the entire CCRC and not just one living unit. Simulation can again be used to obtain results in this case.

References

Actuarial Standards Board (1994). Actuarial Standard of Practice No. 3, Practices Relating to Continuing Care Retirement Communities (Revised Edition), July 1994.

Brace, G.L. (1994). Continuing Care Retirement Communities, *Health Section News*, Society of Actuaries, No. 28, 4–6.

Cumming, R.B. and W.F. Bluhm (1992) CCRC Population and Financial Model, *The Actuary* Vol. 26, No. 3, 3–5.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*, Volume II (Second Edition), Wiley, New York.

Jones, B.L. (1994a). A Stochastic Population Model for High Demand CCRCs, *Insurance: Mathematics and Economics*, to appear.

Jones, B.L. (1994b). Transient Results for a High Demand CCRC Model, University of Iowa, Department of Statistics and Actuarial Science, Technical Report #231.

Kleinrock, L. (1975). *Queueing Systems, Volume 1: Theory*, Wiley, New York.

Ross, S.M. (1983). *Stochastic Processes*, Wiley, New York.

Winklevoss, H.E. and A.V. Powell (1984). *Continuing Care Retirement Communities: An Empirical, Financial, and Legal Analysis*, Richard D. Irwin, Inc., Homewood, IL.