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AN ALTERNATIVE OPTION PRICING MODEL

Joseph D. Marsden, E.A., A.S.A.

ABSTRACT

A European call option pricing model similar to the Black-Scholes equation [1] is derived. Like the Black-Scholes equation, the model is based upon an assumption of a lognormal distribution of the price of a risky, non-dividend-paying security at a given point in the future. However, unlike Black-Scholes, the market's risk preference is recognized through a parameter that permits refinement of the model to agree either with experience or with future advances in the underlying theory.

GENERAL FORM

Derivation of a generalized Black-Scholes equation

The value of a European call option at expiration time $T > 0$ is $\max(0, S_T - X)$, where S_T is the price of the underlying security at time T and X is the exercise price. The value of the call option at time 0 is thus

$$(1) \qquad C_0 = \int_X^\infty PV[(S_T - X) \text{ conditionally payable at time } T]\, f(S_T)\, dS_T .$$

It will be assumed that S_T has a lognormal distribution for which

$$(2) \qquad u = (\ln S_T - \mu)/\sigma\sqrt{T}$$

has the standard normal distribution. Thus

$$(3) \qquad f(S_T) = \frac{1}{S_T \sigma \sqrt{2\pi T}} e^{-(\ln S_T - \mu)^2 / 2\sigma^2 T}, \text{ and}$$

$$(4) \qquad E(S_T) = e^{\mu + \sigma^2 T/2}$$

If an appropriate discounting factor $v(m,\sigma^2 T,S_T)$ can be found, (1) can be rewritten as

$$(5) \qquad C_0 = \int_X^{\infty} [S_T v(m,\sigma^2 T,S_T) - Xe^{-rT}] f(S_T) dS_T,$$

where r is the risk-free rate of return and thus Xe^{-rT} = PV(X payable at time T). The parameter m in $v(m,\sigma^2 T,S_T)$ will serve as a measure of the market's risk tolerance, and it will be further defined later. It will be assumed that

$$(6) \qquad v(m,\sigma^2 T,S_T) = e^{-rT - A(m,\sigma^2 T)[\ln S_T - \mu] - B(m,\sigma^2 T)\,\sigma^2 T/2},$$

where $A(m,\sigma^2 T)$ and $B(m,\sigma^2 T)$ will be chosen to fit certain constraints that will be identified later. Substituting (3) and (6) into (5) gives

$$(7) \qquad C_0 = \int_X^{\infty} S_T\,[e^{-rT - A(m,\sigma^2 T)[\ln S_T - \mu] - B(m,\sigma^2 T)\,\sigma^2 T/2} - Xe^{-rT - \ln S_T}]\,\frac{1}{S_T \sigma \sqrt{2\pi T}}\, e^{-(\ln S_T - \mu)^2/2\sigma^2 T}\, dS_T.$$

Combining (2) and (7) gives

$$(8) \qquad C_0 = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln X - \mu}{\sigma\sqrt{T}}}^{\infty} e^{-rT + \mu - [A(m,\sigma^2 T) - 1][u\sigma\sqrt{T}] - B(m,\sigma^2 T)\,\sigma^2 T/2 - u^2/2}\, du - \frac{Xe^{-rT}}{\sqrt{2\pi}} \int_{\frac{\ln X - \mu}{\sigma\sqrt{T}}}^{\infty} e^{-u^2/2}\, du.$$

If (9) $w = u + [A(m,\sigma^2 T) - 1]\sigma\sqrt{T}$, then

$$(10) \qquad C_0 = \frac{e^{-rT + \mu + \{[A(m,\sigma^2 T) - 1]^2 - B(m,\sigma^2 T)\}\sigma^2 T/2}}{\sqrt{2\pi}} \int_{\frac{\ln X - \mu}{\sigma\sqrt{T}} + [A(m,\sigma^2 T) - 1]\sigma\sqrt{T}}^{\infty} e^{-w^2/2}\, dw - Xe^{-rT}\, N[\frac{\mu - \ln X}{\sigma\sqrt{T}}].$$

where N[ ] is the cumulative standard normal function. This simplifies to

$$(11)\quad C_0 = e^{-rT + \mu + \{[A(m,\sigma^2T) - 1]^2 - B(m,\sigma^2T)\}\sigma^2T/2}$$

$$* N[\frac{\mu - \ln X - [A(m,\sigma^2T) - 1]\sigma^2T}{\sigma\sqrt{T}}] - Xe^{-rT} N[\frac{\mu - \ln X}{\sigma\sqrt{T}}].$$

If the strike price X = 0, then the call option has the same value as the security itself, or

$$(12)\quad S_0 = C_0(X = 0) = e^{-rT + \mu + \{[A(m,\sigma^2T) - 1]^2 - B(m,\sigma^2T)\}\sigma^2T/2}$$, and thus

$$(13)\quad \mu = \ln S_0 + rT - \{[A(m,\sigma^2T) - 1]^2 - B(m,\sigma^2T)\}\sigma^2T/2.$$

Substituting (12) and (13) into (11) gives

$$(14)\quad C_0 = S_0\, N[\frac{\ln S_0 - \ln X + rT - [A^2(m,\sigma^2T) - B(m,\sigma^2T) - 1]\sigma^2T/2}{\sigma\sqrt{T}}]$$

$$- Xe^{-rT} N[\frac{\ln S_0 - \ln X + rT - \{[A(m,\sigma^2T) - 1]^2 - B(m,\sigma^2T)\}\sigma^2T/2}{\sigma\sqrt{T}}].$$

The Black-Scholes equation is the case of (14) in which $A(m,\sigma^2T) = B(m,\sigma^2T) = 0$.

Recognition of the market's risk preference

Let m be defined as the adjustment to the expected rate of return on a risky security based on

$$(15)\quad S_0 = e^{-rT + \mu + [1 - D(m,\sigma^2T)]\sigma^2T/2}.$$

Combining (12) and (15) gives

$$(16) \quad [A(m,\sigma^2 T) - 1]^2 - B(m,\sigma^2 T) = 1 - D(m,\sigma^2 T).$$

$B(m,\sigma^2 T)$ is thus reduced to a function of $A(m,\sigma^2 T)$ and $D(m,\sigma^2 T)$, or

$$(17) \quad B(m\sigma^2 T) = A^2(m,\sigma^2 T) - 2A(m,\sigma^2 T) + D(m,\sigma^2 T).$$

Combining (17) with (14) gives

$$(18) \quad C_0 = S_0 N\left[\frac{\ln S_0 - \ln X + rT - [2A(m,\sigma^2 T) - D(m,\sigma^2 T) - 1]\sigma^2 T/2}{\sigma\sqrt{T}}\right]$$

$$- Xe^{-rT} N\left[\frac{\ln S_0 - \ln X + rT - [1 - D(m,\sigma^2 T)]\sigma^2 T/2}{\sigma\sqrt{T}}\right].$$

BOUNDARY CONDITIONS

In a risk neutral market,

$$(19) \quad S_0 = e^{-rT} E(S_T) = e^{-rT + \mu + \sigma^2 T/2}.$$

If m = 0 represents the risk neutral case, then combining (15) with (19) gives

$$(20) \quad S_0 = e^{-rT + \mu + \sigma^2 T/2} = e^{-rT + \mu + [1 - D(0,\sigma^2 T)]\sigma^2 T/2}, \text{ so}$$

$$(21) \quad D(0,\sigma^2 T) = 0.$$

Also in the risk neutral case, the discounting factor $v(m,\sigma^2 T,S_T)$ should be unrelated to S_T, or

(22) $A(0,\sigma^2 T) = 0.$

SAMPLE SOLUTION

If it is assumed that the risk premium for a security is directly related to the security's variance, or

(23) $(E[S_T]/S_0) - rT = m\sigma^2 T/2$, then

(24) $D(m,\sigma^2 T) = m.$

Combining (18) with (24) gives

(25) $$C_0 = S_0 N\left[\frac{\ln S_0 - \ln X + rT - [2A(m,\sigma^2 T) - m - 1]\sigma^2 T/2}{\sigma\sqrt{T}}\right]$$

$$- Xe^{-rT} N\left[\frac{\ln S_0 - \ln X + rT - (1 - m)\sigma^2 T/2}{\sigma\sqrt{T}}\right].$$

If it is also assumed that all of the risk premium on a risky security is directly due to relative preference for different values of S_T rather than due to any direct bias related to variance, or

(26) $B(m\sigma^2 T) = 0,$

then combining (16), (24), and (26) gives

(27) $[A(m,\sigma^2 T) - 1]^2 = 1 - m,$

or, given the constraint of (22),

(28) $A(m,\sigma^2 T) = 1 - \sqrt{1 - m}.$

Substituting (28) into (25) gives

$$(29)\quad C_0 = S_0 N\left[\frac{\ln S_0 - \ln X + rT - (1 - m - 2\sqrt{1 - m})\sigma^2 T/2}{\sigma\sqrt{T}}\right]$$

$$- Xe^{-rT} N\left[\frac{\ln S_0 - \ln X + rT - (1 - m)\sigma^2 T/2}{\sigma\sqrt{T}}\right].$$

CONCLUSION

There are an infinite number of solutions for $A(m,\sigma^2 T)$ for each $D(m,\sigma^2 T)$, with the only strict requirements for $A(m,\sigma^2 T)$ and $D(m,\sigma^2 T)$ being

(21) $D(0,\sigma^2 T) = 0$ and

(22) $A(0,\sigma^2 T) = 0$.

The selection of $D(m,\sigma^2 T)$ is already the subject of much research related to the Capital Asset Pricing Model (CAPM). Note that CAPM theory indicates that the risk premium of an asset should be related to its nondiversifiable risk rather than to its individual risk. This should be no less true of options, so for all purposes of the model presented here

(30) $\sigma^2 = \beta\sigma_M^2$

should be used, where σ_M^2 is the variance of the return of the market as a whole and β is the covariance of the return of the individual security with the return of the market divided by

σ_M^2[2]. Empirical studies may also refine the selection of $A(m,\sigma^2 T)$.

It is hoped that the flexibility of the general equation (18) with the constraints of (15), (21) and (22) will allow specific equations to be developed that will overcome some of the shortcomings of the Black-Scholes model, such as its tendency to undervalue deep in-the-money calls and overvalue deep out-of-the-money calls [3].

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[1] Black, Fischer; and Scholes, Myron; "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (May/June 1973).

[2] Bodie, Zvi; Kane, Alex; and Marcus, Alan J.; Investments, Second Edition; Homewood, IL: Richard D. Irwin, 1989 and 1993, Ch. 8.

[3] Ibid, p. 696.