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ASSESSING RISK FOR INSURANCE FUNDED BY ZERO COUPONS WITH STOCHASTIC INTEREST RATES

by

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ABSTRACT

The investment risk generated by a stochastic interest rate is recognized as often the greatest risk associated with a whole life or retirement policy. Recent research has investigated the mathematics of this risk when the interest rate follows a stochastic process and the time to death is considered to be random. In these cases the investment instrument is the money market. Though these investigations add considerably to knowledge on the behavior of risk, in practice a company's investment portfolio is made up of various investment instruments. In this paper we focus on the use of the zero coupon bond as an investment instrument to fund a whole life policy. It is unlikely that a company will entirely fund a particular single policy with a zero coupon bond. However, the paradigms presented here provide insight into the behavior and risk associated with this investment strategy when the time to sale of the bond is a contingent function of a life. This paper presents the net single premium amount for funding a life policy using zeroes. It also presents the mathematics for the variance of the loss for such a funding strategy. In conclusion it appears that the value of a zero coupon bond to a company is a function of the time to death random variable as well as the demographic profile of the population of insured.

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I. INTRODUCTION

The purpose of this paper is to examine the additional risk associated with funding a life insurance policy with zero coupon bonds when the interest rate for pricing and valuing the bonds follows a stochastic process. It has been formally recognized for over a decade that the risk associated with interest often dominates other risks in the insurance business. Recent insolvencies and failures of insurance companies have attracted public attention regarding financial assessment of the risk associated with liabilities resulting in increased need to assess interest based risk.

The introduction of the probabilistic approach to life contingencies (Pollard and Pollard 1969) and the stochastic approach to interest rates (Pollard 1971) provides the theoretical framework to examine the various sources of risk associated with the insurance business. The recognition of the need to place the theory of life contingencies on a probabilistic foundation is evidenced by the book by Bowers et al (1986) used by several actuarial societies (see also the review of this book by Dhaene 1989). The result of assuming time to death, for example, as a random variable provides a method of identifying sources of risk associated with the random time to payment or to cessation of payment. As shown by Frees (1988) when time to death is a random variable, prospective and retrospective methods of calculating net single premium do not agree. Here we use the prospective method of calculating net single premium.

Also important to the operations of the insurance company is the risk associated with interest rates that follow some random process. Boyle (1976) explored life contingencies assuming the interest rates follow a lognormal distribution. Waters (1978) explored the calculation of the moments of insurance assuming various discrete mortality models with interest rates that follow an autoregressive process of order 2 (espoused by Pollard 1971). Panjer and Bellhouse (1978, 1980, 1981) explored calculations of life annuities and life insurance assuming a discrete mortality model with interest rates driven by a normal process (with the autoregressive case as an example). Beekman and Fuelling (1990, 1991) examined the problems with insurance product calculations when the mortality followed a Makeham probability distribution and the interest rates were driven by an Ornstein-Uhlenbeck process and its special case, the Wiener process.

In all of these research efforts the type of asset used is essentially the money market (MM) in which the value is accumulated through instantaneous compounding. Often times, however, the insurance company will fund a portion of its insurance liability through non-compounding products such as coupon bonds (which are bonds with periodic payments, often semi-annuals, and with a larger final payment at maturity) or zero coupon (ZC) bonds (which are pure discount bonds with a single payment at maturity). One of the advantages of ZCs is that there is no reinvestment risk until maturity. However, the price movement of ZCs can be very different from MMs. Consequently, using an MM model to assess the effects of risk associated with stochastic interest can be misleading.

When assessing the risk associated with funding using ZCs, it is necessary to have a model for the term structure of yield rates or values of ZCs over time. There are several models for linking the instantaneous interest rate or spot rate with the forward rates used to price ZC bonds (see Heath, Jarrow, and Morton, 1992, for references to various models). The model we consider here is the one factor model (Vasicek, 1977, and Cox, Ingersoll, and Ross, 1985). In essence, this model assumes that the changes in instantaneous rates of returns of ZCs depends only on the current interest rate.

In this paper we examine the cost and risk as measured by variance of loss associated with a series of investment horizons for ZCs used to fund a single premium life insurance. We assume that the interest follows stochastic dynamics with stochasticity generated by the Wiener process such that the mean and covariance kernel are known. We also assume that the horizon, or time to maturity of the ZC can be selected by the investor. Based on these assumptions we determine the required investment for a net single premium using the equivalency principle (see Frees 1988). We also determine the variance of the loss. As expected, both the required investment and the variance are functions of the mortality patterns, the mean and variance of the stochastic process driving the interest, and the pattern of maturity horizons selected.

II. PRELIMINARIES

Define the following:

H_i, $i = 1, 2, \ldots$ is a sequence of investment horizons with $H_i > H_{i-1}$ for all i, and $H_1 > 0$.

$W(t)$ = zero mean Wiener process with incremental variance dt.

$r(t)$ = force of interest at time t.

$P(t,T,r)$ = price of a ZC bond at time $t \leq T$ paying 1 at maturity T given that $r(t)=r$.

$f(x_1, x_2, \ldots, x_k)$=joint density function of r(t) at times $t_1, t_2, \ldots, t_k$.

Let $F(x_1,\ldots,x_k)$ be the joint distribution function of $r(t_1),\ldots,r(t_k)$. We will assume for convenience that the joint density, denoted above as $f(x_1,\ldots,x_k)$, of F exists for each k. The form of $F(x_1,\ldots,x_k)$ is dictated by the dynamics assumed to generate r(t) over time. For example, often r(t) is assumed to be generated by a stochastic differential equation. This can result in a joint distribution of $r(t_1)\ldots r(t_k)$ that is Gaussian and sometimes Markovian. One common one factor model for r(t) is

$$dr(t) = \alpha(t, r(t))\, dt + \beta(t, r(t))\, dW(t)\,, \tag{1}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are real, deterministic functions and $W(t)$ represents the Wiener process at time t.

Vasicek (1977) shows that, under suitable conditions on equation (1), the function $P(t,T,r)$ is of the form

$$\begin{aligned} P(t,T,r) = E[\exp(-\int_t^T r(\tau)d\tau - \frac{1}{2}\int_t^T \{q(\tau,r(\tau))\}^2 d\tau \\ + \int_t^T q(\tau,r(\tau))dW(\tau))\,|\,r(t) = r]\,. \end{aligned} \tag{2}$$

In this conditional expectation, the function $q(\tau,\delta(\tau))$ is the *market value of risk.* Positing a model for $q(\tau, r(\tau))$ is difficult and makes the solution to (2) complicated. One alternative proposed by Vasicek is to assume $q(\tau, r(\tau))$ is a constant, say q_0. As noted by Cox, Ingersoll, and Ross (1981), for continuous time models the *local expectation hypothesis*, which states that q is zero over intervals of positive measure, is the only acceptable model of equilibrium for the one factor model. We give a general solution to the net single premium for an insurance funded with ZC's, assuming a one factor model. We specialize this solution to the model proposed by Vasicek with $q(\tau, r(\tau)) = q_0$. The example at the end of this paper examines only the special case of $q_0 = 0$, the local expectations hypotheses. We note that though the Vasicek model is considered an older model with the undesirable property that it allows negative interest rates, it admits a closed form solution to funding using ZC's for fixed time to death and, consequently, provides a basis for comparing different ZC strategies with an MM strategy. More sophisticated one factor models will result in different magnitudes in the comparisons but the patterns will be similar.

To fund the policy we will assume that a net single premium of amount 1 is invested by purchasing a ZC at the time the policy is issued with a horizon or maturity date of the bond of H_1. The bond yield rate is determined using the known current interest rate and the expected term structure of the bond. Thus the yield at maturity is $P(0,H_1,r(0))^{-1}$. Should death occur at time t prior to H_1, the ZC is sold at its present value determined as $P(t,H_1,r(t))*P(0,H_1,r(0))^{-1}$. Should the life exceed H_1, the maturity value of the bond at H_1 is reinvested at price per unit at maturity of $P(H_1,H_2,r(H_1))$ with a horizon $H_2 - H_1$. The process of reinvesting is continued until death occurs at which time the bond is sold at its present value determined from the forward rate using the then current interest rate.

To calculate the expected single premium, assume the time to death t is known and that the interest rates $r_0, ..., r_{k-1}$ for horizons $H_0=0, ..., H_{k-1}$ are also known. Assume $H_{k-1} \leq t < H_k$. Assume that the interest rate at time t, r(t), is also known. Then the accumulation at time t of an amount of 1 invested at time zero is

$$\begin{aligned} ACCUM = {} & P(H_0,H_1,r_0)^{-1} P(H_1,H_2,r_1)^{-1} \cdots P(H_{k-1},H_k,r_{k-1})^{-1} \\ & P(t, H_k, r(t)) \end{aligned} \tag{3}$$

Conditional on t, $r_0, \ldots, r_{k-1}$, and r(t), the present value of an insurance of 1 payable at t is

$$PV = \frac{1}{ACCUM}. \tag{4}$$

Thus the expected present value is the expectation taken over time and over interest rates. By the monotone convergence theorem we can take expectation in either order.

To calculate the expectation with respect to $r_0, ..., r_k$, r(t), we note that P(t,T,r) is a function of r(t) = r. For fixed, known r, this function has no random component. The random component, for fixed t, is introduced when r is random. Thus expectation of P(t,T,r(t)) is taken with respect to the probability density of r(t). The expected present value, conditional on t of the bond given an initial investment of 1 is

$$E(PV \mid t) = \int \cdots \int P(H_0, H_1, r_0)\, P(H_1, H_2, r(H_1) = x_1) \\ P(H_k, H_{k+1}, r(H_k) = x_k)\, P(t, H_{k+1}, r(t) = x_{k+1})^{-1} \\ f_\delta(x_1, \ldots, x_k, x_{k+1})\, dx_1, \ldots, dx_{k+1}, \tag{5}$$

where $H_k < t \leq H_{k+1}$. Similarly, the second moment is given as

$$E((PV)^2 \mid t) = \int \cdots \int P(H_0, H_1, r_0)^2\, P(H_1, H_2, r(H_1) = x_1)^2 \\ \ldots P(H_k, H_{k+1}, r(H_k) = x_k)^2\, P(t, H_{k+1}, r(t) = x_{k+1})^{-2} \\ f(x_1, \ldots, x_{k+1})\, dx_1 \ldots dx_{k+1}. \tag{6}$$

Equations (5) and (6) are prospective in that they provide the expected present values of accumulation of the investment strategy at time t=0, given time of death at time t is known.

Equations (5) and (6) can be simplified if the process r(t) is a Markov process. In this case, the joint density function f(.) can be written as a product of conditional densities as

$$f(x_1, \ldots, x_{k+1}) = f(x_1 \mid r_0)\, f(x_2 \mid r(H_1) = x_1) \ldots f(x_{k+1} \mid r(H_k) = x_k). \tag{7}$$

For many cases, such as the Vasicek (1977) model and the Cox, Ingersoll, Ross (1985) model, the conditional density is known.

III. RANDOM TIME TO DEATH

With an expression for $P(t,T,r)$ and the Markov density $f(x \mid r = y)$ equations (5) and (6) specify the expected present value of the investment strategy given a fixed time to death t. To determine the expected present value unconditional on the knowledge of t, we integrate over all values of t using the law of total probability.

The relevant mortality variables are defined as follows:

T_x = a random variable representing the future life time of an individual aged x.

μ_x = hazard function or force of mortality for an individual aged x.

${}_tp_x$ = probability an individual aged x survives to x+t.

$g_x(t)$ = density function of T_x.

$$= {}_tp_x \, \mu_{x+t}. \tag{8}$$

As an illustration we will use the Makeham mortality model

$$\mu_x = \mu + \xi\gamma^x \tag{9}$$

with parameters, similar to those given by Bowers, et. al. (1986), i.e.,

$\mu = 0.00059$

$\xi = 0.0000707$

$\gamma = 1.104.$

The following important assumption will allow us to take this unconditional expectation:

ASSUMPTION 1. *The random variable T_x is independent of the random process $r(t)$.*

In this assumption we assume that the likelihood of dying does not depend on the interest rate past, present or future. Although it is unclear how changes in interest rate would directly influence mortality, both the selection process and the likelihood of termination of the plan could both affect the mortality experience indirectly. In this case Assumption 1 would be violated.

Using the principle of equivalency, the net single premium is defined as

$$P = E_T[E(PV \mid t)]. \tag{10}$$

Frees has shown that this equivalent to the premium defined using the prospective loss method for single premium insurances.

IV. INTEREST FOLLOWING A STATIONARY MARKOV GAUSSIAN PROCESS

In this section we illustrate the results of equation (14) when the interest follows a stationary Gaussian Markov or Ornstein/Uhlenbeck (O/U) process (see Iranpour and Chacon, 1988). The interest rate r(t) follows an O/U process if the stochastic dynamics are given by equation (1) with $\alpha(t, r(t)) = \alpha * (\delta_0 - r(t))$ and $\beta(t, r(t)) = \rho$, where α, δ_0 and ρ are constants. In this case, the conditional expectation of the interest at time s, given the rate at time $t < s$ is

$$E(r(s) \mid r(t) = r_0) = \delta_0 + (r_0 - \delta_0)e^{-\alpha(s-t)}. \tag{11}$$

The conditional variance of r(s) given $r(t) = r_0$, $t < s$, is

$$Var(r(s) \mid r(t) = r_0) = \frac{\rho^2}{2\alpha}(1 - e^{-2\alpha(s-t)}). \tag{12}$$

To assess the differences between the MM and the various zero strategies, we solved equation (12) for various values of ρ and α as follows. From (12) as $t \to -\infty$, the unconditional interest rate standard deviation can be expressed as

$$\text{st.dev.}\,(r(s)) = \frac{\rho}{\sqrt{2\alpha}}$$

The level of "memory" of the process is dictated by the conditional variance given in equation (12). Beekman and Fuelling (1990) suggest that α is about 0.17. In this paper we will use a higher value of α for faster reversion of about 0.01 to 0.02. We will use a range of values of ρ between 0.006 and 0.012.

Let the random variable $Y(t,s)$ be defined for fixed t and s, $t < s$, as

$$Y(t,s) = \int_t^s r(\tau)\,d\tau.$$

Then the conditional expectation of $Y(t,s)$ given $r(t) = r_0$ is

$$E(Y(t,s)\,|\,r(t) = r_0) = \delta_0(s-t) + \frac{r_0 - \delta_0}{\alpha}\left(1 - e^{-\alpha(s-t)}\right). \tag{13}$$

The conditional variance is

$$\mathrm{Var}\left(Y(t,s)\,|\,r(t) = r_0\right) = \frac{\rho^2}{\alpha^2}\left((s-t) - \frac{1}{\alpha}\left(1 - e^{-\alpha(s-t)}\right) - \frac{1}{2\alpha}\left(1 - e^{-\alpha(s-t)}\right)^2\right). \tag{14}$$

When $r(t)$ follows an O/U process and $q(\tau, r(\tau)) = q_0$ is a non-zero constant, equation (2) has the solution (see Vasicek 1977)

$$P(t,s,r) = \exp\left\{\frac{1}{\alpha}(1 - e^{\alpha(s-t)})\ (\delta + \frac{\rho q}{\alpha} - \frac{1}{2}\rho^2\alpha^2 - r) - (s-t)\right.$$
$$\left.\left(\delta + \frac{\rho q}{\alpha} - \frac{1}{2}\rho^2\alpha^2\right) - \frac{\rho^2}{4\alpha^3}(1 - e^{-\alpha(s-t)})^2\right\}. \qquad t \le s \tag{15}$$

Equation (15) provides a simple alternative to the local expectations hypothesis. In the development below we will use equation (15) as the function for price. The local expectations hypothesis result is obtained by setting $q = 0$.

To determine a closed form solution to the equations (5) and (6) we note that the random variables $r(s)$ given $r(t) = r_0$ and $Y(s, t)$ given $r(t) = r_0$ are both Gaussian. Thus, with the conditional expectations given in equations (11) - (14) we can use equation (7) to get integral expressions for (5) and (6). Using the fact that $E(e^{\delta r(s)} \mid r(t) = r_0)$ is the moment generating function of a Gaussian random variable, equations (5) and (6) can be expressed explicitly. An amount of algebraic manipulation is involved in this solution. A brief description of these manipulations is given in the appendix. To give the end result, we define the following

$$\begin{aligned}
d(\tau) &= -\frac{1}{\alpha}(1 - e^{-\alpha\tau}) \\
m(\tau) &= -\delta_0[\tau - d(\tau)] \\
\sigma^2(\tau) &= -\frac{\rho^2}{\alpha^2}[\tau - d(\tau) - \frac{\alpha}{2}d^2(\tau)] \\
q(\tau) &= -q_0\frac{\rho}{\alpha}[d(\tau) - \tau] \\
a(\tau) &= \delta_0\alpha d(\tau) \\
b(\tau) &= \frac{\rho^2}{2\alpha}(1 - e^{-2\alpha\tau}) \\
c(\tau) &= e^{-\alpha\tau} \\
\Delta_i &= H_{i+1} - H_i
\end{aligned}$$

For any time of death t, $H_n < t \le H_{n+1}$, define

$$x = H_{n+1} - t$$

$$\tau = t - H_n .$$

Using these definitions, for fixed t, $H_n < t \le H_{n+1}$

$$\begin{aligned}
P = {} & \exp\left\{-m(x) + \frac{1}{2}\sigma^2(x) + q(x)\right\} \\
& \exp\left\{\sum_{i=0}^{n}\left[m(\Delta_i) - \frac{1}{2}\sigma^2(\Delta_i) - q(\Delta_i)\right]\right\} \\
& \exp\left\{r_0\sum_{j=0}^{n}\left[d(\Delta_j)\prod_{i=1}^{j} c(\Delta_{i-1})\right] - d(x)c(\tau)\prod_{i=1}^{n} c(\Delta_{i-1})\right\} \\
& \exp\left\{\sum_{j=0}^{n-1}\left[\sum_{k=0}^{j} d(\Delta_{n-j+k})\prod_{i=1}^{k} c(\Delta_{n-j+i-1})\right.\right. \\
& \qquad \left.\left. - d(x)c(\tau)\prod_{i=1}^{j} c(\Delta_{n-j+i-1})\right] a(\Delta_{n-j-1}) - d(x)a(\tau)\right\} \\
& \exp\left\{\frac{1}{2}d^2(x) + \frac{1}{2}\sum_{j=0}^{n-1}\left[\sum_{k=0}^{j} d(\Delta_{n-j+k})\prod_{i=1}^{k} c(\Delta_{n-j+i-1})\right.\right. \\
& \qquad \left.\left. - d(x)c(\tau)\prod_{i=1}^{j} c(\Delta_{n-j+i-1})\right]^2 b(\Delta_{n-j-1}) + \frac{1}{2}d^2(x)b(\tau)\right\}
\end{aligned} \tag{16}$$

In all cases $\prod_{i=1}^{0} c(\Delta_i) \equiv 1$. Although equation (15) is somewhat involved, its closed form representation provides a simplification for solving (10) since the Makeham mortality law will require a numerical solution.

Equation (4) follows a similar pattern with the appropriate change in the exponents. The expectation for the second moment gives the same result as in equation (16) with the functions $m(\cdot)$, $\sigma(\cdot)$, $q(\cdot)$ and $d(\cdot)$ all being doubled. Since $d(\cdot)$ appears with both powers of one and two, the result of doubling each of these functions is not the same as doubling the exponent.

We use a numerical integral to solve equation (10). Table 1 gives the solution for net single premium for various strategies and various values of ρ and α. In this table q is set to zero. Also in this table we have assumed that the current interest rate, r_0, is equal to the long run average, δ_0.

In Table 1, the column headings refer to the reinvestment strategy horizons. Thus, a horizon of 10 means that a ten year ZC bond is originally purchased for an amount listed in the table. At maturity the ZC bond yield is reinvested in another ten year horizon bond and so forth until death, at which time the bond is sold and used to pay the benefit. All ZC bonds are replaced at maturity with ZC bonds with the same horizon as its predecessor. The numbers quoted in the body of the table refer to the ratio of the single premium investment required under the various strategies to make $EL_T = 0$ relative to the required investment under MM. Thus a value of .99 would mean that the required single premium would be 1% higher than that required under the money market.

As we examine Table 1, we see that, as expected, the net single premium required increases as the horizon increases. However, note that the increase depends on both the process standard deviation and the memory as measured by α. If the interest process standard deviation is larger, the required net single premium for a ZC is relatively larger than that required for an MM strategy. Also, as the memory increases with decreasing α, the relative amount required increases. In the cases considered here where the process standard deviation is small, there is very little difference in the single premium rates as a function of the funding strategy. One would expect larger differences as the standard deviation of the process generating the term structure of interest is increased. In this case however, the Vasicek approximation becomes less reliable, allowing negative interest rates.

V. CONCLUSION

In this paper we have given the general form for calculating the expected value of a single premium payment for a life insurance under a zero coupon (ZC) investment strategy when the instantaneous rate of interest follows a random process. A key assumption regarding pricing a ZC to use to match with the insurance liability is the "market value of risk." Under the one factor model of Vasicek we assumed zero. However, from Jensen's inequality, we know that even under the local expectations hypothesis there is a risk premium implicitly built into the investment strategy. The existence of this implicit risk premium necessitates management decisions regarding the level of risk and the amount risk premium the insurer wishes to take. The actual value of this risk premium seems very small in the examples considered here.

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Table 1

Ratio of net single premium under MM investment stategy to the net single premium under various ZC strategies for various parameters of the O/U process for an individual aged 30. Net single premium is calculated as the expected present value of the risk. Mortality follows the Makeham mortality model and long term average short term interest is set at 0.07.

Horizons

$\alpha = 0.31$	**10**	**15**	**20**	**25**	**30**	**35**
$\rho = .006$	0.999742	0.999170	0.998827	0.998608	0.998467	0.998381
$\rho = .009$	0.999421	0.998135	0.997363	0.996867	0.996551	0.996359
$\rho = .012$	0.998974	0.996686	0.995314	0.994435	0.993873	0.993532

$\alpha = 0.40$	**10**	**15**	**20**	**25**	**30**	**35**
$\rho = .006$	0.999638	0.999352	0.999196	0.999100	0.999040	0.999003
$\rho = .009$	0.999187	0.998543	0.998193	0.997976	0.997842	0.997757
$\rho = .012$	0.998554	0.997409	0.996787	0.996402	0.996164	0.996014

$\alpha = 0.49$	**10**	**15**	**20**	**25**	**30**	**35**
$\rho = .006$	0.999652	0.999495	0.999413	0.999363	0.999333	0.999313
$\rho = .009$	0.999218	0.998863	0.998680	0.998568	0.998499	0.998455
$\rho = .012$	0.998609	0.997979	0.997653	0.997454	0.997333	0.997253

Appendix

To determine the expected present value at time t, for a bond purchased at time zero under the Vasicek model, we proceed as follows. First, assume $0 < t \leq H_1$. In this case

$$
\begin{aligned}
EV_t &= \int P(0,H_1,r_0)^{-1} P(t,H_1,r_1) f(r_1|r_0)\,dr_1 \\
&= P(0,H_1,r_0)^{-1} \exp\left\{-m(x) + \frac{1}{2}\sigma^2(x) + q(x)\right\} \int \exp(-r_1 d(x)) f(r_1|r_0)\,dr_1,
\end{aligned}
\tag{A.1}
$$

where $x = H_1 - t$. Note that the integral is the moment generating function of $f(r_1 | r_0)$ where the argument of the function is $d(x)$. The density $f(r_1|r_0)$ is Gaussian under the Vasicek model. From Vasicek, $P(0, H_1, r_0)$ is given by equation (20) in the text. Putting this into the current notation and recalling the generating function we get, for $0 < t < H_1$

$$
\begin{aligned}
EV_t = \exp\Big\{ & m(H_1) - \frac{1}{2}\sigma^2(H_1) + q(H_1) - m(x) + \frac{1}{2}\sigma^2(x) - q(x) \\
& - d(x)a(t) + \frac{1}{2}d^2(x)b(t) + r_0(d(H_1) - d(x)c(t))\Big\}
\end{aligned}
\tag{A.2}
$$

For $H_1 < t \leq H_2$, define $x = H_2 - t$ and $\tau = t - H_1$. Then

$$
EV_t = P(0,H_1,r_0)^{-1} \int\int P(H_1,H_2,r_1)^{-1} P(t,H_2,r_2) f(r_2|r_1) f(r_1|r_0)\,dr_2\,dr_1 \tag{A.3}
$$

$$
\begin{aligned}
&= P(0,H_1,r_0)^{-1} \exp(-m(x) + \frac{1}{2}\sigma^2(x) + q(x)) \\
&\int\int P(H_1,H_2,r_1)^{-1} \exp(-r_2 d(x)) f(r_2|r_1) f(r_1|r_0)\,dr_2\,dr_1
\end{aligned}
\tag{A.4}
$$

Noting that the inner integral is the generating function associated with $f(r_2|r_1)$ we can rewrite (A.4) as

$$
\begin{aligned}
&= P(0,H_1,r_0)^{-1} \exp\left\{-m(x) + \frac{1}{2}\sigma^2(x) + q(x) - d(x)a(\tau) + \frac{1}{2}d^2(x)b(\tau)\right\} \\
&\exp\left\{m(\Delta) - \frac{1}{2}\sigma^2(\Delta) - q(\Delta)\right\} \int \exp(r_1[d(\Delta) - d(x)c(\tau)]) f(r_1|r_0)\,dr_1,
\end{aligned}
\tag{A.5}
$$

where $\Delta = H_2 - H_1$.

Again we see that the integral expression is the generating function for $f(r_1|r_0)$, but with a more complicated argument. Substituting the expression for the generating function for the integral and using the definition of $P(0, H_1, r_0)$ we get

$$\begin{aligned} EV_T = \exp\Big\{ & m(H_1) - \frac{1}{2}\sigma^2(H_1) - q(H_1) \\ & + m(\Delta) - \frac{1}{2}\sigma^2(\Delta) - q(\Delta) - m(x) + \frac{1}{2}\sigma^2(x) + q(x) \\ & - d(x)a(\tau) + \frac{1}{2}d^2(x)b(\tau) + [d(\Delta) - d(x)c(\tau)]a(\Delta) \\ & + \frac{1}{2}[d(\Delta) - d(x)c(\tau)]^2 b(\Delta) + [d(\Delta) - d(x)c(\tau)]c(\Delta)r_0 \Big\} \end{aligned} \tag{A.6}$$

In general, the solution for any t follows the same pattern. Happily, each integral will in turn reduce to a generating function for $f(r_k|r_{k-1})$ with increasing complexity of the argument. The final result is as given in the text, equation (21).