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THE EFFECT OF THE DEDUCTIBLE ON THE AVERAGE CLAIM SIZE

T. Varga
ÁB-AEGON Insurance Company
Budapest, Hungary

Abstract: In this paper, we study the relationship between deductible and average claim size. Theorems are developed and applied to different claim size distributions.

Key words and phrases: average claim size, claim size distribution, deductible.

Many products in non-life insurance contain a (mandatory or optional) deductible. It is obvious that the introduction of a deductible has two major effects. The amount paid for every claim will drop, resulting in a lower claim burden for the insurance company. There will also be a drop in claim frequency, leading to a decrease in claim handling expenses. However, it is by no means obvious how the average claim size changes with the introduction of a deductible. In this paper, we will study this question. First, we derive some theoretical results then apply them to concrete distributions.

Let us assume that the claim size can be described by the continuous random variable X, having the probability density function (p.d.f.) f(x). Assume that X has a finite first moment. For convenience, we will also assume that $f(x)>0$, for every $x>0$. This constraint is satisfied by most claim size distributions used in practice.

The distribution function of X is denoted by F(x):

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\ dt \ .$$

Let us define the function G(x) by

$$G(x) = 1 - F(x) = P(X > x) = \int_{x}^{\infty} f(t)\ dt \,.$$

We will assume that the deductible is a fixed amount, say M. That means if the amount of a claim is less than M, the insurance company does not pay anything and if it is greater than M, it only pays the amount above M. In other words, if the claim is X, the insurance company pays max(0, X-M). Let us denote the expected value of the claim payment by r(M). What we want to see is whether r(M) is increasing or decreasing in M. We can write

$$r(M) = E(X - M \mid X \ge M) = \frac{\int_{M}^{\infty} (x - M)\ f(x)\ dx}{\int_{M}^{\infty} f(x)\ dx} \,.$$

Using the relationship $G'(x) = -f(x)$, we can integrate by parts in the numerator. Then we get the following result.

Theorem 1. The average claim size can be determined as

$$r(M) = \frac{\int_M^\infty G(x)\ dx}{G(M)}. \tag{1}$$

Example 1. *Exponential distribution.*
The p.d.f. of this distribution is
$f(x) = \lambda \cdot \exp(-\lambda \cdot x)$,
where $\lambda>0$ and $x>0$.
Then we have
$G(x) = \exp(-\lambda \cdot x)$,
so
$r(M) = \frac{1}{\lambda}$.
That means, r(M) is a constant not depending on M, so the average claim size is the same no matter what the deductible is. Actually, this property characterizes the exponential distribution as the following theorem shows.

Theorem 2. Assume that r(M) does not depend on M. Then X has the exponential distribution.

Proof: Let r(M)=c.
Then
$$\int_M^\infty G(x)\ dx = c \cdot G(M).$$
Differentiating both sides with respect to M, we get
$-G(M) = c \cdot G'(M)$.
Thus,
$(\log G(M))' = -\frac{1}{c}$.
So we get
$G(x) = \exp(-\frac{x}{c})$
which proves that X has the exponential distribution.

Example 2. *Pareto distribution with finite first moment.*
The p.d.f. of this distribution is
$f(x) = \alpha \cdot \lambda^\alpha \cdot (\lambda + x)^{-(\alpha+1)}$,
where $\alpha>1$, $\lambda>0$, and $x>0$.
Then we have
$G(x) = (\frac{\lambda}{\lambda + x})^\alpha$,
so
$r(M) = \frac{\lambda + M}{\alpha - 1}$.

Therefore, r(M) is monotone increasing in M; that is, the average claim size is a monotone increasing function of the deductible.

Theorem 1 shows that the average claim size is a monotone increasing (decreasing) function of the deductible if and only if the function $\frac{\int_M^\infty G(x)\,dx}{G(M)}$ is monotone increasing (decreasing) in M. However, this expression is often hard to handle since for most distributions, the integral does not have a nice, closed form (e.g. Weibull distribution, Gamma distribution). Next, we give some other sufficient conditions for the average claim size to be a monotone increasing (decreasing) function of the deductible. These conditions are of a simpler form, so they are easier to apply to many distributions.

Theorem 3. Assume the function

$$s(x) = \frac{f(x)}{G(x)} \tag{2}$$

is monotone decreasing in x. Then r(M) is monotone increasing in M.

Proof: First we prove that the function

$$\frac{G(x+c)}{G(x)}$$

with c fixed is monotone increasing in x.

We have

$$\frac{d}{dx}\frac{G(x+c)}{G(x)} = \frac{G'(x+c)G(x) - G(x+c)G'(x)}{G^2(x)} = \frac{G(x+c)}{G(x)}\cdot\left(\frac{G'(x+c)}{G(x+c)} - \frac{G'(x)}{G(x)}\right)$$

$$= \frac{G(x+c)}{G(x)}\cdot\left(\frac{f(x)}{G(x)} - \frac{f(x+c)}{G(x+c)}\right) \geq 0,$$

thus $\frac{G(x+c)}{G(x)}$ is monotone increasing in x.

Next let us take M and N such that M<N. Then we have

$$\frac{G(M+x)}{G(M)} \leq \frac{G(N+x)}{G(N)}.$$

Integrating on both sides with respect to x, we obtain

$$\frac{\int_0^\infty G(M+x)\,dx}{G(M)} \leq \frac{\int_0^\infty G(N+x)\,dx}{G(N)},$$

thus

$r(M) \leq r(N)$

proving that r(M) is increasing in M.

If s(x) is monotone increasing in x, a similar reasoning leads to the following result.

Theorem 4. Assume the function

$$s(x) = \frac{f(x)}{G(x)}$$

is monotone increasing in x. Then r(M) is monotone decreasing in M.

Example 3. *Weibull distribution.*
The p.d.f. of this distribution is

$$f(x) = c \cdot \gamma \cdot x^{\gamma-1} \cdot \exp(-c \cdot x^{\gamma}),$$

where $c>0$, $\gamma>0$, and $x>0$.
Then we have

$$G(x) = \exp(-c \cdot x^{\gamma}),$$

so

$$s(x) = c \cdot \gamma \cdot x^{\gamma-1}.$$

If $\gamma>1$ then s(x) is monotone increasing in x. Therefore, r(M) is monotone decreasing in M; that is, the average claim size is a monotone decreasing function of the deductible.
If $0<\gamma<1$ then s(x) is monotone decreasing in x. Therefore, r(M) is monotone increasing in M; that is, the average claim size is a monotone increasing function of the deductible.
Note that $\gamma=1$ gives the exponential distribution, which we have already studied in Example 1.

For some distributions, Theorem 3 and Theorem 4 cannot be used since G(x) does not have a closed form (e.g. consider the Gamma distribution). The following theorem can be of help in these cases.

Theorem 5. Assume the function

$$t(x) = \frac{f'(x)}{f(x)} \tag{3}$$

is monotone increasing in x. Then r(M) is monotone increasing in M.

Proof: First we prove that that s(x) is monotone decreasing in x.
Since

$$\frac{d}{dx}\frac{f(x+c)}{f(x)} = \frac{f'(x+c)f(x) - f(x+c)f'(x)}{f^2(x)} = \frac{f(x+c)}{f(x)} \cdot \left(\frac{f'(x+c)}{f(x+c)} - \frac{f'(x)}{f(x)}\right) \geq 0,$$

$\frac{f(x+c)}{f(x)}$ is monotone increasing in x for fixed c.

Next let us take y and z such that $y<z$. Then we have

$$\frac{f(y+x)}{f(y)} \leq \frac{f(z+x)}{f(z)}.$$

Integrating on both sides with respect to x, we obtain

$$\frac{\int_0^\infty f(y+x)\ dx}{f(y)} \le \frac{\int_0^\infty f(z+x)\ dx}{f(z)},$$

thus

$$\frac{G(y)}{f(y)} \le \frac{G(z)}{f(z)},$$

from which

$$s(y) \ge s(z)$$

follows.

So s(x) is decreasing in x. Then Theorem 3 says that r(M) is monotone increasing in M which completes the proof.

The following theorem can be proved similarly.

Theorem 6. Assume the function

$$t(x) = \frac{f'(x)}{f(x)}$$

is monotone decreasing in x. Then r(M) is monotone decreasing in M.

Example 4. *Gamma distribution.*

The p.d.f. of this distribution is

$$f(x) = \frac{\lambda^\alpha \cdot x^{\alpha-1}}{\Gamma(\alpha)} \cdot \exp(-\lambda \cdot x),$$

where $\alpha>0$, $\gamma>0$, and $x>0$.

Then we have

$$f'(x) = \frac{\lambda^\alpha \cdot x^{\alpha-1}}{\Gamma(\alpha)} \cdot \exp(-\lambda \cdot x) \cdot (\frac{\alpha-1}{x} - \lambda),$$

so

$$t(x) = \frac{\alpha-1}{x} - \lambda.$$

If $\alpha>1$ then s(x) is monotone decreasing in x. Therefore, r(M) is monotone decreasing in M; that is, the average claim size is a monotone decreasing function of the deductible.

If $0<\alpha<1$ then s(x) is monotone increasing in x. Therefore, r(M) is monotone increasing in M; that is, the average claim size is a monotone increasing function of the deductible.

Note that $\alpha=1$ gives the exponential distribution.

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