Credit Portfolio Optimization under Condition of Multiple Credit Transition Metrics

Min Han[1]

[1] Master Candidate of School of Management, Fudan University, Shanghai, China. Cell: +86 13816868792; e-mail: muriel.han@gmail.com.

Abstract

Recent years have seen more and more the importance of the management of credit risk for investors, especially institutional investors with large portfolios of corporate bonds, loans or other credit products. Questions such as how to evaluate the credit value-at-risk given large amounts of information (like different ratings and multiple credit metrics issued by different rating companies) and how to build an efficient credit portfolio (having the highest expected return under a certain level of credit risk) have become increasingly difficult to solve using traditional methods and models. Especially for the second question, the rising dimension of the portfolio under limited computational speed calls for leveraging some more robust algorithms for the large portfolio optimization.

In this paper, we choose JP Morgan's CreditMetrics model to evaluate the portfolio's credit value-at-risk for the elaboration of our thesis and try to solve the problem of how to leverage multiple credit metrics (as a major input for the model) issued by different rating firms to largely reduce the negative impact of variation of different sources, for the slightest difference among the metrics might result in a huge deviation in the evaluation of the credit risk. Last we will introduce and exploit an increasingly popular and robust algorithm in today's Large Scale Linear Planning Problem-Simulated Annealing to optimize our credit portfolio.

Generally, the paper can be viewed as applying existing models with some improving methods to better solve today's problem.

Key Words:

C-VaR; Credit transition metrics; credit portfolio optimization; Monte Carlo Simulation; Efficient frontier; Simulated Annealing Algorithm

1. Introduction

In recent years, much attention has been put on credit risk analysis and control, not only in academics but also in industry. What is credit risk? Credit risk is the risk of losses that a creditor may have when the obligator cannot pay back all or part of the debt. This kind of risk could exist in bonds, loans or account receivables. Nowadays, credit risk evaluation models mainly fall into two groups: single factor models like the KMV model based on option theory, JPMorgan's Credit Metrics, CFSP's CreditRisk+ and CreditPortfolioView by McKinsey as well as multiple factor models like Altman's Z-score, Zeta model, etc.

Multiple factor models are based on financial statements and give rates in a statistical way, while single factor models are based on Brown and Poisson processes and describe credit risk by simulating the Markov process. Both ways of rating risk have their strengths and weaknesses. Financial data is a reflection of the past performance but is not a good indication for the future. The single factor model can predict the future well, but too many rely on the credit information provided by rating agencies and pay little attention to the market movement as a whole.

In order to alleviate the negative impact on portfolio decisions brought by multiple rating metrics from different sources, we propose a method that borrows the idea of pessimistic decision to largely reduce the uncertainty. By optimizing the portfolio under this method, that is, choose the lowest rating (worst case) as it is and optimize the portfolio given that rating, we can at least secure our position and this can set us free from worrying about which source of rating is more creditable.

The following paper will cover the general introduction of a few popular credit risk evaluation models and a step-by-step demonstration of JP Morgan's CreditMetrics, our method of dealing with multiple Credit Transition Metrics, general introduction of Simulated Annealing (SA) Algorithm and a step-by-step demonstration of our portfolio's optimization process with SA. Last we will discuss the problems what could affect the effectiveness of this method.

2. Popular Credit Risk Evaluation Models

2.1 Evolution of the Models

Since the 1930s, the development of credit risk evaluation models has gone through comparable analysis, statistical analysis and artificial intelligence. In this section we give a brief introduction of the key assumptions and values of various credit risk evaluation models.

2.1.1 Comparable Analysis in Credit Risk Management

The traditional credit risk evaluation criteria link credit risk with the default event. The key point is data mining the characteristics of both default and non-default companies to establish the identification equations and categorize the samples. The representative model of this stage is 5C analysis—character, capacity, capital, collateral and condition. People try to make a full qualitative analysis about the obligator's willingness and capability of payback from five aspects. Early models usually suffer from the subjective, empiricism and lack of objective assessments.

2.1.2 Statistical Analysis in Credit Risk Management

After Fisher's research on heuristics, there developed quickly and enormously credit risk evaluation models based on statistics, of which most represented is Edward·Ahman's Z-score. Edward·Ahman observed manufacturing companies near or far from bankruptcy in 1968 and took 22 financial ratios to establish the most famous five variable Z-score based on the mathematical statistical screening. These statistic models' identification functions and the premises of the sample distribution can interpret the data as well as the coefficients of the model. Yet the weakness lies in the rigidity of the premise such as data should be normally distributed with known variance, which is not easy to find in reality.

2.1.3 Artificial Intelligence in Credit Risk Management

With the fast development of information technology, recent years have seen large artificial intelligence models that have been incorporated in the credit risk analysis. For instance, neural networks as a self-organizing, self-adapting and self-learning non-parameter method are very robust and accurate in predicting especially when the distribution does not rigidly follow normal.

2.2 CreditMetrics and Other Single Factor Models

In this section we will briefly introduce some single factor models which are based on monitoring the changing process of credit from good to bad and building models on credit rating data. The following are the most famous four models developed in the past two decades.

2.2.1 KMV

The KMV model calculates the Expected Default Frequency (EDF) based on the firm capital structure, the volatility of the assets returns and the current asset value. This model best applies to publicly traded companies for which the value of equity is market determined. The translation of the public information into probabilities of default proceeds in three stages:

- First Stage: Estimation of the asset value and the volatility of asset return
- Second Stage: Calculation of the distance-to-default
- Third Stage: Derivation of the probabilities of default.

2.2.2 CreditRisk+

Unlike the Merton-based approach and CreditMetrics, CFSP's CreditRisk+ methodology is based on mathematical models used in the insurance industry. Instead of absolute levels of default risk—such as 0.25 percent for a triple B rated issuer—CreditRisk+ models default rates as continuous random variables. Observed default rates for credit ratings vary over time, and the uncertainty in these rates is captured by the default rate volatility estimates (standard deviations). Default correlation is generally caused by external factors such as regional economic strength or industry weakness. CFSP argues that default correlations are difficult to observe and are unstable over time. Instead of trying to model these correlations directly, CreditRisk+ uses the default rate volatilities to capture the effect of default correlations and to produce a long tail in the portfolio loss distribution. The minimal data requirements make the model easy to implement, and the analytical calculation of the portfolio loss distribution is very fast.

2.2.3 CreditPortfolioView

Tom Wilson, formerly of McKinsey, developed a credit portfolio model that takes into account the current macroeconomic environment. Rather than using historical default rate averages calculated from decades of data, CreditPortfolioView uses default probabilities conditional on the current state of the economy. Therefore an obligor rated triple B would have a higher default probability in a recession than in an economic boom. The tabulated portfolio loss distribution is conditioned by the current state of the economy for each country and industry segment.

Here is a table that clearly compares the differences among these popular models. We will introduce JP Morgan's CreditMetrics in detail in the following part by a step-by-step demonstration of its calculation, which our later work largely depends on.

TABLE 2-1

Popular Models of Credit Risk Analysis

	CreditMetrics (JP Morgan)	**KMV**	**CreditRisk+ (CFSP)**	**CreditPortfolioView (McKinsey)**
MTM (mark to market) or DM (default method)	MTM	MTM or DM	DM	MTM or DM
Source of the Risk	Normal distributed returns	Normal distributed returns	Poisson distributed default rate	Macro economic variables
Correlation	Share price and transition probability	Volatility of option and share price	Average default rate correlation	Correlations among various macro economic variables
Solution	Algebra or Monte Carlo	Algebra	Algebra	Monte Carlo

2.2.4 CreditMetrics

Since our work is largely based on this model and the theories proposed by JP Morgan in 1997, here we give a step by step instruction of how to calculate credit value at risk. We cover the following topics in this instruction part.

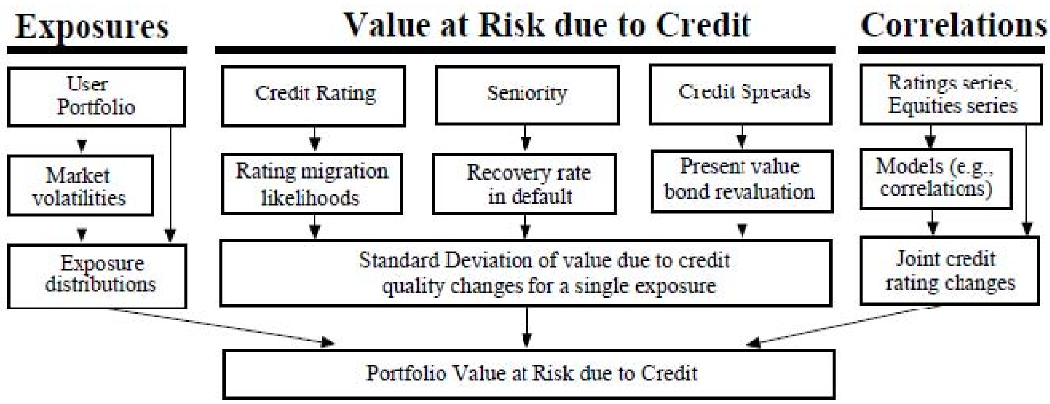
Calculation of the transition probability of different credit ratings and the bond's value at each possible scenario.

- Use of standard deviation as the measurement of credit value at risk of single bond as well as small portfolio.
- Use of Monte Carlo simulation to deal with large portfolio.

CreditMetrics is the product for quantifying credit risk developed by JP Morgan in 1997. Its idea is like Riskmetrics for quantifying market risk published in 1994 in that both are measuring risk by calculating the VaR (value at risk). Here is the logic and framework of the CreditMetrics model:

GRAPH 2-1

CreditMetrics Framework (Source: Moody's Carty & Lieberman [96a] and Standard & Poor's Creditweek [15-Apr-96])



2.3 Calculation of C-VaR of a Two-Bond Portfolio

In this section we will elaborate on how to calculate the credit value at risk of a two-bond portfolio whose composition is like the following:

- Senior Unsecured Bond with initial rating of A, 6 percent coupon and duration of seven years.
- Senior Unsecured Bond with initial rating of B, 5 percent coupon and duration of six years.

We assume that at the end of year 1 there are only three scenarios: rating will change to A, B and D (default). We can get the value of each bond at a specific scenario using the forward interest rate. Detailed calculation and results can be found in the following tables:

TABLE 2-2

Possible Values and Probabilities at End of Year for Bond of Initial Rating A and B

Probability and Value of the Bond (Bond of Initial Rating A)			Probability and Value of the Bond (Bond of Initial Rating A)		
Rating at the End of Year	**Probability**	**Value**	**Rating at the End of Year**	**Probability**	**Value**
A	0.92	109	A	0.03	108
B	0.7	107	B	0.9	98
D	0.1	51	D	0.07	51

Note: for the bond of initial rating A, the mean and standard deviation are 108.28 and 5.78 respectively; and for the bond of initial rating A, the mean and standard deviation are 95 and 12.19 respectively.

At the end of the year, this portfolio can have nine different values for nine different scenarios. For instance, if both bonds remain at their initial rating of A and B, the value of the portfolio is 207（=109+98）.

TABLE 2-3

Possible Values of the Portfolio at End of the Year

Obligator 1 (Initial Rating A)		Obligator 2 (Initial Rating B)		
		A	**B**	**D**
		108	98	51
A	109	217	207	160
B	107	215	205	158
D	51	159	149	102

Note: the entry of the ith row and the jth column in the middle 3*3 matrix is just the sum of corresponding value at the top row and the left column (for instance, 102=51+51).

2.3.1 Joint Probability

We have already have the independent probability of switching from rating 1 (the beginning of the year) to rating 2 (the end of the year) of both bonds; now our problem is to calculate the joint probability of the co-moving of both bonds. Still we simplify it as if the move is independent, thus the joint probability is just the product of the independent probability.

We can derive the joint probability distribution table from Table 2-4.

TABLE 2-4

Joint Probability Distribution

Obligator 1 (Initial Rating A)		**Obligator 2 (Initial Rating B)**		
		A	**B**	**D**
		$p_{21} = p_{B,A} = 3$	$p_{22} = p_{B,B} = 90$	$p_{23} = p_{B,D} = 7$
A	$p_{11} = p_{A,A} = 92$	2.76	82.8	6.44
B	$p_{12} = p_{A,B} = 7$	0.21	6.3	0.49
D	$p_{13} = p_{A,D} = 1$	0.03	0.9	0.07

2.3.2 Standard Deviation

Now we demonstrate how to measure C-VaR of portfolio using standard deviation.

$$V_{m,p} = \sum_{i,j=1}^{3} \pi_{ij} V_{i,j} = 203.29\text{US\$}$$

$$\sigma_{v,p} = \sqrt{\sum_{i,j=1}^{3} (\pi_{ij} V_{i,j}^2) - V_{m,p}^2} = 13.49\text{US\$}$$

If the distribution is normal, we can just use its standard deviation to describe $C - VaR = 1.65\sigma = 1.65 * 13.49 = 22.26$ US$ (at 5% significance level).

In sum the C-VaR of corporate bonds depends on the following factors:

1. Joint probability of transition between each risk scenario.
2. The portfolio value at each this scenario.

Generally speaking, because of the diversification effect on risk, the portfolio C-VaR is smaller than the sum of individual bonds.

2.3.3 Threshold

We can use $\sigma_{v,p}$ to roughly describe the portfolio's C-VaR if the distribution is normal. If not, it would be better to use a preset threshold, say 1 percent of the total value distribution. We must first sort the whole possible values and sum the probability of them from the smallest one until we get 1 percent accumulating probability. For example:

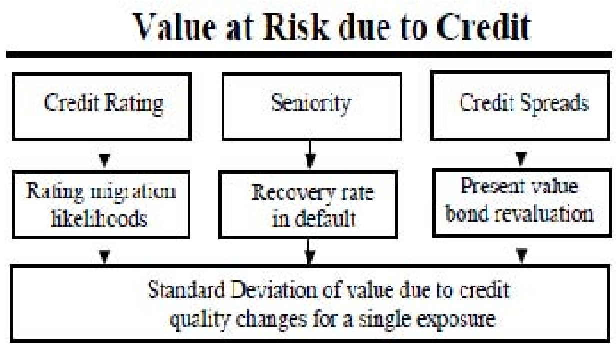
V_{A+B} = {102 US $, 149 US $, 158 US $, 159 US $, …, 217 US $,}
π_{ij} = {0.07, 0.9, 0.49, 0.03, …, 2.76}

Hence, the nearest value when accumulating probability is 1 percent is 149 US $, and the C-VaR is therefore 54.29 US $, （$= V_{m,p} - 149US\$ = 203.29US\$ - 149US\$$）.

In the following graph, we summarize some key points of when to calculate C-VaR.

GRAPH 2-2

Key Steps in Calculating the C-VaR



2.3.4 Calculation of C-VaR of a Two-Bond Portfolio

In reality, it is more common to deal with large portfolios with N assets. Unfortunately, we can't get the joint transition probability matrix easily, not to mention the following standard deviation calculation. Think of a bank with 30 debt portfolios and with joint transition probabilities. In order to better describe the situation and solve the problem of lack of historical data, CreditMetrics tends to use Monte Carlo simulation to get the joint transition probability distribution, and what we need is simply the return distribution on the stork market. Steps in this process include:

1. Get asset return threshold Z matrix for each obligator.
2. Automatically generate a series of asset returns following multiple normal distribution.
3. Get the credit transition information for each asset return by looking for the asset return threshold Z matrix (for instance, company 1 switch from BBB to CCC) and re-calculate the portfolio value at each new rating to get $V_p^{(1)}$.
4. Repeat step 3, for all the simulated return portfolios. We can get a series of

$V_P^{(1)}$. We sort them in ascending order and establish a 1 percent threshold to get the C-VaR.

In the following of optimizing portfolio with multiple assets, we will use this way of calculating C-VaR and demonstrate it in detail.

3. Our Data Selection and Processing

In this section we first interpret the data we use and then we will do some basic processing of the data for the later work. Considering the constraint of our resources and capability, we only establish a visual portfolio with six bonds. The source of all the information is from Yahoo Finance on March 21, 2007.

3.1 Basic Information of the Portfolio

In our portfolio, the initial investment is 10,000 US $ allocating on six senior secured bonds, which are all due to March 2011 and have principal value of 100 US $. Ratings and coupon information are as follows:

TABLE 3-1
Rating and Coupon

	Coupon	S&P	Moody	Fitch
MERRILL LYNCH	7.00%	AA	AA	AA
WALMART	3.38%	AA	AA	AA
BOEING	5.80%	A	A	A
COLA	5.75%	A	A	A
3M	4.20%	BBB	BBB	BBB
TIME WARNER	7.48%	BBB	BBB	BBB

Source：http://finance.yahoo.com/bonds

3.2 Rate of Return in Stock Market

Meanwhile, in order to link to the joint transition probability distribution, we need the rate of return of the stock market for these bonds' issuing companies. Hence we collect the annual return of the stock listed on NYSE from June 14, 1996 to June 14, 2006.

TABLE 3-2

Annual Return of the Stock s of the Bond Issuing Companies

	MERRILL LYNCH	WALMART	BOEING	COLA	3M	TIME WARNER
2006	16.90%	-4.05%	26.48%	-16.20%	4.14%	1.07%
2005	-0.95%	-12.65%	25.37%	-23.60%	-10.50%	-2.99%
2004	16.88%	4.17%	32.33%	34.99%	-40.61%	10.66%
2003	18.89%	-5.17%	-18.52%	-14.71%	1.38%	-5.32%
2002	-44.55%	14.02%	-40.18%	33.66%	4.29%	-112.29%
2001	-62.55%	-10.77%	49.33%	-6.21%	36.07%	-3.95%
2000	54.70%	24.55%	-7.45%	-70.37%	-5.84%	-54.57%
1999	-22.76%	-27.11%	-3.96%	-11.71%	9.70%	6.13%
1998	34.37%	56.96%	-27.69%	52.81%	-19.81%	37.50%
1997	-1.96%	20.30%	-37.36%	-40.05%	36.77%	26.24%

3.3 Yield Curve

In order to get the portfolio value at the end of the year, we need to first calculate the individual bond value by means of discounting all the remaining coupons and possible value at expiration using the forward rate given specific possible rating. For an AAA rating bond, there are a couple of forward rates (we can get them from calculation of its spot rate). These forward rates can be viewed as the best prediction of the future spot rate by the market.

TABLE 3-3

Average Forward Rate of Various Rating Bonds Expiration at March 2011

	Forward Interest Rate (%)			
	F12	F13	F14	F15
AAA	4.1500	4.2340	4.2060	4.1960
AA	5.6740	6.2080	6.2250	6.2440
A	6.2480	6.8990	7.4630	7.0260
BBB	6.5220	7.1890	7.2400	7.7780
BB	7.2910	7.5550	7.6890	8.8940
B	8.7260	8.7360	9.7590	10.2200
CCC	9.7360	10.4570	11.1950	12.2810

F12 means the interest rate from the end of year 1 to the end of year 2.

4. Optimization of Credit Portfolio with Multiple Assets

As for large portfolios, it is not easy to calculate the joint transition probability matrix. Think about N obligators with eight possible rating scenarios. There would be 8^N possibilities! Hence, we use Monte Carlo simulation to roughly generate the value distribution of the portfolio instead of calculating every entry of the matrix. All we need for data input is just the return of the stock. In this section, we will make a step-by-step demonstration of the whole process. In the optimization, we introduce our proposed pessimistic decision method to deal with the multiple transition matrixes as the data input to reduce the negative effect of the uncertainties. Later, we will also leverage a very efficient and powerful algorithm, Simulated Annealing, to get the most optimized weight 8^N of the individual assets in the portfolio.

4.1 Calculation of Credit Value at Risk

4.1.1 Possible Values of the Individual Bond at the End of the Year

Here, we calculate the value of each bond at the end of the year using the famous bond pricing formula which we mentioned in the previous section:

$$V_{X,A} = \sum_{n=1}^{n} \frac{C_X}{(1+F_{A(1,n)})^{n-1}} + \frac{C_X + 100}{(1+F_{A(1,5)})^{n}}$$

$V_{X,A}$ is the market value of bond X at rating A by the end of the year.

C_X is the coupon of the bond.

$F_{A(1,n)}$ is the forward rate of average A -rating bond from the end of year 1 to the end of year n.

Given the yield curve we use in Table 3-3, the value of the bond issued by Merrill Lynch at the end of the year when its rating switches to A-rating would be:

$$\begin{aligned} V_{ML,A} &= \sum_{n=1}^{4} \frac{C_{ML}}{(1+F_{A(1,n)})^{n-1}} + \frac{C_{ML} + 100}{(1+F_{A(1,5)})^{4}} \\ &= 7 + \frac{7}{(1+0.06248)^1} + \frac{7}{(1+0.06899)^2} + \frac{7}{(1+0.07463)^3} + \frac{7+100}{(1+0.07026)^4} \\ &= 106.9095 \end{aligned}$$

With the same way, we can get Merrill Lynch and other five bonds' market values when their rating switches to other possible ratings. Of course, even at default event, investors still can get part of the investment. This is determined by the seniority of the bond. Here we have six senior unsecured bonds; hence the average recover rate is 51 percent, which means we can get 51 percent of the principal when it defaults.

The following table is the market value of individual bonds given ratings:

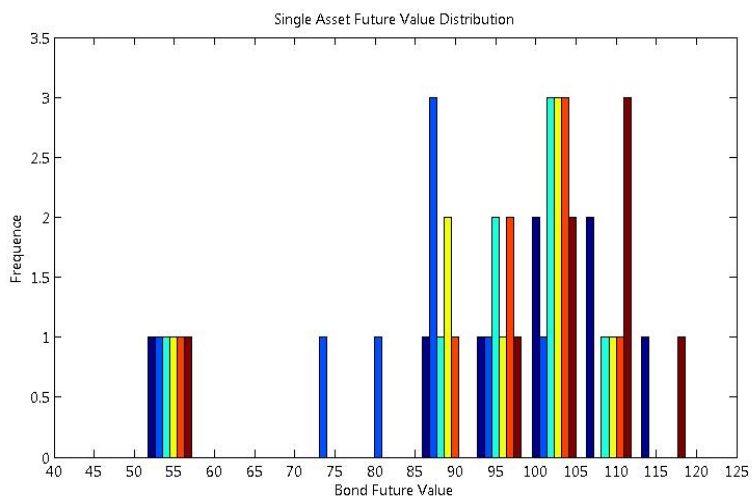
TABLE 4-1

Market Value of Individual Bonds at Given Ratings

	MERRILL LYNCH	WALMART	BOEING	COLA	3M	TIME WARNER
AAA	117.13	100.41	111.59	111.36	111.59	119.34
AA	109.65	93.51	104.31	104.08	104.31	111.78
A	106.91	91.01	101.64	101.42	101.64	109.01
BBB	104.64	88.83	99.40	99.19	99.40	106.73
BB	101.28	85.66	96.11	95.89	96.11	103.35
B	97.15	81.93	92.11	91.90	92.11	99.17
CCC	91.53	76.71	86.63	86.42	86.63	93.49
D	51.00	51.00	51.00	51.00	51.00	51.00

GRAPH 4-1

Market Value Distribution of Individual Bonds at Given Ratings



Until now we only get the individual value, and the portfolio value is determined by:

1. The weight of the individual assets in the portfolio.
2. Joint credit rating transitions of the portfolio assets.

We will cover that in the following part.

4.1.2 Asset Return Threshold Z Matrix

We assume here that the movement in the stock market reflects the change in credit ratings to some extent; hence it can be used as a signal of the credit rating change.

First, we need the annual transition metrics for corporate bonds issued by S&P, Moody and Fitch. The following table is annual corporate credit transition metrics issued by S&P.

TABLE 4-2

S&P One-Year Corporate Bond Credit Transition Metrics

Average	Annual AAA	Global AA	Corporate A	Transition BBB	Matrix BB	1983 B	2002 CCC	D
AAA	96.54	3.31	0.14	0.01	0.00	0.00	0.00	0.00
AA	0.09	90.99	8.47	0.40	0.03	0.02	0.00	0.00
A	0.03	2.50	91.78	5.28	0.24	0.02	0.10	0.05
BBB	0.00	0.25	4.85	89.26	3.97	0.87	0.40	0.40
BB	0.07	0.13	0.20	7.33	79.39	8.06	2.71	2.11
B	0.00	0.00	0.00	0.51	8.08	83.83	5.01	2.57
CCC	0.00	0.00	0.00	0.44	0.00	10.62	58.85	30.09

Source: Roberto Violi；Credit Ratings Transition in Structured Finance (+); CGFS Working Group on Ratings in Structured Finance.

From the above table, we can tell that a BBB rating bond has 0.40 percent chance of default at the end of the year. Then, if we assume the rate of return on the stock market is normally distributed, we can establish a one-to-one relationship linking the stock return with the company's credit rating as follows:

$$\Pr(\text{default}) = \Pr\{R \prec Z_{Def}\} = \Phi(Z_{Def} / \sigma) = 0.40\%$$

Pr is the probability of a specific change in credit rating; R is the annual rate of return on the stock (in order to simplifies our calculation, we standardized the return to make sure it has zero expected return); $\Phi(\ )$ is a standard normal cumulative distribution function; σ is the volatility of the share price of a company with initial rating of BBB. We can use this equation to get Z_{Def}, $Z_{Def} = \Phi^{-1}(0.40\%)\sigma = -2.9677\sigma$.

To make this reverse process, we use MATLAB and get the result immediately: $Z_{Def} = -2.9677\sigma$. Hence, in other words, if we find the rate of return of stock issued by the same company which issued an initial BBB corporate bond decreased by larger than -2.9677σ, this bond may downgrade to default. We can keep calculating other threshold rate of return Z at which the bond credit rating may change. For instance, we observed that this BBB bond has 0.40 percent chance to become a CCC bond after one year, then:

$$\Pr(\text{CCC}) = \Pr\{Z_{\text{Def}} \prec R \prec Z_{\text{CCC}}\} = \Phi(Z_{\text{CCC}} / \sigma) - \Phi(Z_{\text{Def}} / \sigma) = 0.40\%$$
$$\therefore \Phi(Z_{\text{CCC}} / \sigma) = 0.004 + \Phi(Z_{\text{Def}} / \sigma) = 0.008$$
$$\therefore Z_{\text{CCC}} = \Phi^{-1}(0.008)\sigma = -2.8338\sigma$$

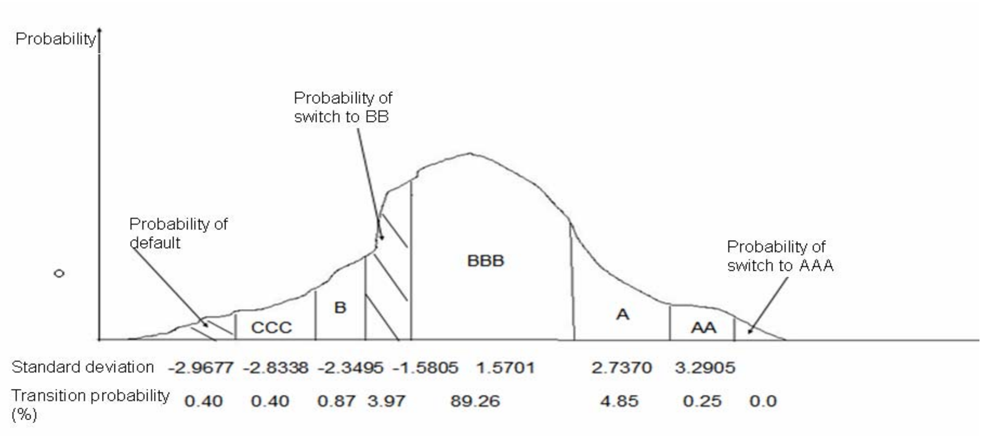
Further, we can consider eliminating this σ（asset return's volatility）, because the joint credit transition probability isn't determined by this σ. Think about two obligators with the same credit ratings yet having different variance of the asset return. Assume they are following this equation. This means the volatility of one of the obligator is twice that of the other. This also means the asset return threshold Z will be twice that of the other. Hence, the probability of transition between different ratings will never change with different $\sigma^* = 2\sigma$. Now, we can use the standardized normal distribution of rate of return which follows N (0, 1).

Hence, $Z_{\text{CCC}} = \Phi^{-1}(0.008)\sigma = -2.8338\sigma = -2.8338$

Similarly, we can get all the threshold asset return Z which triggers rating to switch to other grades as follows:

GRAPH 4-2

Asset Return Threshold Z and the Credit Transition Probability of the Initial BBB Bond



Repeat the above steps, for all bonds with different initial ratings like A, AA, A……CCC, we can get an asset return threshold Z matrix for the credit transition probability metrics issued by S&P as follows:

TABLE 4-3

S&P Asset Return Threshold Z Matrix

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC**
AAA	—	—	—	—	—	—	—
AA	-1.506	2.142	3.195	3.291	3.540	8.210	—
A	-2.478	-1.416	1.986	2.737	3.195	3.353	—
BBB	-3.540	-2.669	-1.625	1.570	2.583	2.929	—
BB	-3.540	-3.090	-2.524	-1.581	1.585	2.452	2.495
B	—	-3.719	-3.012	-2.350	-1.444	1.471	1.930
CCC	—	—	-3.719	-2.834	-2.162	-1.358	1.495
D	—	—	—	-2.968	-2.229	-1.490	-0.704

By this same way we can also get this matrix based on the credit transition probability metrics issued by Moody and Fitch.

4.1.3 Simulate Rate of Asset Return by Monte Carlo

In this section we use Monte Carlo simulation to approximately get the credit portfolio value distribution using the historical data from the stock market and the linkage between stock market and credit rating transition. In this way, we no longer suffer from the dimension problem mentioned previously (think about that!). Besides, we only have 10 years of observations and lack the extreme scenario. Monte Carlo simulation perfectly saves us a lot of time for calculation as well as providing enough data entries. $8^6 = 262144$.

Key Steps of Monte Carlo Simulation

1. Construct and describe the distribution:

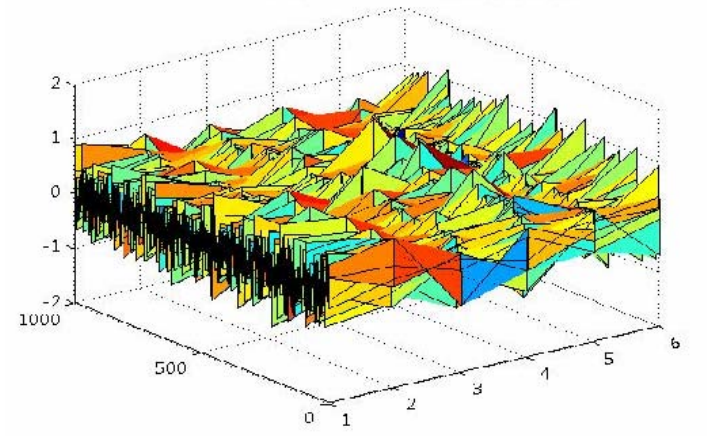
We assume the stock return is normally distributed. Hence, in our Monte Carlo simulation, we use the mean vector as well as covariance of the historical data of those six companies we pick as the input variables.

2. Generate a sample from the establishing distribution:

After we construct our distribution, the key of the Monte Carlo simulation is to generate random variables as our sample from that distribution. We use the random generator in MATLAB to generate 1,000 vectors which follow the distribution we previously set. Each vector consists of six possible asset returns.

GRAPH 4-3

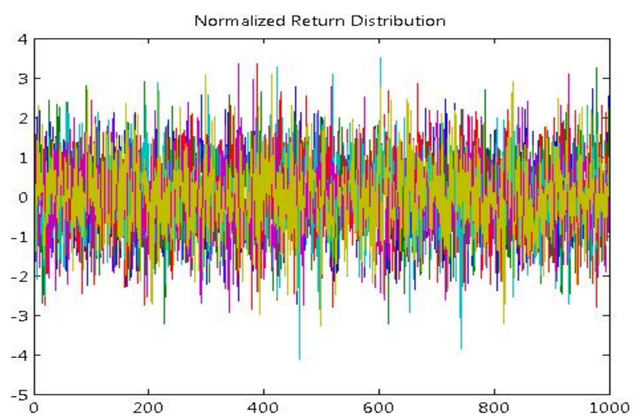
Multiple Normal Distribution of Stock Returns by Monte Carlo Simulation



Again, in order to simplify our later work, we standardized our return by equation and get the standardized normal distribution of return: $X = (X - \mu) / \sigma$

GRAPH 4-4

Standardized Normal Distribution of Return



4.1.4 Determine a Possible Credit Transition

In the last section, we generated 1,000 simulated return vectors and (1.4741, 2.7361, -2.5047, 1.0696, -0.82376, 0.63461) is one of them. We will calculate its credit transition status after one year. These six data points are the stock returns of those six companies issuing the bonds. For instance, $R_{ML} = 1.4741$ is the possible return (standardized) of Merrill Lynch. Moreover, the initial credit ratings of these six bonds are (AA, AA, A, A, BBB, BBB).

From the Asset Return Threshold Z Matrix that we get from the previous section, we can find the triggering point Z that causes the initial AA bond's credit rating to switch to other grades. Because of this, we are confident that the Merrill Lynch bond will remain the same rating at the end of the year.

$$Z_{A} \prec R_{ML} = 1.4741 \prec Z_{AA} .$$

In the same way, we can get the possible credit ratings for the other five bonds after one year. Then we get a credit transition vector after one year such as AA, AAA, BBB, A, BBB, BBB. Comparing with the initial one (AA, AA, A, A, BBB, BBB), we find that Wal-Mart's bond upgraded by one unit and Boeing's downgraded by one unit with all the others remaining the same. That is because the change in the stock returns of Wal-Mart and Boeing exceeded the triggering point we get from asset return threshold Z matrix. Now, we have established a one-to-one relation between stock return volatility and credit rating transition.

4.1.5 Revaluate the Portfolio under New Credit Rating

According to the individual bond value under different credit rating transitions from the initial one, which we calculated in the very beginning of this part, we can immediately get the value of Merrill Lynch's bond with AA rating, which is $V_{ML,AA} = 109.65$. In the same way, we can get the market value of the other five bonds after the credit transition. The new credit rating is AA, AAA, BBB, A, BBB, BBB; therefore the new market value vector is (109.65, 100.41, 99.40, 101.42, 99.40, 106.73).

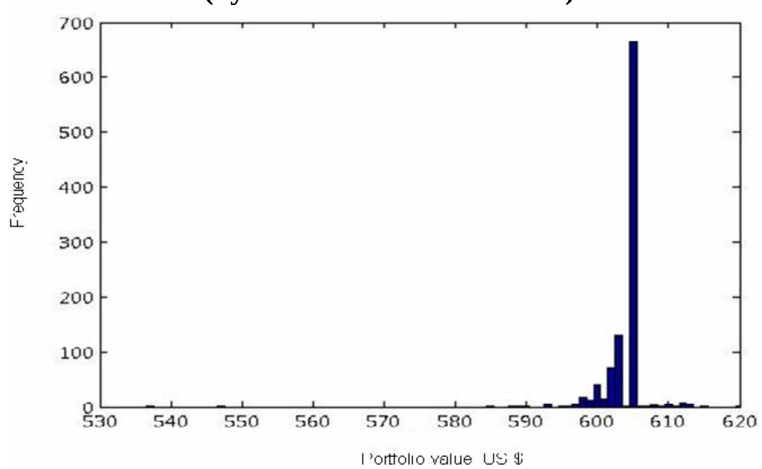
For the portfolio, its total value depends not only on the individual values of the bonds but also their weights. Different weights of combination will affect the overall expected return for the portfolio as well as the credit risk. Therefore, we put lots of effort on how to efficiently and effectively optimize the portfolio given the specific risk preference. Since we will talk about this later, here, for demonstration convenience, we only assume the weights of all six bonds in the portfolio are the same. Thus, the portfolio value is:

$$V_{P} = 109.65 + 100.41 + 99.40 + 101.42 + 99.40 + 106.73 = 617.01 .$$

4.1.6 Portfolio Value Distribution and C-VaR at 5 Percent Significance Level

We have 1,000 possible return combinations and by repeating the revaluating process as in the previous section we can get a distribution of the portfolio value at the end of the year as follows:

GRAPH 4-5
Credit Portfolio Value Distribution at the End of Year
(by Monte Carlo Simulation)



We sort the above 1,000 values in an ascending order. At the top 5 percent, we have our threshold value of 599.43 US $, which is the market value of the 50th data (5%*1000=50). Hence, the portfolio credit value at risk is simply that value at the top 5 percent minus the sample mean: $\text{C-VaR} = V_{P,5\%} - \overline{V_P} = 599.4336 - 604.1141 = -4.6805$. It tells us that, after one year, there is 5 percent chance that the loss of this portfolio because of the credit transition will exceed 4.6805 US $.

We have finished the portfolio credit value at risk calculation part, which covers the following key steps:

1. Look at individual obligator's stock and credit ratings transition; get the asset return threshold Z matrix.
2. Generate a series of return vectors by Monte Carlo simulation.
3. Determine a new credit rating for each return vector and revaluate the portfolio value $V_P^{(1)}$.
4. Repeat step 3, for all the simulated return combinations. We get a series of $V_P^{(i)}$. Sort them in an ascending order. Calculate the credit value at risk at a specific significance level, say 5 percent.

In the following part, we will mainly focus on how to optimize the portfolio. Minimizing the credit value at risk for a given return, as one of the optimization objectives, depends on the calculation of C-VaR, which is basically based on our former work.

4.2 Credit Portfolio Optimization: Our Method and Techniques

Our objective is to find an optimal weight of the individual assets that maximizes the portfolio expected return and meanwhile minimizes its credit risk. This is the principle of the following work, and our job is to try to find an easy and efficient way to solve the following two big problems.

1. How to value the goodness of multiple credit rating information from different rating companies.

   There are only three independent rating firms: S&P， Moody and Fitch. These organizations are independent from each other, issuing ratings for companies and industries based on their own source of information and credit risk models. Even though the results seem not far different from each other, there is still variation, and sometimes this small difference will bring huge negative effects for investment decisions. For example, in our experiments, we find out that compared with S&P and Moody，Fitch seems more "strict," for the issued data are most conservative. This may result from its source of information observed or simply because of its stricter models. Whatever the cause, we try to alleviate this negative effect because of the different ratings for the same company.

2. It is always subject to the time and space constraint to find out the optimal portfolio.

   In our portfolio, the initial investment is 10,000 US $ allocating among six bonds (in the real world, the size of the portfolio may be much larger). The principal of bond is 100 US $, therefore the problem is simply finding the optimal weights and have these weights sum up to 100（10000/100）. Actually, it is much more complex than it looks. If we don't apply any algorithm, just let the computer try every possible combination, the efficiency is, where, $O(m^t O(u*n))$:

   m is the sum of all weights
   t is the number of assets
   $O(u*n)$ is the time need to calculate the C-VaR for one specific combination, where,
   u is the independent rating firms
   n is the random sample generated by Monte Carlo simulation

Take our experiment as an example, assume the processing power of our computer is 1 M flops, therefore the time needed is $t = 100^6 * 15/10^6 = 1.5*10^7 s \approx 173.6$ days (from where we already know the time for calculating the C-VaR for one specific combination is $O(3*1000)$, approximately 15 seconds).

Hence, the goodness of the algorithm significantly affects the processing power of the optimization problem in portfolio investment and restrains the size of the portfolio. Therefore, we use the following method to solve the above two problems.

1. We propose a method that borrows the idea from pessimistic decision to

largely reduce the negative effect by uncertainty of the credibility on multiple rating information.

2. We incorporate a very robust algorithm, Simulated Annealing, to optimize the portfolio.

4.2.1 Pessimistic Decision

When there are more than two scenarios and the probability of them can't be confirmed, we can call this decision problem uncertainty decision. This decision problem has complex constraints and large sets of variables, most of which can't be quantified; hence it's not an easy job to establish the mathematical models. What's more, the variables and correlations among them are uncertain, therefore making it impossible to build the objective function to get the optimal solution.

Popular methods for uncertainty decision problem solving are as follows:

- Maximize the minimum return
- Minimize the maximum regret
- Maximize the maximum return
- Optimistic coefficient.

In our work, we develop the maximizing the minimum return's method and get our pessimistic decision method. What follows is the comparison of the two methods.

- Maximize the minimum return.

  This method needs to first calculate all possible returns under each option and find the option with the maximum return as the optimal solution. This is a more conservative way of decision making, from the perspective of the worst cases.

- Our pessimistic decision (minimize the maximum loss).

  We first find the largest credit value at risk under all credit rating firms' transition metrics. Next, we try to find the optimal weight that minimizes this maximum C-VaR. We stand on the point of the worst case and take this as our optimization objective. In this way, investors can avoid the negative effect by incautiously choosing one rating firm or simply making a weighted average of all the sources.

4.2.2 Simulated Annealing

As its name implies, Simulated Annealing (SA) exploits an analogy between the way in which a metal cools and freezes into a minimum energy crystalline structure (the annealing process) and the search for a minimum in a more general system.

The algorithm is based upon that of Metropolis, which was originally proposed as a means of finding the equilibrium configuration of a collection of atoms at a given temperature. The connection between this algorithm and mathematical minimization was first noted by Pincus, but it was Kirkpatrick who proposed that it form the basis of an optimization technique for combinatorial (and other) problems.

SA's major advantage over other methods is an ability to avoid becoming trapped at local minima. The algorithm employs a random search which not only accepts changes that decrease objective function, but also some changes that increase it. The latter are accepted with a probability.

SA—The Model

SA consists of three parts: solution space, objective function and initial solution.

1. Solution Space:

   It is the group of all possible solutions and it restricts the scope of our choosing the initial solution and the new solution. In many optimization problems, besides objective functions, we also have a set of constraints. Hence, there might be some infeasible solution in the solution space. You can define the solution space exclusive of infeasible solutions or you can allow them by incorporating a penalty function to penalize the occurrence of the infeasible solution.

2. Objective Function:

   It is the mathematical description of the optimization problem. Usually it is constructed as the sum of several optimization targets. The choice of objective function should well reflect the optimization requirement and, as mentioned above, when infeasible solutions are allowed, objective function needs to incorporate a penalty function.

3. Initial Solution:

   It is the starting point of the algorithm. It has been proven that the SA algorithm is robust, and the final solution is independent of the choice of the initial solution.

SA—The Idea

i. Initialize：making initial temperature large enough, setting initial solution S(the starting point of the itinerary of the algorithm)， L（Markov chain length）for the times of itinerary for each temperature value T.

ii. For k=1，……，L do step 3 to step 6.

iii. Generate new solution S′.

iv. Calculate the incremental Δt′=C(S′)-C(S)，where C(S) is the comment function.

v. Find the transition probability of solution according to Metropolis principle. Decide whether to accept the new solution or not. P_t .

$$P_t(i \Rightarrow j) = \begin{cases} 1, & \text{当 } f(j) \le f(i) \\ \exp(\frac{f(i)-f(j)}{t}), & \end{cases}$$

vi. If the stop condition has been satisfied, we replace the current solution as the optimal one and cease the program.

vii. T declines gradually and T>0，return to step 2.

SA—The Pseudocode

```
procedure SIMULATED-ANNEALING;
begin
INITIALIZE (i0, t0, L0);
k:=0;
i:=;  i0
repeat
  for   l: =1 to   do  Lk
        begin
        GENERATE ( j from );  Si
        if f(j) >= f(i) then i:=j
        else
                if exp((f(i) - f(j)) / tk) > random[0,1) then i:=j
                end
        end
        k:=k+1;
        CALCULATE-LENGTH (Lk);
        CALCULATE-CONTROL (tk)
  until stopcriterion
            end;
```

4.2.3 Our Optimization Target and Constraints

Our objective is to find an optimal weight of the individual assets that maximizes the portfolio expected return while minimizing its credit risk. As mentioned above, we first find the largest C-VaR, then find the optimal weights to minimize this maximum C-VaR. Since for a given risk preference there will be a corresponding optimal weight to minimize the credit risk as well as maximize the expected value of the portfolio, our output will be a series of points consisting of an effective frontier line. Our objective function and constraints are as follows:

$$\begin{cases} \mathrm{Max}_X \{\mathrm{Min}_m (CV_m(V_P) + \tau * \mu_m(V_P))\} & m = 1,2,3 \\ V_P = \sum_{i=1}^{6} (X_i V_i) & \\ \sum_{i=1}^{6} X_i = 100 & \\ 95 \succ X_i \succ 0, X_i \in N & i = 1 \sim 6 \end{cases}$$

Where,

V_i is the market value of the ith bond in the portfolio. It is a random variable.

V_P is the market value of the portfolio with a specific set of weights.

CV_m is the credit value at risk based on the transition metrics issued by the mth independent rating firm (we choose 5 percent as our significance level. The range of m is 1=S&P，2=Moody，3=Fitch).

τ is our defined parameter for describing the risk preference of different investors. In order to simulate the effective frontier, we use a function to repeatedly calculate different optimal portfolios at different levels of risk preference $\tau_i = 0.002 * i^2$

μ is the mean

X_i is the weight of the ith asset in the portfolio and.(Our initial investment is 10,000 USD) $\sum_{i=1}^{6} X_i = 100$

4.2.4 The Application of Simulated Annealing in Our Work

1. Solution Space:

   It is the group of all possible solutions, and it restricts the scope of choosing the initial solution and the new solution. Our constraint condition restricts the scope of solution. The solutions must be within the range of 1 to 95 and should be integers with a sum of 100.

   $$S=\{(X_1,X_2,X_3,X_4,X_5,X_6)\mid X_1+X_2+X_3+X_4+X_5+X_6=100,\ X_i\in N\}$$

2. Objective Function:

   It is the mathematical description of the optimization problem. Usually it is constructed as the sum of several optimization targets.

   $$\mathrm{Max}_X\{\mathrm{Min}_m(CV_m(V_P)+\tau*\mu_m(V_P))\}\qquad m=1,2,3$$

   We will find a set of X_i in solution space S to minimize the maximum C-VaR of the portfolio with these weights.

3. Initial Solution:

   It is the starting point of the algorithm. Since the SA algorithm is robust, i.e., the final solution is independent of the choice of the initial solution, we generate six random integers by MATLAB，and use the following rules to standardize them to make sure their sum is 100.

   $$\begin{cases} X_i=\left[\dfrac{100*X_i'}{\sum_{i=1}^{6}X_i'}\right] & \text{for } X_1\sim X_5 \\ X_6=100-X_5-X_4-X_3-X_2-X_1 \end{cases}$$

   Here, our initial solution is（8，8，3，24，45，12）

4. New solution’s generation and acceptance

   Step 1, usually for the sake of convenience and time of calculation, new solution generation ways will make small and simple modifications of the existing solution like swap and so on. We choose the mechanism as follows:

Every time the new solution is generated, we keep the first element of the solution and randomly generate the rest and randomly sort them together. The detailed algorithm is as follows:

```
Temp_x(1,1)=X(1,1);
i=2;
sum= Temp_x (1,1);
while(i<6)
      Temp_x (1,i)=randint(1,1,[1,94-sum+i]);
      sum = sum + Temp_x (1,i);
      i=i+1;
end
Temp_x (1,6)=100-sum;

j=randperm(6);
m=1;
while(m<7)
      X(1,m)= Temp_x (1,j(1,m));
      m=m+1;
end
```

Step 2, recalculate the corresponding objective function value at the new solution:

$$f(P') = \mathrm{Max}_X \{\mathrm{Min}_m (CV_m(V_{P'}) + \tau * \mu_m(V_{P'}))\}$$

Step 3, determine whether or not to accept the new solution by an acceptance principle.

Here we use the most popular Metropolis principle: If $f(P') \succ f(P)$ then accept P' as the new solution，otherwise accept it when $\exp\left(\frac{f(P')-f(P)}{t}\right) \prec \mathrm{random}[0,1)$. Detailed Algorithm is as follows:

```
if(adapt_everbest<adapt_cur)
      adapt_everbest=adapt_cur
      solution_everbest=X
      CVaR_everbest=CVaR_cur
      mu_everbest=mu_cur
end
if(adapt_cur>=adapt_last)
      solution=X;
else
      if(rand>exp(adapt_last-adapt_cur)/T_cur)
            solution = X;
      end
end
```

Step 4, when new solution has been accepted, it should replace the current solution and modify the objective function value at the same time. Thus, we have made an itinerary process and do another round of experiments based on this. When the new solution is rejected by the acceptance principle, we continue the next round of experiments based on the old value until the end of the annealing when the temperature is declined to zero.

4.2.5 Output of the Program

We only consider 50 different risk preferences because of the limit of the processing power. We show these optimal portfolios in the plane of two dimensions of expected value and C-VaR with red dots.

Environment of the Processing:

Hardware:
IBM Corporation Intel (R) , Pentium (R), Processor 1500MHz, 1.50GHz, Memory 760MB

Software:
Microsoft Windows XP Home Edition 2002 edition，Service Pack 2
MATLAB Version 7.0.1.24704 (R14) Service Pack 1 September 13, 2004

Processing Time:

5 hours, 12 minutes 33 seconds

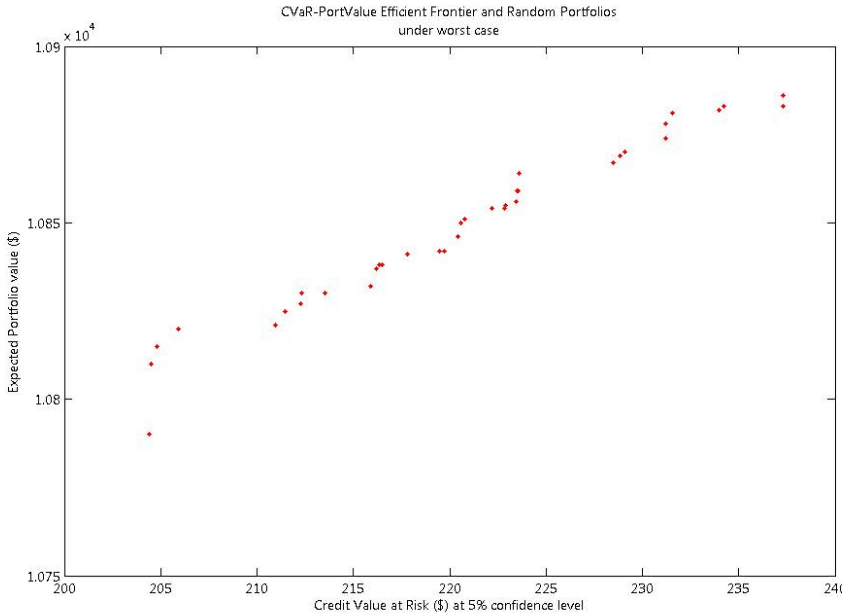
Main Program:

```
t=0.002;
point=1;
while(point<51)
    adapt_everbest=-inf;
    adapt_last=-inf;
    adapt_cur=-inf;
    T_ini=50;
    T_end=1;
    T_cur=T_ini;
    Markov_len=10;
    while(T_cur>T_end)
        n=1;
        while (n<Markov_len+1)
            create;
            adapting;
            keepbest;
            accept;
            n=n+1;
            fprintf('n= %f\n',n);
        end
        annealing;
    end
    CVaR(point,1)=CVaR_everbest;
    Mu(point,1)=mu_everbest;
    point=point+1
    t=0.002*point*point;
end
```

Output of the Program:

GRAPH 4-6

Optimal Credit Portfolios by Different Risk Preferences Under Pessimistic Decision



5. Summary and Further Discussion

In this paper, we chose JP Morgan's CreditMetrics model to evaluate the portfolio's credit value-at-risk for the elaboration of our thesis and tried to solve the problem of the negative impact on portfolio decisions brought by multiple rating metrics from different sources. We proposed a method that borrows the idea of pessimistic decision to largely reduce the uncertainty by minimizing the maximum C-VaR using powerful algorithm Simulated Annealing for the optimization of credit portfolio. Under this method, we can at least secure our position and set us free from worrying about which source of rating is more creditable.

Key steps of the optimization process are as follows:

1. Calculate the C-VaR for a portfolio with specific set of weights under the credit transition metrics issued by all independent rating firms.
2. Find the maximum C-VaR as the input for optimization process, using Simulated Annealing to minimize this C-VaR, getting the optimal portfolio.
3. For different risk preferences, we modify parameter τ and get a series of optimal portfolio allocation points consisting of the effective frontier like Graph 4-6.

Of course, we still have several issues that we haven't touched in our work and new problems generated in our method. For instance, the period of measuring credit risk is one year (because we use annual asset return). However, in reality, some credit instruments have much shorter periods. Also, we haven't talked much about the marginal risk, which is of more concern for the portfolio manager. While we leverage the powerful Simulated Annealing algorithm, we are subjected to the limited new solution generation mechanism instead of trying more options. The length of Markov chains is set at the experience level, all of which might have some negative impacts on our output.

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