**ACTUARIAL RESEARCH CLEARING HOUSE**
**1996 VOL. 2**

# Representative Interest Rate Scenarios

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## Abstract

This paper suggests a possible flexible solution to the time and resource problems of running a large number of stochastic interest rate scenarios, by selecting a representative subset. Each interest rate scenario consists of 30 future spot yield curves, where a reasonable number of points are specified on each curve (such as 12). The distribution of the scenarios is approximated by the subset and each scenario in the subset has equal weight. The method is independent of the interest rate generator used.*


* The author is grateful to Warren Adams, Steve Craighead and Don Sanning for their encouragement, review, and suggestions.

## 1. Introduction

The behavior of interest rates is key to the profitability and to the solvency of insurance companies and other financial institutions. There is much financial literature dealing with the term structure of interest rates, and many models written to fit the various theories. The models usually form the basis of a much larger model that depends on the interest rate scenarios, such as option pricing or asset liability studies. When the model is used for option pricing (hedging choices) an arbitrage-free interest rate generator is required (see Tilley[6]). Arbitrage-free interest rate generators focus on the mean value and their use is appropriate for a short-time frame. New arbitrage-free scenarios need to be created whenever the yield curve changes, since the arbitrage-free condition incorporates the expectations theory of the term structure of interest rates. When interest rate scenarios are used for other purposes, such as reserve adequacy for NY Regulation 126 and/or the new Standard Valuation Law (US) or Dynamic Solvency testing (Canada), then the generator should include a wide range of scenarios covering changes in the level, term structure and sector spread of interest rates. (Canadian Institute of Actuaries[1], Jetton[5]). In either case, it is desirable to run as many scenarios as possible. Steve Craighead [3] begins his study with 10,000 scenarios. Wall street firms regularly run that many scenarios nightly in pricing MBS.

Regardless of the purpose of the final model, the appetite for stochastic interest rate scenarios can easily exceed the budget in terms of time and resources. This is especially true when the scenarios are a small part of a much larger asset/liability model. Thus it is necessary to balance the desirability of the knowledge gained from running a large number of scenarios with the costs of running them. The first efforts to limit the number scenarios were done with arbitrage-free paths created by the binomial tree method. In this case probabilities are assigned to grouped scenarios (see Ho[4]). As Ho's paper notes, these paths were not randomly created, but rather are deterministically created. Multinomial trees, either binomial or trinomial are a deterministic method of creating interest rate scenarios, where it is possible to assign a probability to an upward movement in interest rates, and hence to each node in the tree. The set of paths determined in the linear path space correspond to a single set of interest rate curves that can be used for discounting cashflows. Tilley's approaches to the problem of limited resources is to consider either antithethic variates or stratified sampling of the normal distribution with a shuffling of the deviates and or both. His model is an arbitrage-free, path-based, continuous time model.

## 2. Background

We use the mean-reversionary log normal interest rate generator described by Christiansen [2] to produce stochastic scenarios that cover a wide range of interest rate levels and yield curve shapes. This generator is not arbitrage-free and is used where we have a long-term horizon. The original request for "representative scenarios" came from a product-pricing area which wished to test their pricing over 1000 interest rate scenarios, but their software and time considerations dictated that 50 scenarios was the maximum that they could reasonably expect to run. A simplistic approach was considered, and the scenarios were essential hand-picked. In order to validate the results, the cashflow testing model was run on the 1000 scenarios, and on the 50, and the results were satisfactory. After two more requests for 50 representative scenarios were received, hand-picking did not seem to be the method of choice.

Also, the adoption of new SVL requires aggregation of cash-flow testing results across all lines of business. However, different lines use different software models. For the annuity line, cash-flow testing has been done in APL (home-grown software) on the mainframe, where we were running 100 scenarios (the 7 NY, two shock scenarios and 91 stochastic scenarios). The insurance line utilizes PC based purchased software; and time constraints had always forced us to choose some of the stochastic scenarios. Thus the idea of replacing the 91 stochastic scenarios with 50 stochastic scenarios that were chosen as representatives of 1000 scenarios, should improve the reliability of the mainframe results, while recognizing the problems of time that were constraining those lines of business that used purchased software on the PC, and still enable us to aggregate results across lines of business.

3. **Definition of an Interest Rate Scenario**

For the purpose of this paper, an interest rate scenario consists of a set of spot yield curves. There is a curve for each of the next thirty years (at least one per year). Each curve is specified at the .25, .5, 1, 2, 3, 4, 5, 7, 10. 15, 20 and 30 year maturities. A sample interest rate scenario is graphed in Figure 1. How the curves are obtained is not important, except that all curves produced by the generator should have be bounded at reasonable levels (ie. no negative rates and positive rates consistent with historical levels for the country), and that the rates can be specified at the given points for all curves. If there are any other considerations used in creating the scenarios, such as the shape code in this generator, those considerations can be included in the process of selecting the representative scenarios

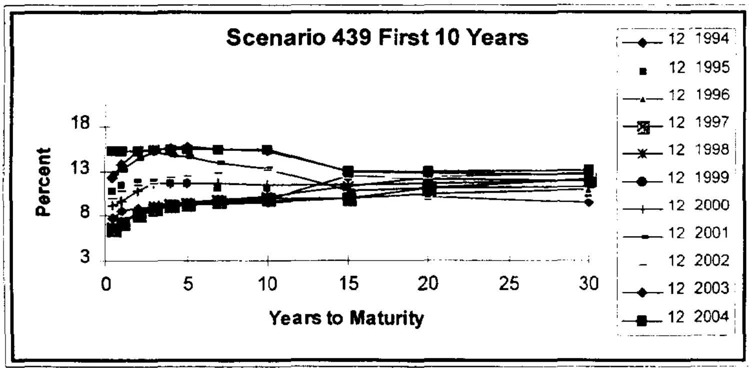


Figure 1a.

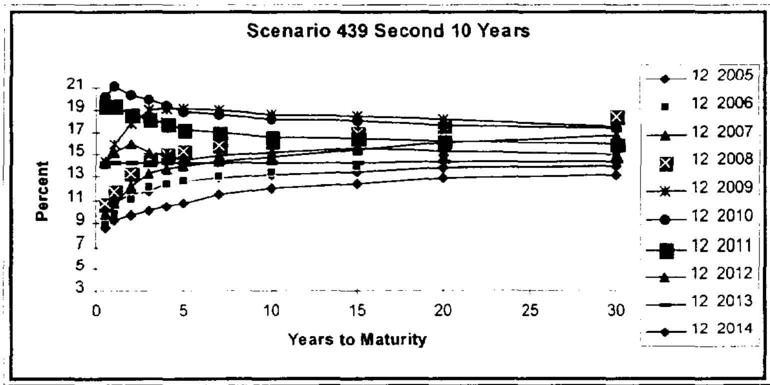


Figure 1b.

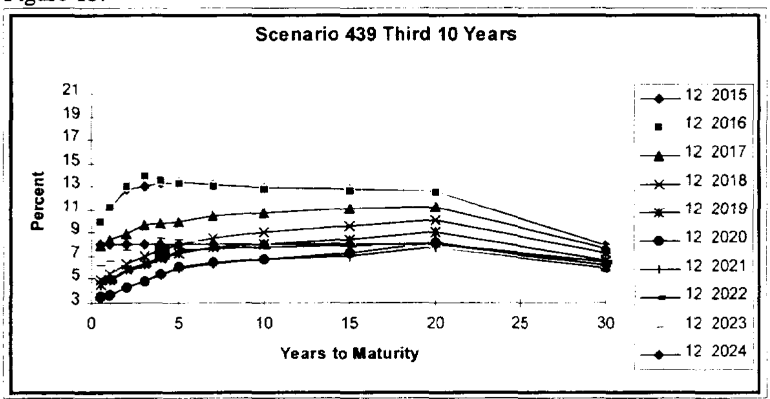


Figure 1c.

### 3. The Goal

The goal was to develop a simple algorithm that was easy to implement where a subset of 50 scenarios was representative of 1000 scenarios. Instead of associating different probabilities with each scenario, the placement of the scenarios reflects the probability distribution and the scenarios are considered to be equally likely. Another part of the goal was to have flexibility in the methodology that would permit changes in priorities, when the method was used for an alternative purpose, such as pricing rather than cashflow testing.

## 4. Considerations in Selecting the Representative Subset

A major consideration is that the algorithm has to be fairly efficient. Since there are $\binom{1000}{50} = 9.64 \times 10^{84}$ possible subsets, the algorithm must limit the number of subsets considered. A second consideration is which interest rates should be used as the basis for the representation. Should they be treasury rates, or should they be the rates that we would expect to earn on assets, ie include a spread? When we generate the rate scenarios, we generate them for different asset classes and then blend the results according to the percentage of each asset class in a given segment. When this method is used to generate scenarios for pricing a product, the rates used are those expected to be earned on the produced and a segment specific blend is used. For cashflow testing for reserve adequacy, we use a company-wide blend of assets.

## 5. Definition of Representative Subset

A representative subset of a set of interest rate scenarios is a subset where for each maturity simultaneously, the subset and the set have approximately the same mean, range and variance.

## 6. The Algorithm

The key to selecting a representative subset is to cut down on the number of subsets considered, while preserving the desired characteristics. The first way that we cut the number of subsets to be considered is to do five runs of 200 scenarios and select 10. In each run there are now $2.245 \times 10^{16}$ possible subsets. The program is set up to generate 1000 scenarios and the random number generator seed is kept, and the first 200 scenarios are generated. To reduce the number of subsets of scenarios considered to a manageable level, the concept of a candidate list is introduced.

## 7. The Candidate List

The candidate list will consist of subsets of ten scenario numbers and only the subsets on this list will be evaluated to see which one best meets the criteria for the representative scenario. The list will consist of at least 12 subsets, but may consist of approximately 200 scenarios.
The process used to build the candidate list begins with the consideration of the three month rate. Since one requirement is that the range is approximately reproduced, begin by determining all of the scenarios where the minimum three month rate at some time during the scenario is the minimum of the three month rates over all of the scenarios. Note that for each maturity we have

$$\begin{pmatrix} i_{1,0}, i_{1,1}, \ldots i_{1,30} \\ i_{2,0}, i_{2,1}, \ldots i_{2,30} \\ \cdots\cdots\cdots\cdots\cdots \\ i_{200,0}, i_{200,1}, \ldots i_{200,30} \end{pmatrix}$$

and the minimum rate for the run can be expressed as $m = \min_s\left(\min_t\{i_{s,t}\}\right)$.

Let $mni = \left\{ s \middle| \min_t(i_{s,t}) = m \right\}$ and similarly define M = $\max_s\left(\max_t\{i_{s,t}\}\right)$ and

$mxi = \left\{ s \middle| \max_t(i_{s,t}) = M \right\}$. Begin to construct the candidate list by listing all combinations $x,y$ where $x \in mni$ and $y \in mxi$, except those where $x = y$. These scenarios will essentially provide the desired range (see [1]). Next we choose the 4 scenarios whose *average* three month rate is closest to $\mu + .85\sigma$, $\mu - .85\sigma$, $\mu + .65\sigma$, and $\mu - .65\sigma$, (being careful not to include any scenarios which are in *mni* or *mxi*), and add these scenarios to *each* combination of $x,y$. Thus if these four scenarios are $a,b,c,d$ and $x_1, x_2 \in mni$ and $y_1, y_2 \in mxi$ then the candidate list contains $\{x_1, y_1, a, b, c, d\}$, $\{x_1, y_2, a, b, c, d\}$, $\{x_2, y_1, a, b, c, d\}$ and $\{x_2, y_2, a, b, c, d\}$. Now each candidate has six distinct scenarios and has an average three month rate of $m_{s,k}$ for k=1,2,3,4...

Now for subset k find the 4 scenarios whose average three month rate is closest to $\frac{10\mu - 6m_{s,k}}{4}$, but which are not already in subset k. These bring candidate k up to a full complement of 10 scenarios, with an average 3 month rate which is close to $\mu$. Repeat this process with all of the other maturities, adding the completed candidates to the current list. Once this process has been completed for the 30 year rate, the candidate list is final. No other subsets will be considered.

## 8. Selecting the subset from the candidate list:

Now that the candidate list is complete, it is time to choose the "best" candidate. The best candidate will have a minimum weighted least squares deviation from the run based on the means in all of the maturities and any other category available. In our case, since we had shape codes for every scenario, we included the mean of the shape codes in the selection of the scenarios. The weights are *arbitrary* and depend on the relative importance of the maturity to the *purpose* for which the scenarios are created. For example if the scenarios are created for pricing a product, where most of the asset or liability cashflows will be in the 5-7 year range, the 5-7 year rates would be weighted most important with a weight of 4. If there were no assets/liabilities that extended beyond 10 years, then the 15, 20 and 30 year maturities would get a weight of 1 (least important). Now the 1, 2, 3, 4, and 10 year maturities would probably be relatively important and would get a weight of 3, while small variations in the 3 or 6 month rate might be considered relatively unimportant would receive a weight of 2. The shape code would also receive a weight consistent with its relative importance. For cashflow testing for the company as a whole, the longer rates would probably be more important than a 1, except possibly for the 30 year rate.

Once the weights have been selected for each subset k determine

$fit_k = \sum_t w_t \left(m_{k,t} - \mu_t\right)^2$, where t ranges over all of the maturities and any other criteria, and select the subset k where $fit_k$ is a minimum. Keep track of the scenario numbers for subset k. We use $200 \times (run - 1) + s_{k,1}, s_{k,2} \cdots s_{k,10}$ so that these scenarios can be reproduced. Now repeat the algorithm using the random numbers for the second (and subsequent) set of 200 scenarios.

9. **Finishing and determining the fit**:

Recreate the 50 scenarios that are on the list. Combine all of the statistics from the 5 runs, so that the data is there for the entire run of 1000 scenarios. No assumptions are made as to independence for the variance (since rates were modified for maximum annual permitted change and bounds). For each run and maturity the sum of squares is backed out from the standard deviation and the mean, and a total sum of squares is calculated and the mean is calculated as the average of the means for the runs. From these the variance and standard deviation are obtained. Minimum, maximum and median are determined also. The data for the representative subset is determined directly.

10 **Sample results**

The following two tables illustrate sample results that we had for interest scenarios created beginning with the June 30, 1995 interest rate curves, using a blend of the various asset classes to create the initial curve. Table 1 shows the results for all 1000 scenarios and table 2 gives the results for the representative 50. Figure 2 graphically illustrates the quality of the fit obtained to all of the statistical measures on all of the maturities.

| | MEAN | MEDIAN | STD | MIN | MAX |
|---|---|---|---|---|---|
| SHAPE | 4.528 | 4.000 | 2.431 | 1.000 | 11.000 |
| 3 MO | 6.214 | 5.761 | 2.375 | 3.500 | 25.000 |
| 6 MO | 6.294 | 5.898 | 2.337 | 3.500 | 25.000 |
| 1 YR. | 6.603 | 6.278 | 2.307 | 3.500 | 25.000 |
| 2 YR. | 7.072 | 6.704 | 2.275 | 3.500 | 25.000 |
| 3 YR. | 7.363 | 7.014 | 2.212 | 3.500 | 25.000 |
| 4 YR. | 7.585 | 7.264 | 2.162 | 3.500 | 25.000 |
| 5 YR. | 7.757 | 7.450 | 2.113 | 3.500 | 25.000 |
| 7 YR. | 7.937 | 7.668 | 2.071 | 3.500 | 25.000 |
| 10 YR. | 8.071 | 7.805 | 2.045 | 3.500 | 25.000 |
| 15 YR. | 8.241 | 8.131 | 2.032 | 3.525 | 24.994 |
| 20 YR. | 8.475 | 8.345 | 2.044 | 3.611 | 24.393 |
| 30 YR. | 8.671 | 8.544 | 2.099 | 3.740 | 24.886 |

Table 1.

| | MEAN | MEDIAN | STD | MIN | MAX |
|---|---|---|---|---|---|
| SHAPE | 4.602 | 4.000 | 2.650 | 1.000 | 11.000 |
| 3 MO | 6.312 | 5.823 | 2.611 | 3.500 | 25.000 |
| 6 MO | 6.380 | 5.940 | 2.579 | 3.500 | 25.000 |
| 1 YR. | 6.645 | 6.296 | 2.570 | 3.500 | 25.000 |
| 2 YR. | 7.048 | 6.704 | 2.579 | 3.500 | 25.000 |
| 3 YR. | 7.324 | 6.867 | 2.571 | 3.500 | 25.000 |
| 4 YR. | 7.530 | 7.017 | 2.546 | 3.500 | 25.000 |
| 5 YR. | 7.689 | 7.134 | 2.522 | 3.518 | 25.000 |
| 7 YR. | 7.882 | 7.412 | 2.498 | 3.621 | 25.000 |
| 10 YR. | 8.037 | 7.716 | 2.482 | 3.727 | 25.000 |
| 15 YR. | 8.283 | 8.011 | 2.484 | 3.796 | 24.994 |
| 20 YR. | 8.505 | 8.273 | 2.513 | 3.858 | 24.393 |
| 30 YR. | 8.679 | 8.489 | 2.557 | 3.862 | 24.886 |

Table 2.

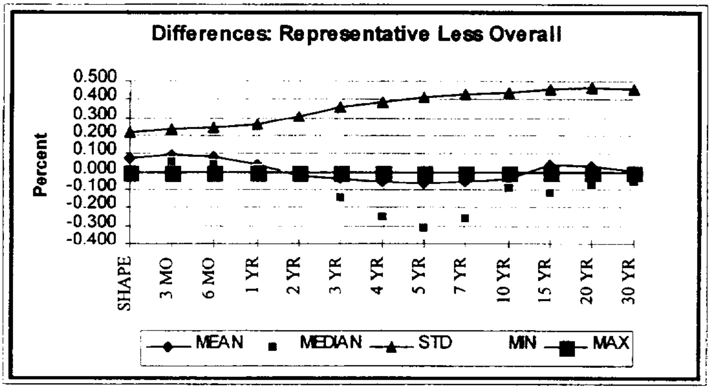


Figure 2.

## 11. Conclusions

In conclusion the algorithm successfully provides a method of reducing a 1000 interest rate scenarios that are three dimensional, in that they have complete yield curves specified for the next today and the next 30 years, to 50 representative scenarios. All of the simple statistical measures used to define the distribution agree for all maturities to within 50 basis points, and the means agree to within 10 basis points in this sample run. Since the comparison data is generated automatically, it is possible to determine whether or not the fit is acceptable before accepting the representative scenarios. In our experience this has been a robust procedure and the fit from other runs has been within the same general tolerances, with the possible exception of the maximum at the longer maturities, especially if the long maturities are given a low weight in the weighting process. In this case the maximum may be off by about 2%. If a closer fit for the extremes is desired, they could be added into the weighted fit formula. The results are quite sensitive to the choice of the weights.

## 12. Proposed Modification for Arbitrage -Free Scenarios

Most studies that use arbitrage-free scenarios tend to call them paths and really work based on a two dimensional approach, since they tend to equate maturity with time in the future (considering time to be time). Due to that fundamental equation (time = time) they generally do not generated complete yield curves because the latest time in the future plus the longest maturity generated has to be less than or equal to the longest maturity on the original treasury spot yield curve. They generally produce matrices of forward rates which after adjustment to make them arbitrage-free get converted into one spot yield curve per scenario which is used for discounting all cashflows at time 0. However, Tilley has proposed a method of determining complete yield curves for 30 years, based on extending the original yield curve flat from time 30 to time 60, and taking "rolling sequences" of forward rates. Thus the time 0 curves would use the forward rates from time 0 through time 359, and the first year curve would use the one month forwards from time 12 to time 371, etc. These could be converted to spot curves, the above method applied, and the forward rates underlying the representative scenarios could then go through the arbitrage-free process. The results would then clearly be arbitrage free. I have not had the opportunity to test this, which assumes that the two processes (choosing a representative subset, and the arbitrage free adjustment) are commutative.

13. References:


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**Actuarial Education and Research Fund**
**1994 Annual Practitioners' Award Submission**

Originally published in ARCH 1995.2 without exhibits.
Text and exhibits included in this volume.