

**STOCHASTIC INVESTMENT MODELS:
UNIT ROOTS, COINTEGRATION, STATE SPACE AND GARCH MODELS
FOR AUSTRALIAN DATA**

by

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This Version

18 June, 1996

**A version of this paper can be downloaded on the WWW from the URL
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Acknowledgment:

The authors would like to acknowledge financial support from an Institute of Actuaries of Australia 1995 Research Grant. Michael Sherris would like to acknowledge support from a Macquarie University Research Grant and thank Andrew Leung for research assistance.

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BACKGROUND

The development of stochastic investment models for actuarial applications has become a major area of interest to actuaries. Stochastic investment models have applications to pension funds, to life assurance, to general insurance and investment management. Geoghegan et al (1992) and Daykin & Hey (1989, 1990) discuss some of these applications.

Wilkie (1986) first developed a stochastic investment model for the UK using annual data extending over long periods of time. Extensions to the model and analysis of additional aspects were presented in Wilkie (1995). Jon Carter (1991), using quarterly data and employing the same methodology used by Wilkie, derived a stochastic investment model for the Australian experience. The approach used by these authors is based on time series techniques that use differences of data to obtain stationary series without any formal tests of whether or not the series are stationary. Transfer functions are used to determine the structure of the model. These models assume unidirectional causality for the variables used in the models and do not formally test for this causality or incorporate feedback mechanisms (bi-directional causality), that are likely to exist in investment data. Research by Huber (1996) raises other aspects of weaknesses in these models.

The development of these models has taken place at the same time as significant advances in time series and econometric analysis have occurred. These developments include methods for the non-stationary analysis of economic relationships (unit roots), allowing for long-run equilibrium relationships between the series (co-integration), and models that allow for heteroscedasticity such as GARCH models. Harris (1994, 1995) develops and applies a stochastic investment model based on Australian data that allows for heteroscedasticity that he refers to as ERCH. His model uses annual data. Harris (1994) does not formally analyse the stationarity of the series used but does have a model structure that allows for feedback.

INTRODUCTION

The main aim of this paper is to apply some of the techniques of modern time series and econometric analysis to analyse investment data in order to better understand the nature of long run relationships in investment series typically used in stochastic investment models. The analysis is based on Australian data. Other stochastic investment modelling studies for actuarial applications have not formally investigated such relationships yet these are fundamental to the structure of any model to be used in actuarial applications.

The paper attempts to identify and address fundamental issues that need to be considered before developing a particular stochastic investment model. It has identified many structural features of investment models that should be included in such a model. Many of these features are not found in published stochastic investment models for actuarial applications. The paper does not present a stochastic investment model. The detail required for a stochastic investment model will depend on the application. Parameter estimates and initial values for a stochastic investment model will generally need to be based on the most recent data.

Transfer functions were used to fit models to Australian investment data. These were found not to be appropriate for investment modelling analysis for the Australian asset returns series data since there was evidence of feedback relationships between many of the series. State space models were then fitted to the asset returns data and the inflation series allowing for feedback between inflation and asset class returns. The fitted models are reported in this paper.

Based on the results of the research discussed in this paper any model used for stochastic investment modelling will need to incorporate time variation in means and variances and allow for cointegration between interest rates and the CPI. Transfer function models and constant mean and variance models often used in practice were not found to be a satisfactory representation of the return generating process.

The analysis in this paper uses quarterly data. Stochastic investment models are used in practice to establish strategic asset allocations and to examine solvency and capital adequacy. Long run asset allocation strategies are often determined using an annual model on the assumption that cash flows occur at annual intervals. In practice this will be a crude approximation to the timing of cash flows and a higher frequency model will be preferred. Similarly the capital requirements for meeting a solvency test will be much less stringent when solvency is tested at annual time intervals than at quarterly intervals. In this context the difference between annual and quarterly models has yet to be investigated but in order to do this it will be necessary to develop an appropriate quarterly model.

There are many important issues in stochastic investment modelling that require further investigation. These include modelling structural changes that have occurred in the economy using regime-shifting models (Garcia and Perron, 1996 and van Norden and Vigfusson, 1996) and allowing for other time varying components of the series such as heteroscedasticity. Parameter estimation and stability of parameters require further investigation. Parameter and model uncertainty also needs to be incorporated. This research has considered some of these issues including heteroscedasticity.

Structural time series models provide a framework for analysing and developing stochastic investment models. Such models can be developed using a state space formulation. Such an approach provides a number of significant advantages over more dated time series techniques. These are:

- Models reflect the salient characteristics of the data.
- Model parameters are interpretable.
- Feedback mechanisms (bi-directional causality) are included.
- Stationarity does not have to be assumed.
- On-line model maintenance and updating on receipt of additional information.

Further research is required in this area. It is hoped that this paper will provide a foundation for that research.

DATA

The structure of a model needs to be based on economic and financial theory as far as possible. Often theory will rely on empirical data for justification. The model structure and the parameters of a stochastic model will need to reflect historical data. Parameter estimation will usually be based on historical data. A statistical analysis of historical data will also provide useful insights into the features of past experience that the model will need to capture.

The data used for the empirical analysis in this research is taken from the Reserve Bank of Australia Bulletin database. The study uses quarterly data in contrast to most other studies in this area that use annual data. The reasons for using quarterly data, rather than monthly or some other higher frequency, is that this is the highest frequency for which many of the main economic and investment series are available. It is also a frequency that is suited to most practical applications and will be more realistic than annual data as currently used.

Different series are available over different time periods. The longest time period for which data was available on a quarterly basis for all of the financial and economic series was from September 1969. Individual series were available for differing time periods. The series considered were Consumer Price Index - All groups (CPI), the All Ordinaries Share Price Index (SPI), Average Weekly Earnings - adult males (AWE), share dividend yields, 90 day bank bill yields, 2 year Treasury bond yields, 5 year Treasury bond yields, and 10 year Treasury bond yields. An index of dividends was constructed from the dividend yield and the Share Price Index series. Logarithms and differences of the logarithms are used in the analysis of the CPI, SPI, AWE and dividends. The differences in the logarithms of the level of a series is the continuously compounded equivalent growth rate of the series.

Appendix A sets out summary statistics for the series used. It is important to note that none of the series is well represented by an assumption of independent and identically distributed normal variables. Mean-variance optimisation models that are often used in determining optimal asset allocation strategies by fund managers and asset consultants assume that the returns are correlated but serially independent and identically distributed normal variables. It is clear from the data that such assumptions are inappropriate and that mean-variance models should be used with caution.

Appendix B provides time series plots of the series, the logarithm of the series and their differences. An examination of the plots for the CPI, AWE, SPI and the Dividend Index series shows exponential growth and the plot of the logarithms of these series suggests that the series could be fluctuations around a linear trend in the logarithms. Such a series is referred to as trend stationary. The analysis in this research indicates that the trend is in fact stochastic and the logarithms of these series can best be modelled using first differences or the rate of continuous growth of the series. An examination of the plot of the differences of the logarithms of these series also appears to indicate a non-constant variance or heterogeneity. This is tested for in this research and found to be the case.

The interest rate series all show a changing level as interest rates rose during the 1970's and 1980's. Models of interest rates that incorporate mean-reversion are often used. These models assume that the level of interest rates is stationary. This is not obvious from an examination of the time series plots of the interest rates. The differences in the levels of the interest rates do however appear to fluctuate around a constant value but the series appears to be heteroscedastic.

UNIT ROOTS AND STATIONARY SERIES

Many of the series used in stochastic investment modelling are non-stationary. For example the level of the Consumer Price Index, the level of the Share Price Index and the level of a dividend index can be seen to be non-stationary by simple inspection of a time series plot. It is less clear whether or not interest rates have stationary distributions and this cannot easily be determined by inspection of a time series plot. Rates of changes in indices or rates of return have been used in stochastic investment models and this might be justified because they can be considered as "natural" variables to use.

Because the level of the Consumer Price Index, the level of the Share Price Index and the level of a dividend index are non-stationary this has lead researchers to difference the data, or some transformation of the data such as a logarithmic transform, in order to obtain a stationary series for modelling. Wilkie (1986) and Carter (1991) used differenced data, as does Harris (1994, 1995) for equity and inflation series. In Carter (1991) the order of differencing was decided using more traditional time series techniques based on the sample autocorrelations. FitzHerbert (1992) fits a deterministic trend to various index levels instead of taking differences. Neither FitzHerbert (1992) nor Harris (1994, 1995) conduct formal tests for stationarity of the series used in their models.

If the level of a series is non-stationary but the difference of the series is stationary then the series is said to contain a "unit root", be "integrated or order 1", or be "difference stationary". It is important to understand that the existence of unit roots determines the nature of the trends in the series. If a series contains a unit root then the trend in the series is stochastic and shocks to the series will be permanent. If the series does not contain a unit root then the series is "trend stationary". The trend in the series will be deterministic and shocks to the series will be transitory. This has major implications for investment models in actuarial applications.

The other aspect of unit roots is that if they exist in a series and differences are not used in model fitting and parameter estimation then the statistical properties of the parameter estimates for the model will not be standard and the use of standard results for model identification and parameter estimation can result in an incorrect model structure and unreliable parameter estimates. Insignificant parameters are more likely to be accepted as being significant.

These are all significant reasons that make testing for unit roots in a series for use in developing an investment model critical. Formal statistical tests for unit roots have been developed over the past decade in the econometric literature. Unit root tests are described in many articles and books including Dickey and Fuller (1979) and Mills

(1993). These procedures are implemented in various statistical and econometric packages such as Shazam (1993).

In order to test for unit roots in the series x_t the following regressions are fitted:

$$\Delta x_t = \alpha_0 + \alpha_1 x_t + \varepsilon_t \quad (1)$$

$$\Delta x_t = \alpha_0 + \alpha_2 t + \alpha_1 x_t + \varepsilon_t \quad (2)$$

where ε_t are assumed to be independent and identically distributed and $\Delta x_t = x_t - x_{t-1}$ is the first difference in the series. These are referred to as Dickey-Fuller regressions. If the value of α_1 is equal to zero then x_t is integrated or order 1. In this case (1) defines x_t as a random walk with drift and (2) defines x_t as a random walk around a non-linear time trend. This can be seen by substituting $\alpha_1 = 0$ and rearranging to get:

$$y_t^* = y_{t-1}^* + \varepsilon_t$$

with

$$y_t^* = x_t - \alpha_0 t \text{ for (1), and}$$

$$y_t^* = x_t - \left[\alpha_0 + \frac{\alpha_2}{2} \right] t - \frac{\alpha_2}{2} t^2 \text{ for (2).}$$

In the case that the ε_t are not i.i.d. then the following regressions are used:

$$\Delta x_t = \alpha_0 + \alpha_1 x_t + \sum_{j=1}^p \gamma_j \Delta x_{t-j} + \varepsilon_t \quad (3)$$

$$\Delta x_t = \alpha_0 + \alpha_1 x_t + \alpha_2 t + \sum_{j=1}^p \gamma_j \Delta x_{t-j} + \varepsilon_t \quad (4)$$

where p is selected to ensure the errors are uncorrelated. These are referred to as augmented Dickey-Fuller regressions.

The procedure for testing for unit roots and determining the order of integration uses the t-statistic of the coefficient α_1 of x_{t-1} in the regressions given by (3) and (4). The null hypothesis is that the series is non-stationary with $\alpha_1 = 0$ and the alternative is that $\alpha_1 < 0$. The t-statistic under the null hypothesis has a non-standard distribution and is compared with the table of critical values found in Fuller (1976, p. 373). If the null is rejected then this is evidence that x_t is a stationary series. If the null is not rejected then differences of the series are taken and the differences tested for a unit root. When the null is eventually rejected the level of differencing required to reject the null determines the order of integration of the series. Usually this requires only one order of differencing for financial and economic series.

The critical values used for testing for a unit root depend on whether or not α_2 is zero in the regression given by (4). If α_0 and α_2 are zero then the t-statistic for α_1 has the non-standard limiting distribution as in Dickey and Fuller (1979). This distribution applies if α_0 is non-zero. If α_2 is non-zero then the limiting distribution is standard normal. It is therefore necessary to determine if α_2 is non-zero in order to determine critical values for testing for unit roots. Dickey and Fuller (1981) provide critical values for a range of tests as follows:

- ϕ_1 using Equation (3) with Null $\alpha_0=0, \alpha_1=0$
- ϕ_2 using Equation (4) with Null $\alpha_0=0, \alpha_2=0, \alpha_1=0$
- ϕ_3 using Equation (4) with Null $\alpha_0 \neq 0, \alpha_2=0, \alpha_1=0$

The procedure for testing for unit roots can be set out as follows (Holden and Perman, 1994):

Step 1: Estimate the regression (4).

Step 2: Use ϕ_3 to test the null hypothesis $\alpha_0 \neq 0, \alpha_2=0, \alpha_1=0$ against the alternative $\alpha_0 \neq 0, \alpha_2 \neq 0, \alpha_1 \neq 0$. If the null can not be rejected then go to Step 5.

Step 3: Need to determine if $\alpha_2 \neq 0, \alpha_1=0$, or $\alpha_2=0, \alpha_1 \neq 0$ or $\alpha_2 \neq 0, \alpha_1 \neq 0$. Test for $\alpha_1=0$. If this null is not rejected then conclude that $\alpha_2 \neq 0, \alpha_1=0$ and the series has a unit root and linear trend.

Step 4: If the null is rejected in Step 3 then there is no unit root and the series is stationary. A conventional t-test for $\alpha_2=0$ is used to test for a trend. If this null is rejected then the series is stationary with a linear trend. A conventional t-test is used to test for a constant $\alpha_0 \neq 0$.

Step 5: In this case the series has a unit root with no trend and possibly with drift. This can be confirmed using the non-standard critical values for the null $\alpha_1=0$.

Step 6: To test for non-zero drift use ϕ_2 since this tests the null $\alpha_0=0, \alpha_2=0, \alpha_1=0$ and previous tests have not rejected $\alpha_2=0, \alpha_1=0$. If the null is not rejected then the evidence suggests that the series is a random walk without drift. If the null is rejected then the series is a random walk with drift.

Step 7: Regression (3) is used and ϕ_1 used to test the null $\alpha_0=0, \alpha_1=0$. This will confirm the results of earlier steps.

Table 1 sets out the unit root test statistics using the augmented Dickey-Fuller procedure for Australian quarterly data over the period September 1969 to December 1994. Table 2 gives the parameter estimates and t statistics for regressions (3) and (4) for this data.

Applying the procedure to the unit root test statistics in Tables 1 and 2 gives the following results:

logCPI - logarithm of the Consumer Price Index

ϕ_3 does not reject the null hypothesis $\alpha_0 \neq 0, \alpha_2=0, \alpha_1=0$ so the series has a unit root. ϕ_1 rejects the null $\alpha_0=0, \alpha_1=0$ suggesting the drift is significant. This is confirmed by the regression (3) where the estimate of α_0 is 0.0244 with a t statistic of 2.94 which is significant at the 0.4% level.

Δ LogCPI - first difference of logCPI

ϕ_3 rejects the null hypothesis $\alpha_0 \neq 0, \alpha_2=0, \alpha_1=0$ for the differences. The τ_2 test statistic rejects the hypothesis that $\alpha_1=0$ so there is evidence that the differences are stationary.

Conclusion

On the assumption of i.i.d. errors, the logarithm of the CPI is integrated of order 1 and differences in the logarithm of the CPI is a stationary series with drift.

LogSPI - logarithm of the Share Price Index

ϕ_3 does not reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ so the series has a unit root with no trend. ϕ_2 does not reject the null $\alpha_0 = 0, \alpha_2 = 0, \alpha_1 = 0$ so that this is evidence of no drift. Using the more powerful test with ϕ_1 does not reject the null $\alpha_0 = 0, \alpha_1 = 0$. From the regression (3) the estimate of α_0 is 0.0547 with a t statistic of 0.52 which is not significant suggesting that the drift is zero.

Δ LogSPI

ϕ_3 rejects the null hypothesis that $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ for the differences. The τ_2 test statistic rejects the hypothesis that $\alpha_1 = 0$ so there is evidence that the differences are stationary. From regression (4) the estimate of α_0 is 0.0128 and of α_2 is 0.0002 and neither of these is significant.

Conclusion

The logarithm of the SPI is integrated of order 1 and differences in the logarithm of the SPI is a stationary series. The results suggest that the drift in the difference of $\log(\text{SPI})$ is not significantly different from zero. This could be because the test used has little power against an alternative of a positive drift close to zero.

LogAWE - logarithm of Average Weekly Earnings

ϕ_3 does not reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ so the series has a unit root with no trend. ϕ_2 does not reject the null $\alpha_0 = 0, \alpha_2 = 0, \alpha_1 = 0$ so this does not reject the zero drift hypothesis. Using the more powerful test with ϕ_1 does reject the null $\alpha_0 = 0, \alpha_1 = 0$ suggesting that the drift is significant. From the regression (3) the estimate of α_0 is 0.1182 with a t statistic of 3.04 which confirms this.

Δ LogAWE

ϕ_3 does not reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ for the differences which suggests that the differences have a unit root. However the τ_2 test statistic rejects the hypothesis that $\alpha_1 = 0$ and this is evidence that the differences are stationary. Note that if the differences in the logarithm of AWE are not stationary then this means that a random shock to the continuously compounding growth rate of AWE would be permanent. A model with this feature would not be sensible since it would allow the continuously compounding growth rate to become arbitrarily large or small.

Conclusion

The logarithm of AWE is most likely integrated of order 1 although the statistical tests suggest it could be of higher order.

LogSDiv - logarithm of the Share Index Dividend Series

ϕ_3 does not reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ so the series has a unit root with no trend. ϕ_2 does not reject the null $\alpha_0 = 0, \alpha_2 = 0, \alpha_1 = 0$ so this does not reject the zero drift hypothesis. Using the more powerful test with ϕ_1 does not reject the null $\alpha_0 = 0, \alpha_1 = 0$ suggesting that the drift is not significant. From the regression (3) the estimate of α_0 is 0.0940 with a t statistic of 1.188 which confirms this.

Table 1 Test Statistics for Unit Roots - ADF Regressions

Variable	n	τ_1	ϕ_1	τ_2	ϕ_2	ϕ_3
<i>10% Critical Value</i>		(-2.57)	(3.78)	(-3.13)	(4.03)	(5.34)
LogCPI	96	-2.5394	5.0773*	-0.14401	3.3658	3.2153
Δ LogCPI	95	-1.9654	1.9387	-3.2620*	3.8593	5.7811*
LogSPI	101	-0.36670	1.0593	-2.5374	2.9977	3.4468
Δ LogSPI	93	-4.0574*	8.2403*	-4.0353*	5.5106*	8.2573*
LogAWE	94	-2.9235*	5.0615*	-1.1134	3.3407	4.2320
Δ LogAWE	94	-1.6971	1.5320	-3.1763*	3.4963*	5.1460
LogSDiv	91	-0.94375	3.2947	-2.2644	3.8328	2.7697
Δ LogSDiv	97	-3.6085*	6.5145*	-3.5827*	4.3077*	6.4580*
SDyields	94	-2.6065*	3.4282	-2.4752	2.4863	3.6984
Δ SDyields	93	-4.2322*	8.9565*	-4.3210*	6.2671*	9.3998*
BB90	95	-2.0651	2.1326	-1.8133	1.4128	2.1190
Δ BB90	93	-4.3252*	9.3662*	-4.5146*	6.8251*	10.225*
TB2	98	-2.1987	2.4757	-2.2523	1.7769	2.6071
Δ TB2	97	-3.5883*	6.4738*	-3.5303*	4.3129*	6.4337*
TB5	98	-1.9812	2.0334	-1.8892	1.3792	1.9985
Δ TB5	96	-3.5858*	6.4526*	-3.6100*	4.4730*	6.6862*
TB10	101	-1.8629	1.9083	-1.3939	1.3380	1.8353
Δ TB10	98	-4.8847*	11.930*	-4.9430*	8.2041*	12.306*

* indicates significant at 10% level

Table 2 Tests for Unit Roots - Parameters of ADF Regressions
(t statistics in brackets beneath the estimate)

Variable	Regression 3			Regression 4			
	p	α_0	α_1	p	α_0	α_1	α_2
LogCPI	5	0.02438 (2.94)	-0.00446 (-2.539)	5	0.017387 (0.5417)	-0.00175 (-0.1440)	-0.00006 (-0.2255)
Δ LogCPI	5	0.00390 (1.744)	-0.20633 (-1.965)	5	0.01406 (3.262)	-0.41302 (-3.262)	-0.00011 (-2.724)
LogSPI	0	0.05472 (0.5161)	-0.00594 (-0.3667)	0	0.54693 (2.536)	-0.10166 (-2.537)	0.00251 (2.598)
Δ LogSPI	7	0.02421 (1.803)	-1.2959 (-4.057)	7	0.01283 (0.4519)	-1.3302 (-4.035)	0.00021 (0.4551)
LogAWE	7	0.11815 (3.041)	-0.01577 (-2.923)	7	0.12763 (1.513)	-0.01766 (-1.113)	0.00005 (0.1267)
Δ LogAWE	6	0.00520 (1.204)	-0.29492 (-1.697)	6	0.03463 (2.952)	-0.80799 (-3.176)	-0.00034 (-2.684)
LogSDiv	10	0.09404 (1.188)	-0.00092 (-0.9437)	10	0.99212 (2.332)	-0.14792 (-2.264)	0.00361 (2.147)
Δ LogSDiv	3	0.01354 (1.789)	-0.67085 (-3.609)	3	0.01587 (1.043)	-0.67512 (-3.583)	-0.00004 (-0.1772)
SDyields	7	0.84903 (2.610)	-0.18277 (-2.607)	7	0.91397 (2.719)	-0.17543 (-2.475)	-0.00176 (-0.7951)
Δ SDyields	7	0.01025 (0.1704)	-1.3678 (-4.232)	7	0.13634 (0.9379)	-1.4646 (-4.321)	-0.00224 (-0.9529)
BB90	6	1.3958 (1.987)	-0.12360 (-2.065)	6	1.4177 (1.963)	-0.11965 (-1.813)	-0.00121 (-0.1463)
Δ BB90	7	0.00596 (0.02885)	-1.5594 (-4.325)	7	0.58113 (1.167)	-1.7217 (-4.515)	-0.01032 (-1.268)
TB2	3	0.74802 (2.203)	-0.06935 (-2.199)	3	0.71168 (2.061)	-0.07914 (-2.252)	0.00255 (0.6407)
Δ TB2	3	0.03630 (0.3556)	-0.71139 (-3.588)	3	0.10645 (0.4620)	-0.73037 (-3.530)	-0.00130 (-0.3400)
TB5	3	0.61876 (2.011)	-0.05561 (-1.981)	3	0.61260 (1.977)	-0.06071 (-1.889)	0.00111 (0.3311)
Δ TB5	4	0.03165 (0.3752)	-0.79979 (-3.586)	4	0.16815 (0.8454)	-0.86536 (-3.610)	-0.00249 (-0.7582)
TB10	0	0.54248 (1.951)	-0.04698 (-1.863)	0	0.54105 (1.939)	-0.04031 (-1.394)	-0.00134 (-0.4767)
Δ TB10	2	0.02894 (0.3953)	-0.84539 (-4.885)	2	0.15870 (0.9776)	-0.88785 (-4.943)	-0.00240 (0.8958)

$\Delta\text{LogSDiv}$

ϕ_3 does reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ which is evidence that the differences in the series are stationary without trend. The τ_2 test statistic rejects the hypothesis that $\alpha_1 = 0$ so this is further evidence that the differences are stationary.

Conclusion

The logarithm of the dividend series is integrated of order 1 and the differences in the series have no drift.

SDyields - Dividend yields on the Share Price Index

ϕ_3 does not reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ so the series has a unit root with no trend. ϕ_2 does not reject the null $\alpha_0 = 0, \alpha_2 = 0, \alpha_1 = 0$ so this does not reject the zero drift hypothesis. Using the more powerful test with ϕ_1 does not reject the null $\alpha_0 = 0, \alpha_1 = 0$ suggesting that the drift is not significant. However regression (3) shows an estimate for α_0 of 0.849 and this is significant.

$\Delta\text{SDyields}$

ϕ_3 does reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ for the differences of the series which is evidence that the differences are stationary. The τ_2 test statistic rejects the hypothesis that $\alpha_1 = 0$ so this is further evidence that the differences are stationary.

Conclusion

The dividend yield series is integrated of order 1 and the differences in the series are stationary with zero drift.

Interest rates - BB90 - 90 day bank bill yields, TB2 - 2 year Treasury bond yields, TB5 - 5 year Treasury bond yields, TB10 - 10 year Treasury bond yields

The same test statistics are significant for all of the interest rate series. ϕ_3 does not reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ so this is evidence that each of the series has a unit root with no trend. ϕ_2 does not reject the null $\alpha_0 = 0, \alpha_2 = 0, \alpha_1 = 0$ so this does not reject the zero drift hypothesis. Using the more powerful test with ϕ_1 does not reject the null $\alpha_0 = 0, \alpha_1 = 0$ suggesting that the drift is not significant.

$\Delta\text{Interest rates} - \Delta\text{BB90}, \Delta\text{TB2}, \Delta\text{TB5}, \Delta\text{TB10}$

ϕ_3 does reject the null hypothesis $\alpha_0 \neq 0, \alpha_2 = 0, \alpha_1 = 0$ for all interest rate series which is evidence that the differences of each of the series is stationary without trend. The τ_2 test statistic rejects the hypothesis that $\alpha_1 = 0$ so this is further evidence that the differences of the series are stationary.

Conclusion

Each of the interest rate series is integrated of order 1 so the changes in yields are stationary with no drift.

Table 3 Test Statistics for Unit Roots -Phillips-Perron Tests (n=101)

Variable	τ_1	ϕ_1	τ_2	ϕ_2	ϕ_3
<i>10% Critical Value</i>	<i>(-2.57)</i>	<i>(3.78)</i>	<i>(-3.13)</i>	<i>(4.03)</i>	<i>(5.34)</i>
LogCPI	-3.8378*	119.83*	2.2660	98.752*	13.073*
Δ LogCPI	-5.2039*	13.482*	-6.3676*	13.538*	20.307*
LogSPI	-0.3098	0.9029	-2.8153	3.4695	4.3031
Δ LogSPI	-9.8017*	48.052*	-9.8160*	32.128*	48.178*
LogAWE	-4.1848*	48.665*	0.0434	33.830*	9.6581*
Δ LogAWE	-8.4424*	35.636*	-9.9542*	33.037*	49.544*
LogSDiv	-1.1221	6.5966*	-1.5004	4.8385*	1.4252
Δ LogSDiv	-10.607*	56.279*	-10.613*	37.587*	56.374*
SDyields	-3.1195*	4.9308*	-3.0255	3.3472	4.9644
Δ SDyields	-9.1122*	41.529*	-9.4328*	27.813*	41.712*
BB90	-2.7201	3.7059	-2.6124	2.4444	3.6589
Δ BB90	-10.773*	58.023*	-10.775*	38.702*	58.047*
TB2	-1.9289	1.9504	-1.7487	1.2882	1.8453
Δ TB2	-9.1718*	42.071*	-9.1772*	28.106*	42.152*
TB5	-1.8815	1.8962	-1.5613	1.2723	1.7848
Δ TB5	-9.2738*	43.005*	-9.3308*	29.027*	43.541*
TB10	-4.8920*	1.9395	-1.4685	1.3388	1.8608
Δ TB10	-9.0689*	41.130*	-9.1662*	28.013*	42.018*

* indicates significant at 10% level

Phillips and Perron (1988) have proposed non-parametric procedures for testing for unit roots with more general assumptions concerning ϵ_t than the i.i.d. assumptions for the Dickey-Fuller and augmented Dickey-Fuller tests. Table 3 sets out the equivalent test statistics to those in Table 1 using the Phillips-Perron test procedures. These were calculated using the procedures in Shazam (1993).

The conclusions already drawn for the SPI and for the interest rate series are supported by these test statistics. There are however some differences apparent for the other series. For the logCPI series the Phillips-Perron ϕ_3 rejects the null hypothesis $\alpha_0 \neq 0$, $\alpha_2 = 0$, $\alpha_1 = 0$ and the hypothesis that $\alpha_1 = 0$ is not rejected. The conclusion is that the series has a unit root with a trend. The differences in the series are stationary. Similar conclusions are reached for logAWE using the Phillips-Perron statistics. In the case of LogSDiv, the logarithm of the dividends series, the Phillips-Perron statistics suggest that this series is difference stationary with drift. Dividend yields are difference stationary without drift.

So far the data period used has been common to all the series covering the period September 1969 to December 1994. Some of the series are available for longer time periods. Tables 4 and 5 report unit root test statistics for these longer time periods for the relevant series.

Table 4 Test Statistics for Unit Roots - ADF Regressions

Various Periods

Variable	n	τ_1	ϕ_1	τ_2	ϕ_2	ϕ_3
<i>10% Critical Value</i>		<i>(-2.57)</i>	<i>(3.78)</i>	<i>(-3.13)</i>	<i>(4.03)</i>	<i>(5.34)</i>
Data from March 1939 to March 1995						
LogSPI	223	-0.0369	3.2802	-2.7539	4.9204*	3.9974
Δ LogSPI	210	-4.4210*	9.7739*	-4.4454*	6.5980*	9.8960*
LogAWE	210	-0.6899	2.1214	-2.4390	3.3084	3.0349
Δ LogAWE	210	-2.3994	2.9010	-2.3545	1.9627	2.9216
Data from March 1958 to December 1994						
LogCPI	179	0.18249	3.3767	-1.6925	3.3323	1.5961
Δ LogCPI	180	-3.2500*	5.2817*	-3.2444*	3.5113	5.2666
LogSPI	185	-0.29941	2.8000	-2.4794	3.9967	3.1611
Δ LogSPI	172	-4.0135*	8.0617*	-4.0168*	5.4166*	8.1171*
LogAWE	172	-0.07160	1.4983	-2.6340	3.4014	3.5482
Δ LogAWE	173	-2.6096*	3.7562	-2.6057	2.5520	3.4787
Data from March 1958 to December 1994						
LogCPI	147	-0.51786	1.3032	-2.6342	3.1287	3.4721
Δ LogCPI	141	-2.1290	2.2675	-1.9831	1.5080	2.2608
LogSPI	146	-0.60057	2.3429	-2.1793	3.0718	2.3939
Δ LogSPI	138	-4.3209*	9.3540*	-4.3673*	6.3799*	9.5508*
LogAWE	134	-1.0694	2.1296	-1.4421	1.9822	1.4069
Δ LogAWE	134	-1.6881	1.4316	-1.6724	1.2924	1.9319
TB10	143	-1.4655	1.2807	-1.6998	1.2059	1.6020
Δ TB10	142	-4.4264*	9.8343*	-4.4309*	6.6185*	9.8903*

* indicates significant at 10% level

For the period March 1939 to March 1995 Tables 4 and 5 provide support for the hypothesis that logSPI is difference stationary with drift. This longer period of data provides a more reliable estimate of the drift so the conclusion is that the logSPI is difference stationary with positive drift. There is evidence in Table 4 that logAWE and logCPI are integrated of a higher order than 1 but the results in Table 5 suggest that they are integrated of order 1.

Because the data used is quarterly it is necessary to test for seasonal integration. In quarterly data there could be a bi-annual or annual frequency seasonal unit root as well as the quarterly unit root tested for already. Hylleberg et al (1990) develop tests and test statistics for seasonal unit roots. Shazam (1993) provides procedures for implementing these tests. These procedures were used to obtain Table 6. Bi-annual and annual unit roots are convincingly rejected for all of these series.

It is worth noting that structural breaks in any series can result in a stationary series appearing to have a unit root. This will lead to differencing the data when a model using the levels of the data and explicitly capturing the structural break would be more appropriate. Differencing series results in the loss of information about the long run level of the series so that care has to be taken to ensure that the series is not stationary.

It could be argued that deregulation of financial markets during the 1980's resulted in a structural break in many of the series. For instance the method used to sell government securities changed during this period and the bond market became more active. The requirements for life insurance companies, superannuation funds and banks to hold government securities were also relaxed. During this period an imputation tax system was introduced for share investments. All of these factors could well have resulted in structural changes in rates of return and the levels of the series used in this study. This issue is briefly addressed later in this paper.

**Table 5 Test Statistics for Unit Roots - Phillips-Perron Tests
Various Periods**

Variable	τ_1	ϕ_1	τ_2	ϕ_2	ϕ_3
<i>10% Critical Value</i>	<i>(-2.57)</i>	<i>(3.78)</i>	<i>(-3.13)</i>	<i>(4.03)</i>	<i>(5.34)</i>
Data from March 1939 to March 1995 (n=224)					
LogSPI	-0.066319	3.1281	-2.8455	4.9236*	4.2426
Δ LogSPI	-14.151*	100.13*	-14.143*	66.678*	100.02*
LogAWE	0.020279	57.654*	-1.0626	38.751*	0.58240
Δ LogAWE	-11.358*	64.474*	-11.332*	42.781*	64.171*
Data from March 1948 to December 1994 (n=186)					
LogCPI	0.23384	73.724*	-0.64481	49.144*	0.28228
Δ LogCPI	-5.5568*	15.382*	-5.5431*	10.185*	15.278*
LogSPI	-0.32714	2.6717	-2.5693	4.0070*	3.3852
Δ LogSPI	-12.851*	82.575*	-12.827*	54.846*	82.268*
LogAWE	-0.96316	48.872*	-0.52415	32.424*	0.52388
Δ LogAWE	-10.584*	55.994*	-10.604*	37.463*	56.192*
Data from March 1958 to December 1994 (n=147)					
LogCPI	1.5225	75.700*	-2.5431	57.601*	5.4467*
Δ LogCPI	-5.1516*	13.206*	-5.2291*	9.1058*	13.655*
LogSPI	-0.61547	2.2874	-2.2381	3.0804	2.5254
Δ LogSPI	-11.650*	67.861*	-11.611*	44.949*	67.421*
LogAWE	-0.42646	39.283*	-0.52395	26.044*	0.19468
Δ LogAWE	-10.069*	50.672*	-10.041*	33.597*	50.395*
TB10	-1.3495	1.1305	-1.3746	0.90826	1.1456
Δ TB10	-10.920*	-59.631	-10.913*	39.706*	59.559*

* indicates significant at 10% level

Table 6 Seasonal Unit Root Tests for Quarterly Data

Variable	Unit Root -Zero Frequency		Bi-annual unit root		Annual unit root	
	No seasonal dummies	Seasonal dummies	No seasonal dummies	Seasonal dummies	No seasonal dummies	Seasonal dummies
<i>10% Critical Values</i>	-2.58	-2.63	-1.60	-2.63	2.35	5.56
logCPI	-2.5558	-2.4495	-2.3806*	-2.8796*	13.3244*	16.7450*
Δ logCPI	-1.6454	-1.6184	-1.9934*	-2.6699*	7.7801*	11.5590*
logSPI	-0.2028	-0.2029	-3.5139*	-3.7428*	9.1280*	9.5253*
Δ logSPI	-4.0574*	-4.0689*	-3.0922*	-3.4052*	6.7530*	7.5579*
logAWE	-2.9666*	-2.9198*	-4.1878*	-4.2243*	9.2549*	9.7714*
Δ logAWE	-1.4787	-1.4798	-3.6134*	-3.7350*	6.6096*	7.1982*
logSDiv	-0.9518	-0.9242	-2.4735*	-2.3999	13.6728*	13.8382*
Δ logSDiv	-2.8230*	-2.8009*	-2.1389*	-2.1051	9.0294*	9.1264*
SDyields	-2.5334	-2.4800	-2.8599*	-2.8753*	13.4955*	13.7299*
Δ SDyields	-4.2322*	-4.2173*	-2.5347*	-2.6077	10.1068*	10.5368*
BB90	-1.8145	-1.8794	-3.6654*	-3.6088*	5.0402*	6.4094*
Δ BB90	-4.3252*	-4.2128*	-3.3461*	-3.3806*	4.5974*	5.7972*
TB2	-2.0141	-2.0619	-3.4886*	-3.9141*	10.0680*	10.1565*
Δ TB2	-3.7605*	-6.7620*	-3.4435*	-3.9788*	9.3317*	10.2047*
TB5	-2.0343	-2.0590	-4.1606*	-4.4070*	11.6016*	11.6886*
Δ TB5	-3.1579*	-3.1463*	-3.7562*	-4.0800*	9.0354*	9.7720*
TB10	-2.0179	-2.0250	-3.7420*	-3.8649*	11.7289*	11.8837*
Δ TB10	-2.9494*	-2.9253*	-3.3636*	-3.5422*	8.6545*	9.1863*

*values significant at 10% critical level

COINTEGRATION

The differencing operation used to achieve stationarity, often used in developing stochastic investment models for actuarial applications, involves a loss of information about long-run movements in the series. The theory of cointegration explains how to study the inter-relationships between the long-term trends in the series. These long-term trends are differenced away in the standard Box-Jenkins approach. The inter-relationships between the long-term trends in the series can be interpreted as equilibrium relationships between the series.

Empirical studies have demonstrated that financial markets generally move quickly to an equilibrium since informed investors act quickly on new information particularly when transaction costs are low and markets are liquid. Financial markets can be out of apparent equilibrium as evidenced by “speculative bubbles” that occur when the share market booms and subsequently “crashes” even though these events are consistent with rational expectations. Economic systems are less likely to be in equilibrium since friction and price stickiness in goods and labour markets can cause the adjustment process to equilibrium to occur over an extended time frame. This suggests that if equilibrium relationships exist between financial and economic variables then these will only be detected by examining data over long time periods.

Rates of return on different investments would be expected to have long run equilibrium relationships determining their relative values. For example the spread between the return on a short term investment and the return on a longer term investment should fluctuate around some long term relationship that reflects the risk premium investors require for the longer term investment over the shorter term investment. If a long term relationship does hold then the difference between the returns should have a stationary distribution. The rates of return themselves might not be stationary but a linear combination of them will be stationary if such a long run equilibrium holds. Rates of return adjusted for expected rates of inflation, referred to as “real” rates of return as compared with nominal rates of return, might also be expected to have a stationary distribution.

Similarly the level of the share price index (SPI) and the level of an inflation index (CPI) could have an equilibrium such that they do not “wander” too far away from each other even though each is non-stationary. Thus if there is excessive inflation then it is often argued for a variety of reasons that the level of the share market should eventually increase in line with inflation and vice versa. In some cases a well developed theory might not exist to specify the nature of the equilibrium relationships between different series or there might be conflicting theories. In this case it will be the empirical relationships in the data that will support one or the other theories.

Most actuaries assume that there is a relationship between equity returns and inflation. This assumption is usually implicit in the use of “real” rates of return for projecting asset values and for valuation purposes. If a constant “real” rate of return is used then this implicitly assumes that asset returns are perfectly correlated with inflation. The Wilkie model uses inflation as the main factor driving asset returns. Investment model studies by Carter (1991) and Harris (1995) include results derived from fitting Wilkie’s model to Australian data and find no statistically significant empirical relationship between equity returns and rates of inflation. This conflict between often used actuarial assumptions and empirical results clearly requires investigation since it will be fundamental to investment modelling and modelling the interaction between liabilities and assets of insurance companies and pension (superannuation) funds.

It is important to recognise that equity values and inflation can have a long-run equilibrium relationship and for there to be no statistically significant relationship between equity returns and rates of inflation. This could be the case if the series are co-integrated. Since rates of inflation and equity (capital) returns are differences in the logarithm of the level of the inflation index and differences in the logarithm of the equity index respectively, these rates of change in the levels of the indexes might appear to have no statistical relationship even though the levels of the indexes might be co-integrated with a long run equilibrium relationship. Each of the index series would be difference stationary containing a unit root consistent with the notion of market efficiency and with studies of Australian data such as Carter (1991), Harris (1994) and the results in this paper.

As another example consider the level of the share market index, dividends and interest rates. The level of the index (SPI) will equal the present value of future dividends

(DIV) on the index and, assuming dividends grow at constant rate g and are present valued at constant rate i , can be written as:

$$SPI_t = \frac{DIV_t}{i_t - g_t}$$

so that

$$\log(SPI_t) = \log(DIV_t) - \log(i_t - g_t)$$

In this case the logarithm of the share price index, the logarithm of the dividends and interest rates will be co-integrated.

If variables are non-stationary but an equilibrium relationship represented by a linear combination of the variables exists such that this linear combination is stationary then the variables are said to be co-integrated. Engle and Granger (1987) suggested the concept of cointegration and developed tests for cointegration. The concept of cointegration captures the notion that two or more series “move together” in some fashion. Each series, if looked at individually, need not have a long run equilibrium but their relative values might. The series have common stochastic trends.

Testing for cointegration between any two series, where there is only one co-integrating linear combination determining the equilibrium relationship between the series, requires only the unit root tests used earlier to determine the order of stationarity of the investment data. Consider two series x_t and y_t that are integrated of order 1 so that they are difference stationary. If a long term (linear) relationship exists between these then $x_t - \beta y_t$, for some constant β , will be stationary. If x_t is regressed on y_t and there is a long run equilibrium relationship between them, then the residuals from this regression will not have a unit root. Thus for these residuals the null hypothesis of a unit root should be rejected if the series are co-integrated. Otherwise there is no evidence of cointegration.

Table 7 reports the results of unit root cointegration tests for bi-variate series from Australian data using Augmented Dickey Fuller tests and Table 8 reports the results using Phillips-Perron tests. From the tests carried out earlier in this paper all of the series used were previously found to be integrated of order 1. The results in Tables 7 and 8 were calculated using procedures in Shazam (1993). They consider each of the bi-variate series over the longest time period available and also for shorter time periods.

There is no evidence that any of the bi-variate series considered, other than the 90 day bank bill yield and the 10 year Treasury bond yield, are co-integrated. In all cases other than for these two interest rates the test statistics for both ADF and Phillips-Perron tests given in Tables 7 and 8 do not reject the null hypothesis of a unit root. Thus there is no evidence that the SPI and the CPI “move together” nor that share index dividends and the CPI “move together”. It is encouraging to find that the long and short interest rate are co-integrated since this is supported by the results of Ang and Moore (1994).

Table 7 Test Statistics for Co-integration - ADF Regression Tests

Various Periods

Variable	n	τ_1	ϕ_1	τ_2	ϕ_2	ϕ_3
<i>10% Critical Value</i>		<i>(-2.57)</i>	<i>(3.78)</i>	<i>(-3.13)</i>	<i>(4.03)</i>	<i>(5.34)</i>
Data from September 1948 to March 1995						
RSC (SPI-CPI)	186	-2.2308	2.4892	-2.2755	1.7733	2.6588
Δ RSC	186	-3.9832*	7.9760*	-3.9460*	5.2886*	7.8902*
Data from March 1958 to December 1994						
RSC (SPI-CPI)	147	-2.2356	2.5917	-2.2261	1.7159	2.4817
Δ RSC	138	-4.1852*	8.7753*	-4.1839*	5.8690*	8.7864*
R10C(TB10-CPI)	144	-1.8067	1.6823	-1.7801	1.2556	1.8333
Δ R10C	147	-4.5473*	10.377*	-4.5761*	7.0730*	10.571*
Data from September 1967 to December 1994						
RSC (SPI-CPI)	109	-2.1755	1.5808	-1.8020	1.1877	1.7770
Δ RSC	100	-3.5151*	6.1879*	-3.7956*	4.8270*	7.2305*
RSD (SPI-DIVS)	102	-1.9911	1.9934	-2.0944	2.0728	3.0978
Δ RSD	100	-3.8000*	7.2352*	-4.1384*	5.7208*	8.5658*
RDC (DIVS-CPI)	99	-2.0153	2.0389	-2.0644	1.4597	2.1813
Δ RDC	99	-4.5093*	10.167*	-4.4580*	6.7409*	10.111*
Data from September 1969 to December 1994						
RSC (SPI-CPI)	101	-2.1646	2.3788	-2.2573	2.1308	3.1598
Δ RSC	93	-3.9151*	7.6758*	-4.0427*	5.4993*	8.2371*
RSD (SPI-DIVS)	94	-2.2297	2.4956	-2.4545	2.5298	3.7847
Δ RSD	93	-4.2881*	9.1979*	-4.3814*	6.4889*	9.7294*
RDC (DIVS-CPI)	91	-1.9483	1.9080	-2.0843	1.5118	2.2576
Δ RDC	97	-3.7729*	7.1208*	-3.7797*	4.8360*	7.2507*
RB90T10 (BB90-TB10)	95	-3.0086*	4.6227*	-3.3381*	3.7971	5.5976*
Δ RB90T10	93	-4.5139*	10.189*	-4.4986*	6.7474*	10.120*
RB90C (BB90-CPI)	95	-1.9732	1.9849	-1.9528	1.5621	2.3052
Δ RB90C	93	-4.3734*	9.5769*	-4.5307*	6.8770*	10.302*
RT10C (TB10-CPI)	101	-1.4796	1.0999	-1.6161	1.3038	1.9502
Δ RT10C	98	-4.9972*	12.486*	-5.0212*	8.4723*	12.708*

* indicates significant at 10% level

RSC are the residuals from the regression $\log \text{SPI} = \alpha_0 + \alpha_1 \log \text{CPI}$

RSD are the residuals from the regression $\log \text{SPI} = \alpha_0 + \alpha_1 \log \text{SDiv}$

RDC are the residuals from the regression $\log \text{SDiv} = \alpha_0 + \alpha_1 \log \text{CPI}$

RB90T10 are the residuals from the regression $\text{BB90} = \alpha_0 + \alpha_1 \text{TB10}$

RB90C are the residuals from the regression $\text{BB90} = \alpha_0 + \alpha_1 \log \text{CPI}$

RT10C are the residuals from the regression $\text{TB10} = \alpha_0 + \alpha_1 \log \text{CPI}$

Table 8 Test Statistics for Co-integration - Phillips-Perron Tests

Variable	Various Periods					
	n	τ_1	ϕ_1	τ_2	ϕ_2	ϕ_3
<i>10% Critical Value</i>		(-2.57)	(3.78)	(-3.13)	(4.03)	(5.34)
Data from September 1948 to March 1995						
RSC	187	-2.3507	2.7666	-2.3937	1.9552	2.9318
Δ RSC	186	-12.372*	76.536*	-12.345*	51.800*	76.200*
Data from March 1958 to December 1994						
RSC	148	-2.3100	2.7567	-2.3011	1.8269	2.6546
Δ RSC	147	-11.440*	65.442*	-11.400*	43.330*	64.991*
R10C	148	-1.5317	1.2507	-1.5376	1.0544	1.5053
Δ R10C	147	-11.194*	62.657*	-11.207*	41.872*	62.807*
Data from September 1967 to December 1994						
RSC	110	-1.8159	1.6544	-1.8381	1.2285	1.8384
Δ RSC	109	-10.128*	51.297*	-10.122*	34.158*	51.277*
RSD	110	-2.2646	2.5663	-2.2649	1.7906	2.6852
Δ RSD	109	-10.423*	54.332*	-10.413*	36.145*	54.207*
RDC	110	-1.9553	1.9130	-1.9500	1.2701	1.9015
Δ RDC	109	-11.337*	64.269*	-11.283*	42.449*	63.674*
Data from September 1969 to December 1994						
RSC	102	-2.1995	2.4551	-2.2783	2.1503	3.1902
Δ RSC	101	-9.6874*	46.938*	-9.7916*	31.968*	47.938*
RSD	102	-2.7287	3.7547	-2.7584	2.8158	4.1945
Δ RSD	101	-9.8873*	48.908*	-9.9629*	33.111*	49.644*
RDC	102	-1.8420	1.6966	-1.8487	1.1802	1.7688
Δ RDC	101	-10.755*	57.858*	-10.703*	38.215*	57.318*
R9010	102	-4.2355*	8.9906*	-4.5131*	6.8119*	10.201*
Δ R9010	101	-11.167*	62.349*	-11.111*	41.160*	61.726*
R90C	102	-2.6822	3.6009	-2.7375	2.6313	3.9422
Δ R90C	101	-10.789*	58.196*	-10.782*	38.750*	58.119*
R10C	102	-1.5535	1.2136	-1.6759	1.3390	2.0035
Δ R10C	101	-9.2265*	42.570*	-9.2868*	28.755*	43.130*

* indicates significant at 10% level

RSC are the residuals from the regression $\log\text{SPI} = \alpha_0 + \alpha_1 \log\text{CPI}$
RSD are the residuals from the regression $\log\text{SPI} = \alpha_0 + \alpha_1 \log\text{SDiv}$
RDC are the residuals from the regression $\log\text{SDiv} = \alpha_0 + \alpha_1 \log\text{CPI}$
R9010 are the residuals from the regression $\text{BB90} = \alpha_0 + \alpha_1 \text{TB10}$
R90C are the residuals from the regression $\text{BB90} = \alpha_0 + \alpha_1 \log\text{CPI}$
R10C are the residuals from the regression $\text{TB10} = \alpha_0 + \alpha_1 \log\text{CPI}$

These tests consider only bi-variate series. There could be co-integrating relationships amongst more than just two of the series and these would not be detected using the procedure adopted so far. Johansen and Juselius (1990) developed maximum likelihood test statistics for determining the number and form of co-integrating equilibrium relationships amongst several series. These test statistics can also be applied to bi-variate series as well. The results of applying the Johansen procedure using the trace test statistic for the bi-variate series confirm that there are no co-integrating vectors for these series other than for the interest rates.

Based on the earlier discussion co-integrating relationships could be expected to exist between all of the interest rate series, possibly including the CPI variable to obtain an adjustment for inflation, and also amongst the SPI, the dividends on the SPI and interest rates. Table 9 sets out the results of the Johansen trace statistic for these variables.

The Johansen test statistics for the interest rate series indicate the existence of a co-integrating relationship between the interest rates for different maturities. They also indicate a co-integrating relationship when the CPI is included with the interest rates. This provides evidence that, although the interest rate series do not appear to be stationary, the levels of interest rates for different maturities “move together” maintaining an equilibrium relationship over time. Note that non-stationary interest rates means that they do not have a mean-reversion property. The rate of inflation, which is equal to the difference in the logarithm of the CPI, is stationary and interest rates appear to be co-integrated with the logarithm of the CPI.

Table 9
Johansen Trace test statistics
Quarterly data September 1969 to December 1994

Variables: BB90, TB2, TB5, TB10									
No. of lags in VAR= 4					No. of lags in VAR= 8				
Number of co-integrating relationships									
0	1	2	3	0	1	2	3		
59.281*	29.084	12.161	3.959	46.157	21.313	10.950	4.918		
Variables: BB90, TB2, TB5, TB10, LCPI									
No. of lags in VAR = 4					No. of lags in VAR = 8				
Number of co-integrating relationships									
0	1	2	3	4	0	1	2	3	4
74.419*	43.025	25.825	14.061	5.254	83.145	44.626	23.963	11.207	4.974
Variables: LSPI, LCPI, LSDIV, BB90									
No. of lags in VAR = 4					No. of lags in VAR = 8				
Number of co-integrating relationships									
0	1	2	3	0	1	2	3		
44.352	26.139	13.009	4.172	48.944	28.994	15.499	4.391		

* significant at 10% critical value

The results for the share price index, dividends, the CPI and the bank bill yield are not as positive and the hypothesis that these have no co-integrating relationship is not rejected.

The conclusions that can be drawn from this analysis of co-integrating, or “long-run” equilibrium, relationships in the Australian returns data is that, with the exception of the interest rate series, the analysis finds no strong evidence that such equilibrium relationships exist between the series analysed. This has implications for the structure of stochastic investment models since it will be important to incorporate an equilibrium structure for interest rates (and inflation) in the model but differences in the logarithms of the SPI, CPI and dividends can be used in the model as stationary variables without

the need to incorporate any specific equilibrium between these series. This also adds to the empirical evidence that there are no strong relationships between inflation and equity returns.

ERROR CORRECTION MODELS

If a number of series are co-integrated then they have common stochastic trends and move together through time following a long-run equilibrium. This long run equilibrium is disturbed by random shocks that are short term or temporary effects. The series eventually adjusts for these. This short term adjustment process is referred to as an error correction process. Engle and Granger (1987) proved that for any co-integrated series an error correction model exists. The error correction model captures both the short term departures from the long run equilibrium and the long run equilibrium in the model structure.

The series that appear to be co-integrated are the interest rate series and the CPI so that a multi-variate error-correction model can be estimated for these series. An investment model should incorporate such an error-correction structure. This will ensure an equilibrium exists in the model between the levels of the interest rates. Otherwise if differences in interest rates are modelled as stationary variables with no error-correction structure then the levels of interest rates will have stochastic trends and “shocks” to interest rates will be permanent. In such a model interest rates could “wander” off to arbitrarily high and low levels in a manner inconsistent with the historical data.

If the interest rate at time t is denoted by r_t and the logarithm of the CPI by F_t , both of which are non-stationary, then the previous analysis indicates that a regression of r_t on F_t results in stationary errors (under standard unit root tests). This means that $r_t - \alpha_1 F_t$, where α_1 is the regression coefficient, is a stationary process representing the long run (linear) relation between the two series. It is necessary to incorporate this equilibrium relationship between the series into any model. This is incorporated using an error correction model as $\Delta r_t = a + b(r_t - \alpha_1 F_t) + e_t$. Standard stationary time series modelling techniques will not detect the long-run relationship captured in the error correction term given by $b(r_t - \alpha_1 F_t)$. A similar model for the rate of inflation in error correction form will also need to be estimated. It is worth stressing that ignoring any cointegrating relationships ignores important information about long-run equilibrium relationships between the series.

Research suggests the need for at least two and preferably three interest rates to be included in an investment model (Sherris, 1995). Thus a short interest rate such as a 3 month yield (the 90 day Bank Bill yield), a medium term interest rate (either the 2 or 5 year Treasury yield) and a longer term yield (the 10 year Treasury yield) should be used in a model. This will allow the model to capture parallel shifts in the yield curve, changes in the slope of the yield curve and changes in the curvature of the yield curve. For liabilities that are sensitive to the yield curve such a model will be essential for developing realistic immunisation strategies.

It is somewhat of a concern that the equity and CPI index were not found to be co-integrated. This could be a result of a lack of power of the statistical tests used and the

relatively short period of data available for the quarterly time interval. The reason that this is of concern is that the level of the SPI will have a stochastic trend and there will no equilibrium relationships in the model that will constrain the relative value of the index. This is not consistent with studies that have detected long run mean-reversion in equity values. This aspect of the results needs further investigation.

STATE SPACE MODELS

Wilkie (1986, 1995) and Carter (1991) use transfer functions to develop their models. This approach allows the estimation of a cascade structure for a stochastic investment model where causality in one direction is assumed. The main driving variable in these models is the rate of inflation.

Transfer functions were examined in this research. The results are not reported in any detail here since it was found that after fitting these models there was evidence of feedback between the different variables. This means that transfer functions will not adequately capture the relationship between the different series since they impose a uni-directional causality that is not supported by the empirical data.

An alternative model is the Vector Autoregressive or VAR model. These models are used in practice for asset models and have the advantage that they allow for feedback. VAR models were fitted and it was found that too many lags were required and the models were difficult to interpret. Introducing a moving average term into these models is equivalent to an infinite number of auto-regressive terms so that a Vector Autoregressive Moving Average (VARMA) model should provide a more parsimonious model than a VAR model.

State space models provide a more succinct method of representing a stochastic investment model. They have an equivalent (VARMA) representation which has fewer lagged variables than the VAR models. Transfer function models are nested in the VARMA models. A state space model can be written as a state equation:

$$\mathbf{z}_{t+1} = \mathbf{F} \mathbf{z}_t + \mathbf{G} \mathbf{e}_{t+1}$$

and an observation equation:

$$\mathbf{y}_t = \mathbf{H} \mathbf{z}_t$$

where \mathbf{y}_t are actual observations at time t , \mathbf{z}_t is the state of the model at time t , \mathbf{F} , \mathbf{G} and \mathbf{H} are matrices of parameters and \mathbf{e}_t is a vector of mean zero, serially uncorrelated disturbances with covariance matrix Σ . The statistical package SAS was used to fit state space models using its state space procedure that selects the best model using the AIC model selection criteria. We assume that the state of the system is observed without error and that the series used in the model are the relevant state variables.

Returns and inflation

To examine the relationships between asset returns and inflation, state space models were fitted using each of the individual asset returns series and inflation. Models were fitted to the return on the equity index, the growth rate of dividends, the 10 year bond rate and the rate of inflation since actuaries often focus on rates of return and inflation when assessing premiums and liabilities. These models also allow a comparison with the transfer function models fitted by others.

EQUITY INDEX (SPI) AND INFLATION (CPI)

The following state space model for equity index and inflation rates was fitted as the “best” model using the quarterly data from September 1948 to March 1995:

$$z_t = \begin{bmatrix} y_t \\ x_t \\ y_{t+1|t} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0.905 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.432 & 0 \end{bmatrix}, e_{t+1} = \begin{bmatrix} n_{t+1} \\ e_{t+1} \end{bmatrix},$$

$$\text{with } \Sigma = \text{var} \begin{bmatrix} n_{t+1} \\ e_{t+1} \end{bmatrix} = \begin{bmatrix} 8.39 \times 10^{-5} & -7.18 \times 10^{-5} \\ -7.18 \times 10^{-5} & 8.84 \times 10^{-3} \end{bmatrix}.$$

where the variables are the differences in the logarithms of the series, or the continuously compounding returns, adjusted for the mean of the series as follows:

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} (1 - B)Y_t - 0.0153 \\ (1 - B)X_t - 0.0161 \end{bmatrix}$$

with $Y_t = \log(\text{CPI})_t$, $X_t = \log(\text{SPI})_t$, and $y_{t+1|t} = y_{t+1} - n_{t+1}$ is the “predicted” value for time $t+1$ conditional on information at time t . Note that the vectors e_{t+1} are assumed to be a sequence of independent normally distributed random vectors with mean $\mathbf{0}$ and covariance matrix Σ .

From the covariance matrix the standard deviation of the residuals after fitting the model are 0.0092 for the quarterly continuously compounding rate of inflation and 0.094 for the quarterly continuously compounding rate of growth of the SPI with a correlation between the residuals of -0.0834.

The parameter estimates were:

Parameter	Estimate	Std. Error	t value
F(3,3)	0.905	0.041	22.326
G(3,1)	0.432	0.066	6.563

For simulation studies such as in asset-liability modelling it is important to recognise that this model does not capture parameter or model uncertainty and has been calibrated to historical data over the time period September 1948 to March 1995. The variances in asset returns and rates of inflation are assumed to be homoscedastic in this model whereas the analysis carried out later in this paper provides evidence of heteroscedasticity.

The model can be written as an equivalent VARMA model as follows:

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} 0.905 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix} + \begin{bmatrix} -0.473 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_t \\ n_t \end{bmatrix}$$

Note that using this modelling approach the best model for $\log(\text{SPI})_t$ is a random walk with drift and an ARMA model is required for $\log(\text{CPI})_t$. This model is very different to that suggested by Wilkie.

EQUITY DIVIDENDS AND INFLATION (CPI)

The following state space model for equity dividends and inflation was fitted as the “best” model using the quarterly data from September 1967 to December 1994:

$$z_t = \begin{bmatrix} x_t \\ y_t \\ y_{t-1|t} \\ y_{t-2|t} \\ y_{t-3|t} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0.030 & 0.340 & -0.236 & 0.481 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0.258 \\ -0.021 & 0.382 \\ 0 & 0.440 \end{bmatrix},$$

$$e_{t+1} = \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix}, \Sigma = \text{var} \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix} = \begin{bmatrix} 4.26 \times 10^{-3} & 4.93 \times 10^{-5} \\ 4.93 \times 10^{-5} & 6.22 \times 10^{-5} \end{bmatrix}$$

with

$$x_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} (1 - B)X_t - 0.0208 \\ (1 - B)Y_t - 0.0178 \end{bmatrix}$$

and $X_t = \log(\text{DIVS})_t$, $Y_t = \log(\text{CPI})_t$.

The parameter estimates were:

Parameter	Estimate	Std. Error	t-value
F(5,1)	0.030	0.010	3.038
F(5,2)	0.340	0.121	2.800
F(5,3)	-0.236	0.082	-2.864
F(5,4)	0.481	0.150	3.208
G(3,2)	0.258	0.091	2.837
G(4,1)	-0.021	0.010	-2.057
G(4,2)	0.382	0.089	4.290
G(5,2)	0.440	0.089	4.971

An equivalent VARMA model can be readily developed from the above state space model.

10-YEAR TREASURY BOND RATES (TB10) AND CPI

Using the quarterly series from March 1958 to December 1994 the state space model was:

$$\mathbf{z}_t = \begin{bmatrix} x_t \\ y_t \\ y_{t+1|t} \\ y_{t+2|t} \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.675 & -0.308 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0.126 \\ 0.382 & 0.182 \end{bmatrix},$$

$$\mathbf{e}_{t+1} = \begin{bmatrix} \mathbf{e}_{t+1} \\ \mathbf{n}_{t+1} \end{bmatrix}, \Sigma = \text{var} \begin{bmatrix} \mathbf{e}_{t+1} \\ \mathbf{n}_{t+1} \end{bmatrix} = \begin{bmatrix} 7.15 \times 10^{-5} & 5.09 \times 10^{-6} \\ 5.09 \times 10^{-6} & 7.79 \times 10^{-5} \end{bmatrix}$$

with

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} (1-B)\mathbf{X}_t - 0.0217 \\ (1-B)\mathbf{Y}_t - 0.0147 \end{bmatrix}$$

and $\mathbf{X}_t = (\text{TB10})_t$, $\mathbf{Y}_t = \log(\text{CPI})_t$.

The parameter estimates were:

Parameter	Estimate	Std. Error	t-value
F(4,3)	0.675	0.042	16.146
F(4,4)	-0.308	0.057	-5.448
G(3,2)	0.126	0.056	2.245
G(4,1)	0.382	0.073	5.232
G(4,2)	0.182	0.052	3.516

The equivalent VARMA model fit is given by the following equation.

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -0.308 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.675 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.434 \end{bmatrix} \begin{bmatrix} e_t \\ n_t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0.382 & -0.454 \end{bmatrix} \begin{bmatrix} e_{t-1} \\ n_{t-1} \end{bmatrix}$$

These bi-variate models capture the relationship between these variables and assist in understanding the nature of these series and their interrelationships. However it can be seen that the models for the inflation series differ in each of the above models. This suggests that a model incorporating all of the series could provide more information about the best model for inflation since it will incorporate the interrelationships between the series. Such a model was fitted and the resulting model was complex and difficult to interpret so it has not been set out in this paper.

Other models

Models were also fitted to the SPI and the dividend series as well as the SPI and the 10 year bond yield to examine the relationships between these series.

EQUITY INDEX (SPI) AND EQUITY DIVIDENDS

The best state-space model using the quarterly series from September 1967 to December 1994 was found to be:

$$\mathbf{z}_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{e}_{t+1} = \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix},$$

$$\Sigma = \text{var} \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix} = \begin{bmatrix} 4.26 \times 10^{-3} & 5.67 \times 10^{-4} \\ 5.67 \times 10^{-4} & 1.28 \times 10^{-2} \end{bmatrix}.$$

where

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} (1-B)\mathbf{X}_t - 0.0208 \\ (1-B)\mathbf{Y}_t - 0.0177 \end{bmatrix}$$

and $\mathbf{X}_t = \log(\text{DIVS})_t$, $\mathbf{Y}_t = \log(\text{SPI})_t$.

The model indicates that these series are random walks with drifts and correlated errors. Note that the model is for an equity dividend index and not for a dividend yield. The dividend yield is given by the difference in the dividend index divided by the value of the share price index. Models that assume that the dividend yield is stationary and mean-reverting will not necessarily be consistent with this fitted model.

EQUITY INDEX (SPI) AND 10-YEAR TREASURY BOND RATES (TB10)

The best state-space model for the quarterly series from March 1958 to December 1994 was:

$$\mathbf{z}_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix}, \mathbf{F} = \begin{bmatrix} 0.982 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{e}_{t+1} = \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix},$$

$$\Sigma = \text{var} \begin{bmatrix} e_{t+1} \\ n_{t+1} \end{bmatrix} = \begin{bmatrix} 2.71 \times 10^{-6} & -3.67 \times 10^{-5} \\ -3.67 \times 10^{-5} & 1.05 \times 10^{-2} \end{bmatrix}.$$

where

$$\mathbf{x}_t = \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} (1-B)\mathbf{X}_t - 0.0217 \\ (1-B)\mathbf{Y}_t - 0.0177 \end{bmatrix}$$

and $\mathbf{X}_t = (\text{TB10})_t$, $\mathbf{Y}_t = \log(\text{SPI})_t$.

The parameters were:

Parameter	Estimate	Std. Error	T-Value
F(1,1)	0.982	0.016	63.029

Note that the fitted 10 year bond yield model is close to a random walk with drift and the equity index is a random walk with drift.

Summary

These models have all been bi-variate models. Ignoring heteroscedasticity, they provide support for the random walk model for the equity index. They also provide support for modelling the difference in interest rates as a stationary series and not the level of the series. These features are not found in many of the stochastic models that have details available in the public domain. For asset liability studies it will be important to have a model to project the equity returns, dividends, inflation and interest rates as a multi-variate system. As noted earlier such a model appears to be rather complex and difficult to interpret. This is an important issue that requires further investigation. The analysis of the variances of the series indicates the need for models that incorporate heteroscedasticity. In the state space approach the parameters in F and G can be allowed to be time varying. The models can incorporate parameter uncertainty. The Kalman filter maximum likelihood approach can be used with state space models to estimate these model parameters.

It should also be noted that tests of the assumptions of the model concerning the residuals have not been performed for these models. The results are based on the assumptions of i.i.d. and normally distributed errors. The analysis of ARCH/GARCH models reported later in this paper cast doubt on the validity of these assumptions.

STRUCTURAL BREAKS AND HETEROSCEDASTICITY

STRUCTURAL BREAKS

Chow (1960) suggests a test for structural change in parameters. The test divides the time period of the data into two sub-periods and fits a model to each of the sub-periods. The test determines whether using different parameters for the two sub-periods is significantly different from using constant parameters for the whole time period. If there is a significant difference then this suggests evidence of a structural change in the series.

The Chow tests were applied to models that fitted a constant mean and variance to the continuously compounding quarterly return on the SPI (March 1939 to March 1995), the continuously compounding quarterly return on the CPI (September 1948 to March 1995), the continuously compounding quarterly growth in the Dividend index (September 1967 to December 1994), the 90 day bank bill yield and changes in the 90 day bank bill yield (September 1969 to December 1994), the 2 year Treasury bond yield and changes in the 2 year Treasury bond yield (September 1964 to December 1994), the 5 year Treasury bond yield and changes in the 5 year Treasury bond yield (September 1969 to December 1994), and finally the 10 year Treasury bond yield and changes in the 10 year Treasury bond yield (March 1958 to December 1994).

The results of applying this test to the data were as follows:

SPI

the mean quarterly (continuously compounding) return on the SPI did not have a statistically significant structural break during the data period.

CPI

the mean quarterly rate of inflation appears to have a structural break at December 1950; December 1956; March 1963; September 1975; June 1986; and March 1993 since the mean rate in the period prior to these dates is significantly different to the mean rate after these dates.

Dividends

the mean quarterly growth rate of the Dividend index in the period prior to the following dates is significantly different to the mean rate after these dates - June 1988 and September 1990. The June 1988 date is consistent with the introduction of dividend imputation.

90 day bank bill yields and changes in yields

90 day bank bill yields appear to have different means prior to and after both September 1984 and June 1989. Changes in 90 day bank bill yields appear to be constant except for when September 1974 is included for the first time.

2 year Treasury bond yields and changes in yields

2 year Treasury bond yields appear to have different means prior to and after March 1987 and December 1990. Changes in 2 year Treasury bond yields appear to be constant.

5 year Treasury bond yields and changes in yields

5 year Treasury bond yields appear to have different means prior to and after September 1987 and March 1990. Changes in 5 year Treasury bond yields appear to have different means prior to and after December 1971, December 1973, December 1981 and December 1983.

10 year Treasury bond yields and changes in yields

10 year Treasury bond yields appear to have different means prior to and after March 1989. Changes in 10 year Treasury bond yields appear to have a constant mean.

These results suggest that rates of inflation have been changing during the time period of the data analysis. They also suggest that the introduction of dividend imputation had an impact on dividends in the 1980's and this needs to be allowed for in any investment model that estimates parameters based on historical data. The data also suggest that if a model is based on the level of interest rates then this model will have to allow for structural changes particularly in the 1980's. These changes are consistent with the deregulation that occurred in Australian financial markets in the 1980's. The need to allow for structural changes in interest rates does not appear as important when differences in interest rates are modelled.

MODELLING VARIANCES - ARCH AND GARCH

The plots in Appendix B indicated the possibility of heteroscedasticity. There are various forms of heteroscedasticity. Models that have found favour and have received empirical support are the ARCH models introduced by Engle (1982) and the GARCH models introduced by Bollerslev (1986). The ARCH model allows the variance, h_t , to be a function of past "shocks", ϵ_t , using the equation:

$$h_t = \alpha_0 + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2$$

where q is the order of the ARCH process. The model does not allow the variance to depend on the level of the series.

The GARCH model allows the current variance to depend on previous values of the variance as well as past shocks using the equation:

$$h_t = \alpha_0 + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2 + \sum_{j=1}^p \phi_j h_{t-j}$$

which is referred to as a GARCH(p,q) process.

The models can also allow the variance to be a function of an another variable or the level of the variable being modelled. This type of model is the basis of the ERCH model developed by Harris (1995) where the real rate of GDP growth has the role of an exogenous variable. In this case the variance can be modelled using the ARCH or GARCH process along with the exogenous variable. An example of such an exogenous variable model for the variance based on the ARCH process would be:

$$h_t = \alpha g_t + \alpha_0 + \sum_{j=1}^q \alpha_j \varepsilon_{t-j}^2$$

Various ARCH and GARCH models were fitted to the uni-variate series to determine the need to model the variance. The models fitted assumed a constant mean. After fitting the ARCH/GARCH models the residuals were tested for normality using the Jacques-Bera Lagrange Multiplier test. The results are set out and discussed below. The tables show the parameter estimates along with t-statistics. They also show the log-likelihood of the model and the Jacques-Bera Lagrange Multiplier (JB LM) test statistic for the model residuals.

Rates (continuously compounded) of inflation

Variable: September 1948 to March 1995

ΔLogCPI

Model	ARCH(1) Parameters		Mean t-statistic	0.0136 17.88	Loglikelihood	Residuals
Variance equation	α_0	α_1			559.323	JBLM (d.f.=2)
t-statistic	0.0000902	0.48039				26.8721
	5.979	3.167				

Model	GARCH(1,1) Parameters		Mean t-statistic	0.0104 17.74	Loglikelihood	Residuals
Variance equation	α_0	α_1	α_2	ϕ_1	570.299	JBLM (d.f.=2)
t-statistic	0.0000106	0.28700		0.65127		3.8338
	2.063	3.211		7.948		

This shows that a GARCH(1,1) process provides a reasonable fit to the differences in the logarithm of the CPI confirming that there is heteroscedasticity in the rate of inflation over this period. The model residuals do not exhibit significant non-normality. Various authors, including Wilkie (1995), have suggested the use of GARCH models in stochastic investment models for actuarial applications. In fact the earliest application of these models was to the modelling of inflation - Engle (1982).

Rates (continuously compounded) of return on the SPI

Variable: March 1939 to March 1995

ΔLogSPI

Model	ARCH(1) Parameters		Mean t-statistic		Loglikelihood	Residuals
			0.0146 2.914			
Variance equation	α_0	α_1			244.639	JBLM (d.f.=2) 53.8019
t-statistic	0.0049172 7.736	0.36074 2.821				

Model	GARCH(1,1) Parameters		Mean t-statistic		Loglikelihood	Residuals
			0.0103 2.486			
Variance equation	α_0	α_1	ϕ_1		253.013	JBLM (d.f.=2) 44.9515
t-statistic	0.0005377 2.196	0.34470 3.637	0.64138 8.628			

Model	GARCH(1,2) Parameters		Mean t-statistic		Loglikelihood	Residuals
			0.0103 2.512			
Variance equation	α_0	α_1	ϕ_1	ϕ_2	253.120	JBLM (d.f.=2) 45.2157
t-statistic	0.0004826 2.045	0.31818 2.751	0.78166 2.324	-0.10819 -0.4342		

Despite the GARCH(1,1) model giving an improved log-likelihood the residuals of the model are not normally distributed after allowing for this form of heterogeneity. The residuals remain negatively skewed and fat-tailed. This model requires further investigation and the use of an exogenous variable such as real GDP growth is a possible alternative. There is however evidence of heteroscedasticity in SPI returns and this must be included in a stochastic investment model since the variance will have a significant impact on asset allocation strategy and on solvency assessments.

Further investigation is required to determine the form of heteroscedasticity for the share price index series. The variance could be allowed to depend on the level of the series, the absolute value of the shocks or on an asymmetric function of the shocks. It would also be possible to assume that the residuals have a conditional t-distribution rather than a conditional normal distribution. These issues require further investigation.

One concern with adding the complexity of ARCH and GARCH to stochastic investment models is that the number of parameters to be estimated increases dramatically if all of the series are allowed to be heteroscedastic. The other point to note is that the conditional variance is deterministic in these models and a stochastic variance model could be more appropriate.

Rates (continuously compounded) of growth in dividends on the SPI

Variable: September 1967 to December 1994

ΔLogSD

Model	ARCH(1) Parameters		Mean t-statistic	0.02288 3.98	Loglikelihood	Residuals
Variance equation	α_0	α_1			146.873	JBLM (d.f.=2)
	0.0031814	0.23854				1.0665
t-statistic	5.577	1.663				

Model	GARCH(1,1) Parameters		Mean t-statistic	0.02347 4.39	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1	147.374	JBLM (d.f.=2)
	0.0041067	0.28427		-0.24097		0.8086
t-statistic	3.426	1.935		-1.363		

Model	GARCH(1,2) Parameters		Mean t-statistic	0.0328 10440	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1 ϕ_2	153.076	JBLM (d.f.=2)
	0.0053923	0.21581		-0.16664 -0.37923		0.2347
t-statistic	3.848	2.288		-3.528 -1.852		

The dividend growth rate shows evidence of heteroscedasticity and a GARCH(1,2) model appears to fit the series variance adequately. The assumption that the model residuals are normally distributed can not be rejected using the JBLM statistic.

90 day bank bill yields

Variable: September 1969 to December 1994

BB90

Model	ARCH(1) Parameters		Mean t-statistic	11.191 47.16	Loglikelihood	Residuals
Variance equation	α_0	α_1			-274.547	JBLM (d.f.=2)
	3.7416	0.83263				9.624
t-statistic	3.254	4.247				

Model	GARCH(1,1) Parameters		Mean t-statistic	10.797 44.37	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1	-273.986	JBLM (d.f.=2)
	5.4778	0.81864		-0.12508		7.5378
t-statistic	3.351	4.271		-2.929		

These results provide evidence that bank bill yields are heteroscedastic. The residuals after fitting the model are skewed but the kurtosis is not significant.

The earlier analysis in this paper provided evidence that changes in interest rates are stationary rather than their levels. Since this suggests that the differences in interest rates are the variables that should be included in a stochastic investment model it is preferable to examine heteroscedastic models for the differences of the interest rate series.

Differences of 90 day bank bill yields

Variable: September 1969 to December 1994

$\Delta BB90$

Model	ARCH(1) Parameters		Mean			
			t-statistic	0.145		
				0.791	Loglikelihood	Residuals
Variance equation	α_0	α_1			-210.865	JBLM (d.f.=2)
	2.9923	0.2785				110.4545
t-statistic	5.385	1.720				

Model	GARCH(1,1) Parameters		Mean			
			t-statistic	-0.076		
				-0.460	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1	-208.917	JBLM (d.f.=2)
	0.65957	0.30228		0.58907		156.0198
t-statistic	1.736	1.980		3.779		

There is evidence of heteroscedasticity for the differences in the 90 day bank bill yields. However the residuals after fitting the model are far from normally distributed. This is because of both positive skewness and positive kurtosis.

Differences of 2 year Treasury bond yields

Variable: September 1964 to December 1994

$\Delta 2yrTB$

Model	ARCH(1) Parameters		Mean			
			t-statistic	0.05706		
				0.695	Loglikelihood	Residuals
Variance equation	α_0	α_1			-160.691	JBLM (d.f.=2)
	0.76162	0.10080				11.6134
t-statistic	6.487	0.9900				

Model	GARCH(1,1) Parameters		Mean			
			t-statistic	0.0547		
				0.690	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1	-159.085	JBLM (d.f.=2)
	0.27822	0.05076		0.58226		8.5677
t-statistic	0.6949	0.7131		1.034		

The variances for the 2 year bond rate do not appear to be heteroscedastic or at least the ARCH/GARCH models do not appear to capture any heteroscedasticity. None of the parameters are significant. The residuals after fitting the model have significant kurtosis and the assumption that they are normally distributed is rejected.

ARCH/GARCH models do not appear to be satisfactory for modelling the differences in the short term interest rates.

Differences of 5 year Treasury bond yields

Variable: June 1969 to December 1994

$\Delta 5yrTB$

Model	ARCH(1) Parameters		Mean t-statistic	0.0472 0.596	Loglikelihood	Residuals
Variance equation	α_0	α_1			-121.958	JBLM (d.f.=2) 6.7868
t-statistic	0.58092	0.13269				
	5.744	1.085				

Model	GARCH(1,1) Parameters		Mean t-statistic	0.0367 0.5402	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1	-118.184	JBLM (d.f.=2) 2.7205
t-statistic	0.11730	0.30900		0.56834		
	1.516	2.095		3.181		

Model	GARCH(1,2) Parameters		Mean t-statistic	0.0579 0.897	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1	-116.309	JBLM (d.f.=2) 1.4678
t-statistic	0.11057	0.19754		1.1606	-0.49587	
	3.062	2.107		6.802	-4.300	

Heteroscedasticity in the differences in the 5 year Treasury bond yields appears to be well modelled using a GARCH(1,2) process.

Differences of 10 year Treasury bond yields

Variable: March 1958 to December 1994

$\Delta 10yrTB$

Model	ARCH(1) Parameters		Mean t-statistic	0.01564 0.03367	Loglikelihood	Residuals
Variance equation	α_0	α_1				JBLM (d.f.=2) 101.6980
t-statistic	0.28507	0.26768			-131.279	
	7.100	2.063				

Model	GARCH(1,1) Parameters		Mean t-statistic	0.03203 1.232	Loglikelihood	Residuals
Variance equation	α_0	α_1		ϕ_1		JBLM (d.f.=2) 33.0265
t-statistic	0.0053803	0.32538		0.75591	-102.652	
	1.603	3.572		14.17		

There were problems finding the maximum likelihood when fitting the GARCH models to the differences of the 10 year Treasury bond yields. The GARCH(1,1) parameters are not reliable and higher order models suffered similar problems. There is evidence of heteroscedasticity but even after fitting the GARCH models the residuals still exhibit high kurtosis.

Although the differences of the interest rates appear to exhibit heteroscedasticity the results of fitting the ARCH/GARCH process to the differences in the interest rate series are not very encouraging. Even after fitting ARCH/GARCH models the residuals still remain skewed and "fat-tailed". This is an important issue that requires further

investigation. The results do however demonstrate the need to allow for heteroscedasticity.

CONCLUSIONS

This paper has set out the results of research into the structural features of a stochastic investment model for actuarial applications using Australian data. This analysis is fundamental to the construction of a soundly based model. It has analysed Australian investment data using a quarterly time period. It has formally tested for stationarity of all the series and tested to see which series are co-integrated and therefore maintain a long run equilibrium relationship. It has also examined the appropriateness of transfer function models that assume one way causality between series using Australian investment data. Finally the nature of heteroscedasticity in the series has been explored.

The results of the research suggest that the stationary variables in the Australian investment data are the rate of (continuously compounding) growth in the Share Price Index (SPI), the rate of (continuously compounding) growth in the Consumer Price Index (SPI), the rate of (continuously compounding) growth in a Dividend index representing the dividends on the SPI, and differences in the interest rate series. The statistical analysis did not provide evidence that the interest rate levels were stationary.

The cointegration tests indicated a long-run equilibrium relationship exists between the interest rates and the logarithm of the inflation index (CPI) whereas there was no evidence to support such a relationship between equity values, as measured by the SPI and a Dividend index, and the level of the inflation index (CPI).

Transfer functions models were fitted to the various series and inflation but they were not found to capture the relationships between these series. Thus it was necessary to use state space models to allow for feedback between the different series. State space models for the different series and inflation were fitted to provide a comparison with transfer function models fitted by other researchers.

Finally, heteroscedasticity for the different stationary series was examined using ARCH/GARCH models. The investment series display heteroscedasticity but the standard ARCH/GARCH models did not always capture the correct form of the heteroscedasticity.

This research demonstrates that further investigation is required in order to understand the appropriate structure for a stochastic investment model. It does highlight some important lessons for those wishing to construct and use stochastic investment models. It indicates the type of variables that should be used in these models and the nature of the relationships that should be built into them. It also highlights the need to include time varying parameters in these models particularly for the variances. These matters are fundamental to the construction of stochastic investment models.

APPENDIX A

AUSTRALIAN INVESTMENT DATA - SUMMARY STATISTICS

Different series were available for different time periods. In the analysis in the paper the longest time period available for the series have been used where possible. Statistics for these series over different time periods are summarised in this Appendix.

(1) Consumer Price Index - all groups (CPI) and All Ordinaries Share Price Index (SPI)

Quarterly data was available for these series for the period September 1948 to March 1995. Table A1 sets out summary statistics for these indices, the logarithm of the index and the change in the logarithm.

Table A1
Summary statistics of CPI and SPI, logarithm of CPI and SPI and first differences of logarithms of CPI and SPI.

Statistics	N	Min	Max	Mean	St Dev	Skewness	Kurtosis
CPI	187	6.70	114.70	39.1064	34.0734	1.0008	-0.4688
log(CPI)	187	1.9021	4.7423	3.2964	0.8573	0.3615	-1.2894
$\Delta\log(\text{CPI})$	186	-0.0087	0.0704	0.0153	0.0132	1.1935	2.0125
SPI	187	84.60	2238.70	561.5283	567.3915	1.3879	0.6055
log(SPI)	187	4.4379	7.7137	5.8806	0.9354	0.3983	-0.9342
$\Delta\log(\text{SPI})$	186	-0.5728	0.2613	0.0161	0.0940	-1.6993	8.7947

Note the negative skewness and high kurtosis for the continuously compounding return on the SPI - given by the variable $\Delta\log(\text{SPI})$.

(2) Consumer Price Index - all groups (CPI) and Share Dividends (DIVS)

Quarterly data for the period September 1967 to December 1994 was available for the CPI and dividend yields. The Dividend yield series is the Melbourne weighted (M.W.) series from September 1967 to December 1982. This was merged with the Australian dividend yield (A.Y.) series that is available from September 1983 to March 1995 by taking $2/3\text{M.W.} + 1/3\text{A.Y.}$ for March 1983 and $1/3\text{M.W.} + 2/3\text{A.Y.}$ for June 1983. The share dividend series (DIVS) is derived as the product of the SPI and the dividend yield for each quarter. It represents an annualised amount of dividends paid over the prior 12 months. Table A2 sets out summary statistics for these series, the logarithms of the series and the differences in the logarithms of the series.

Table A2
Summary statistics of CPI and DIVS, logarithm of CPI and DIVS and first differences of logarithms of CPI and DIVS.

Statistics	N	Min	Max	Mean	St Dev	Skewness	Kurtosis
CPI	110	16.20	112.80	56.9036	33.2526	0.3283	-1.3353
log(CPI)	110	2.7850	4.7256	3.8406	0.6678	-0.2351	-1.3759
$\Delta\log(\text{CPI})$	109	-0.0046	0.0566	0.0178	0.0116	0.7168	1.0666
DIVS	110	747.40	9398.25	3526.76	2603.95	0.8094	-0.6369
log(DIVS)	110	6.6166	9.1483	7.8808	0.7802	0.0526	-1.3005
$\Delta\log(\text{DIVS})$	109	-0.1987	0.2132	0.0208	0.0653	-0.3005	1.2734

(3) Interest Rates

Quarterly interest rate data is available for 90-day Bank Bills (BB90) from September 1969 to December 1994. The summary statistics of BB90, $\log(\text{BB90})$ and $\Delta\log(\text{BB90})_t = \log(\text{BB90})_t - \log(\text{BB90})_{t-1}$ are given in Table A3. Data for 5-year Treasury Bonds (TB5) is available for the period June 1969 to December 1994. The summary statistics of TB5 and $\Delta(\text{TB5})_t = \text{TB5}_t - \text{TB5}_{t-1}$ are also given in Table A3. Data for 10-year Treasury Bonds (TB10) is available for the period March 1958 to December 1994. The summary statistics of TB10 and $\Delta(\text{TB10})_t = \text{TB10}_t - \text{TB10}_{t-1}$ are also given in Table A3.

Table A3
Summary statistics of Interest rates and first differences of Interest rates.

Statistics	N	Min	Max	Mean	St Dev	Skewness	Kurtosis
BB90	102	4.45	19.95	10.9093	4.1029	0.3310	-0.8313
log(BB90)	102	1.4929	2.9932	2.3148	0.3981	-0.2784	-0.8812
$\Delta\log(\text{BB90})$	101	-0.4002	0.6213	0.0034	0.1712	0.6058	1.4652
TB5	103	0.0128	0.0394	0.0253	0.0072	-0.0909	-1.0977
$\Delta(\text{TB5})$	102	-0.0060	0.0050	0.0001	0.0019	-0.1966	1.2405
TB10	148	0.0106	0.0394	0.0216	0.0085	0.2654	-1.3368
$\Delta(\text{TB10})$	147	-0.0056	0.0048	0.00008	0.0014	-0.1179	3.7768

(4) All series

Quarterly data for all series was available from September 1969 to December 1994. Table A4 provides summary statistics for this time period. The data are CPI - Consumer Price Index, LogCPI - logarithm of (CPI), AWE - Average Weekly Earnings, LogAWE - logarithm of (AWE), SPI - Share Price Index, LogSPI - logarithm of (SPI), SDyields - Share dividend yields, SDiv - Share dividends series, BB90 - 90-day bank bills yields, TB2 - 2-year treasury bond yields, TB5 - 5-year treasury bond yields, TB10 - 10-year treasury bond yields.

Table A4
Summary statistics of all series
Quarterly Data from September 1969 to December 1994

Variable	Mean	St.Dev.	Max	Min	Median	Mode	Skewness	Kurtosis
CPI	60.074	32.462	112.80	17.000	55.300	107.60	0.2375	-1.3631
LogCPI	3.9220	0.62386	4.7256	2.8332	4.0128	4.6784	-0.3408	-1.2288
AWE	776.57	398.30	1364.3	176.90	796.33	1000.8	-0.0145	-1.3769
LogAWE	6.4806	0.64460	7.2184	5.1756	6.6800	6.9086	-0.6544	-0.8507
SPI	865.01	595.05	2238.7	194.30	603.40	2238.7	0.6797	-1.0008
LogSPI	6.5177	0.71137	7.7137	5.2694	6.4026	7.7137	0.1667	-1.4523
SD yields	4.4506	1.1496	7.7300	2.0700	4.5000	5.8500	0.2237	-0.1128
SDiv	3741.5	2584.0	9398.3	861.74	2877.4	9398.3	0.7365	-0.7603
BB90	10.909	4.1029	19.950	4.4500	10.350	15.450	0.3310	-0.8313
TB2	10.185	3.2623	16.400	4.6000	9.9400	15.150	0.0137	-1.1443
TB5	10.465	2.9845	16.400	5.2000	10.030	13.850	-0.0775	-1.0843
TB10	10.648	2.8299	16.400	5.7500	10.180	9.5000	-0.0997	-1.0091

Table A5
Jarque-Bera Asymptotic LM Normality Test
September 1969 - December 1994
Chi-squared 2DF 5% Critical Value 5.99

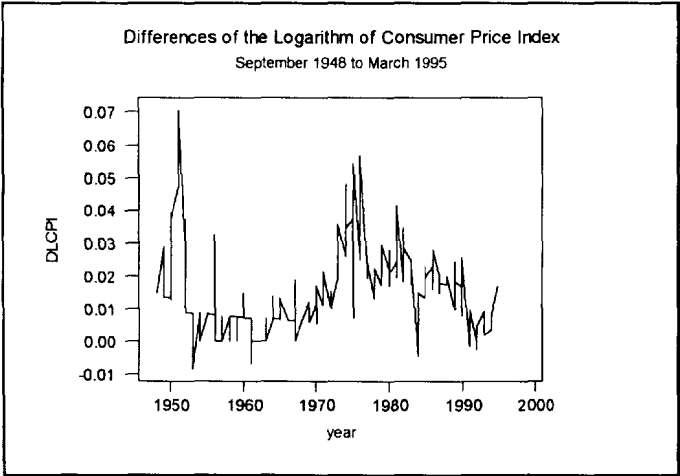
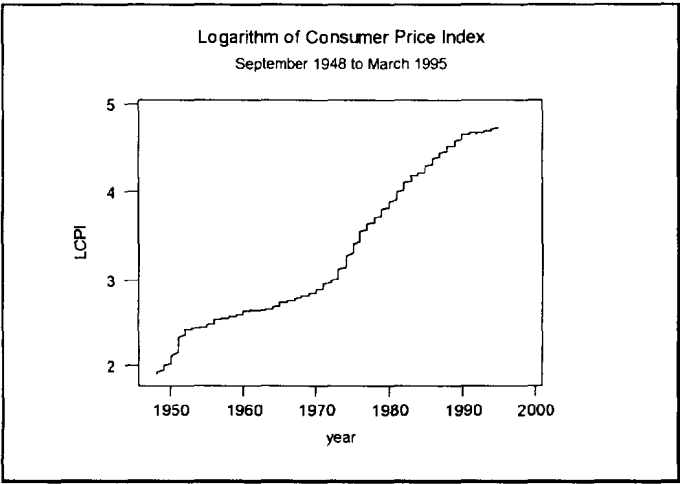
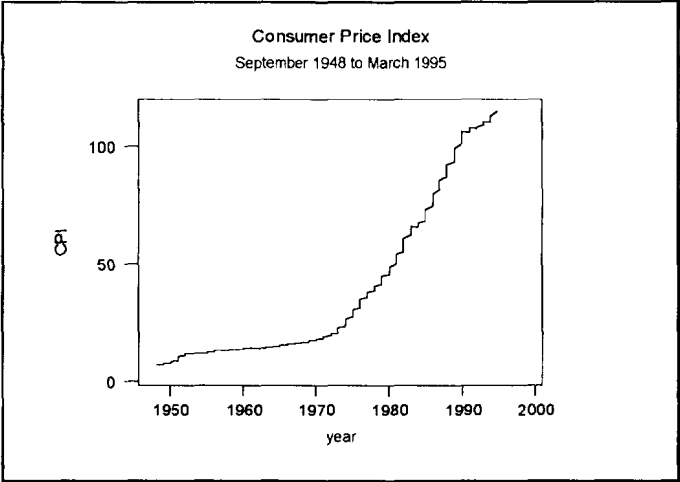
Variable	Chi-Square Statistic
CPI	8.74*
LogCPI	8.32*
AWE	7.96*
LogAWE	10.27*
SPI	11.97*
LogSPI	9.28*
SD yields	9.47*
SDiv	11.55*
BB90	4.87
TB2	5.60
TB5	5.15
TB10	4.57

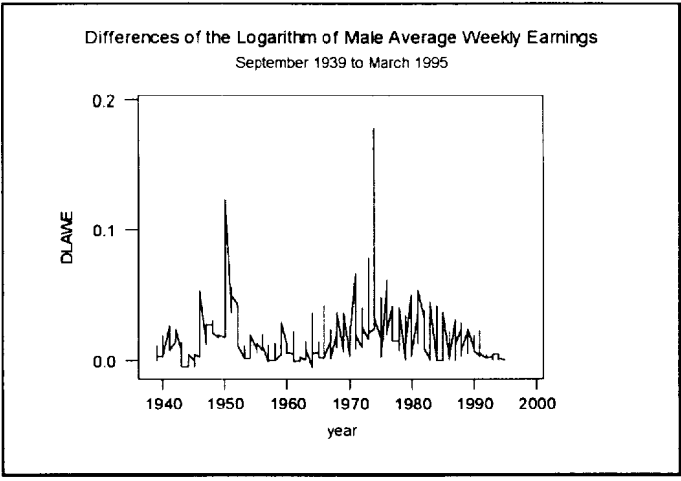
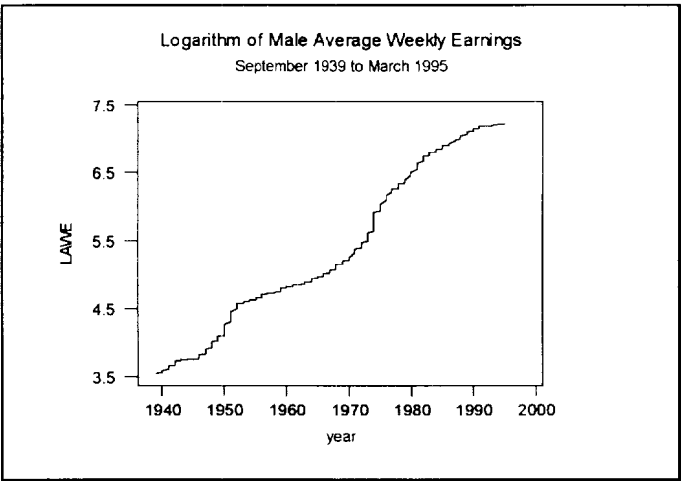
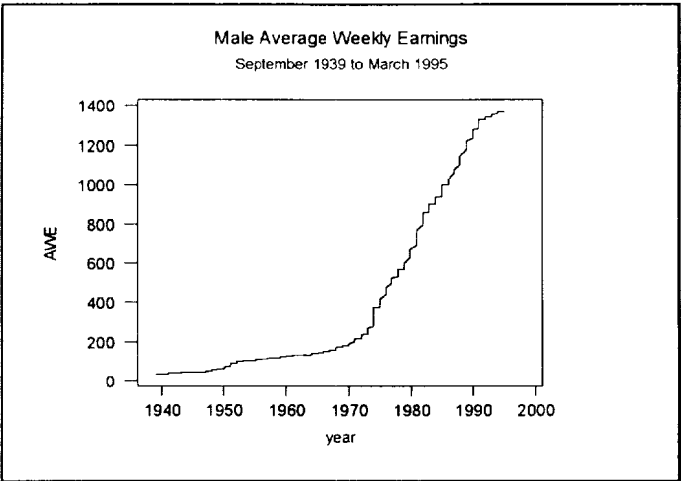
*significant at 5% level

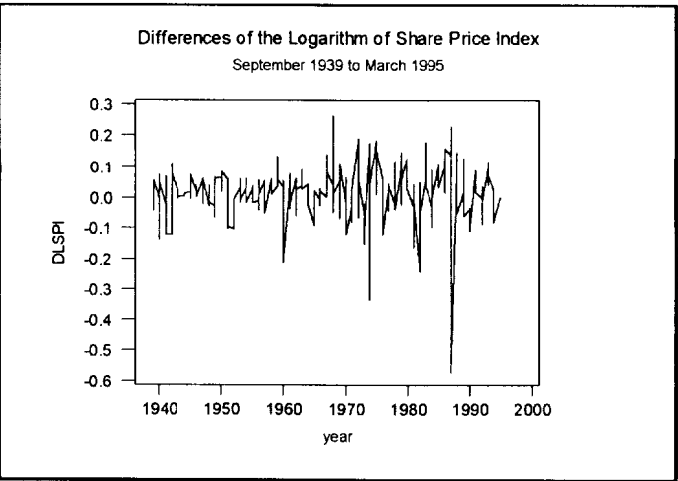
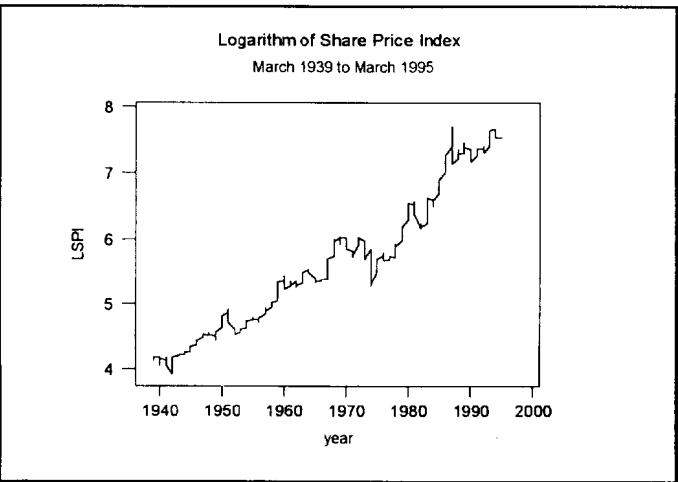
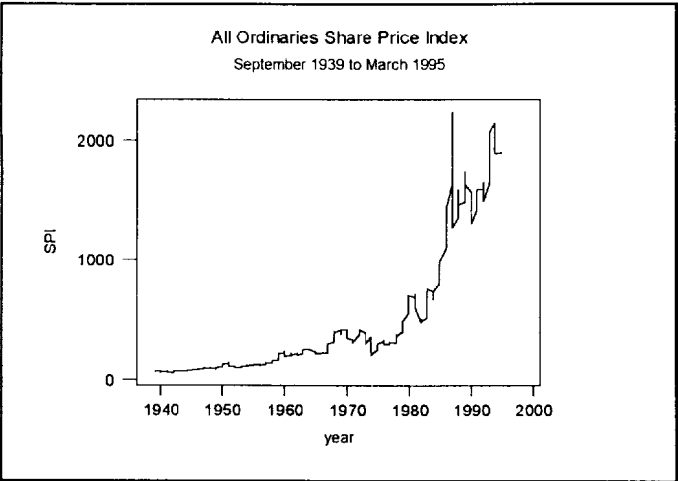
APPENDIX B

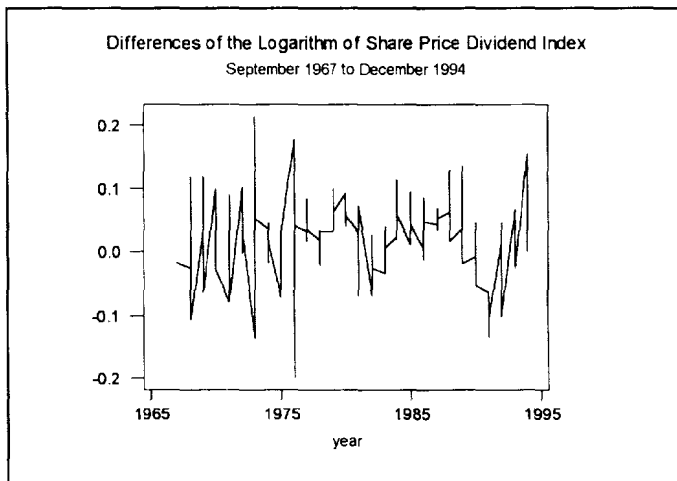
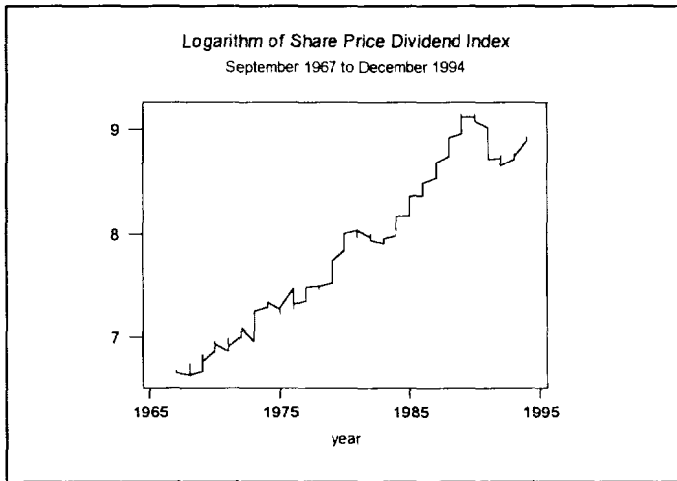
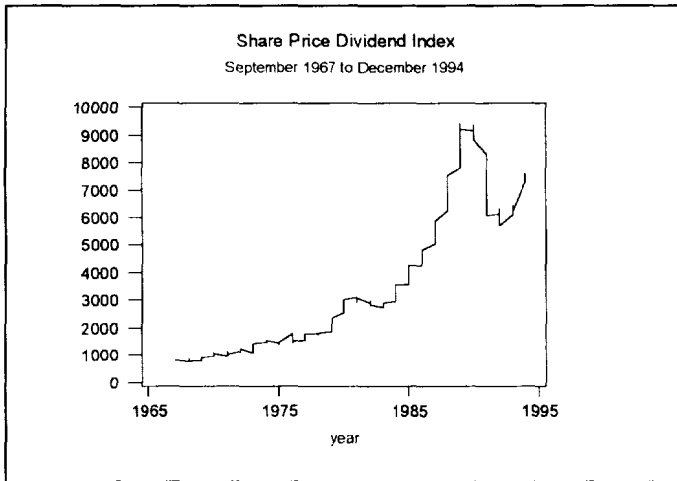
TIME SERIES PLOTS

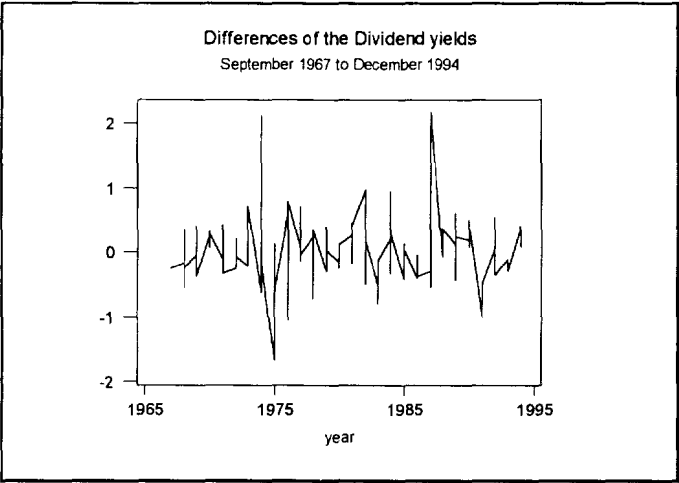
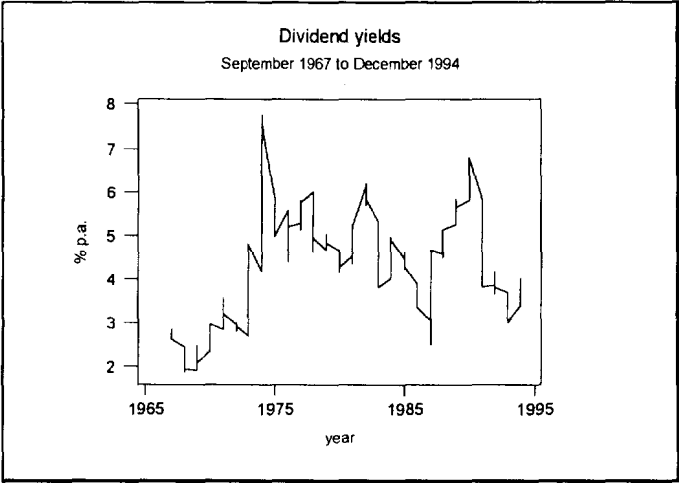
This Appendix provides time series plots of all the series and the first differences of the series for the quarterly Australian data used in this research.

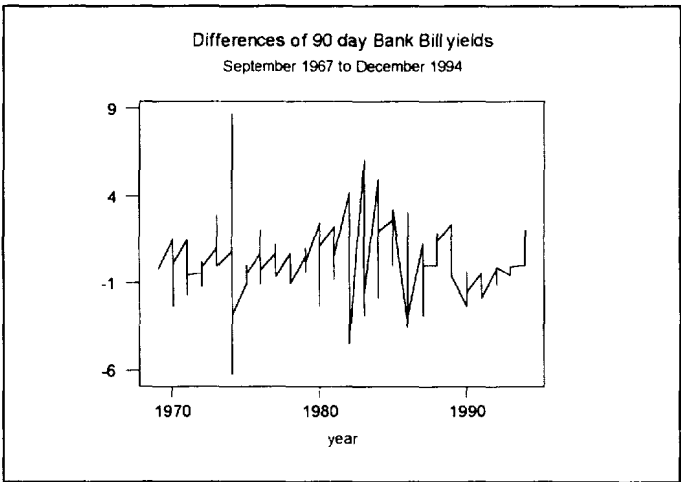
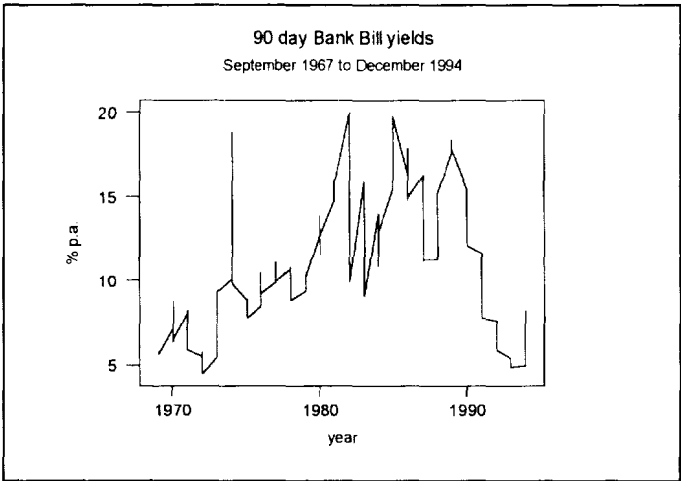


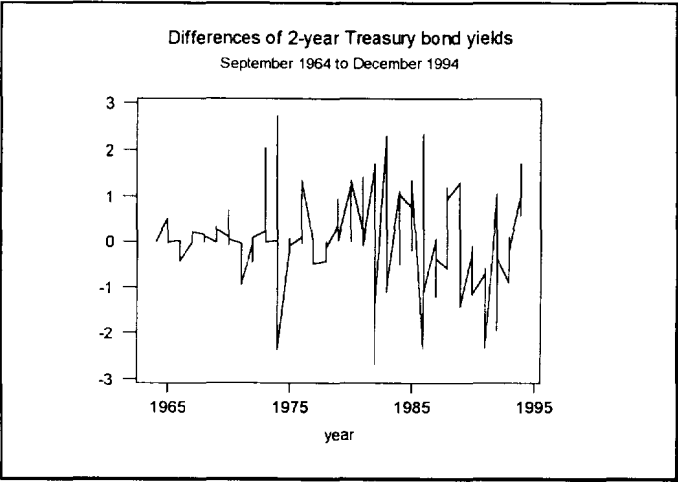
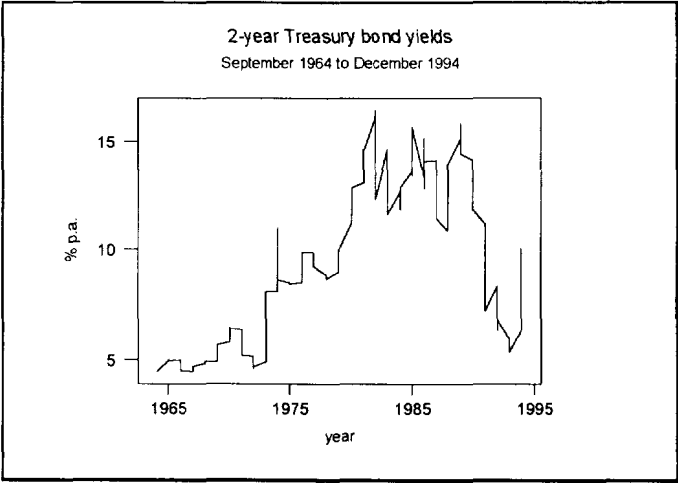


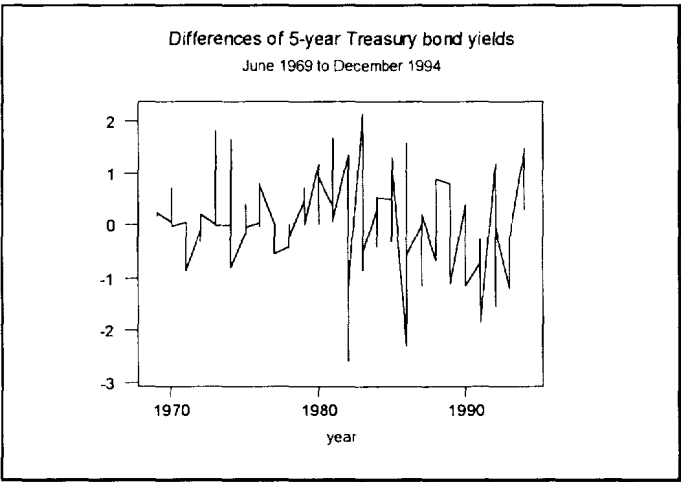
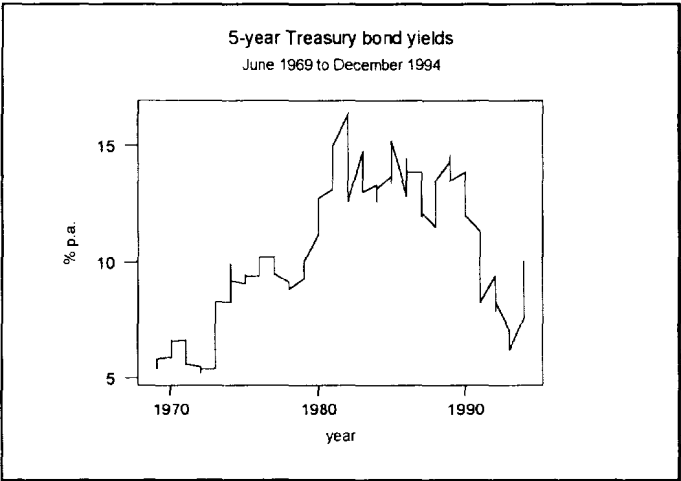




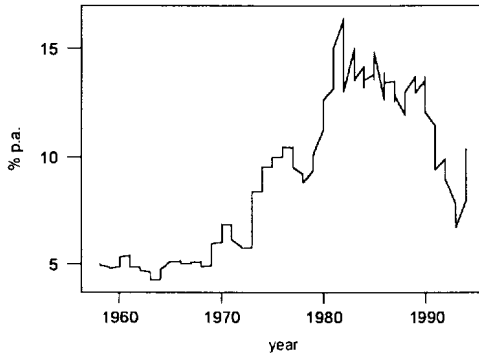




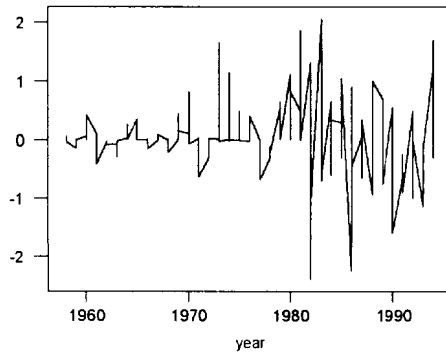




10-year Treasury bond yields
March 1958 to December 1994



Differences of 10-year Treasury bond yields
March 1958 to December 1994



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