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DECISION MAKING UNDER CONFLICTING CRITERIA FOR ACTUARIAL ASSUMPTIONS: AN EXPECTED UTILITY MODEL

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ABSTRACT

Shapiro (1990) documented some of the criteria used by actuaries when selecting the assumptions for actuarial valuations. He noted, however, that many of these criteria are often were in conflict with one another. This paper uses expected utility theory to model this process of choosing actuarial assumption when faced with potentially conflicting criteria. The three criteria considered are "prudence," "best estimate" and "conservativeness."

Keywords: Actuarial Assumptions, Criteria, Expected Utility, Prudence, Best Estimate, Conservativeness.

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1. Introduction

Defined benefit pension plan valuations must be performed periodically by the plan actuary. However, as noted by Shapiro (1990), many of the criteria underlying the choice of assumptions for these valuations are often in conflict with one another. That observation is not surprising, of course, since the actuary must balance his or her preferences and judgements and those of a number of self-interest groups, including employees, the employer, and tax and labor authorities.

This paper uses expected utility theory to model this process of choosing actuarial assumption when faced with potentially conflicting criteria. To keep the model simple, only three criteria are considered: "prudence," "best estimate" and "conservativeness." This is a first attempt at modeling this type of actuarial decision making. However, it is hoped that the essence of the concept is sufficiently well developed so as to promote future research which will enhance the decision model and include more of the criteria.

2. The Criteria

The first criterion, prudence, is satisfied if the contribution which results is in the range of prudent contributions, that is, contributions which would be developed by prudent actuaries in similar circumstances. The context considered is the one where tax authorities are concerned with the possibility of overfunding to escape current taxation and consequently define a deductible contribution as one which is below a certain upper limit.[1] Since excise taxes and other penalties may result if deductions are taken for nondeductible contributions, one limit on the range of prudent assumptions is that they produce a contribution no larger than that which would result if all the assumptions used were the most generous allowed under IRS standards. However, it is assumed that whether the plan contribution falls above the upper limit is not of concern if the plan can meet a facts and circumstance test. Moreover, this test is characterized in terms of the relationship between the actual contribution and the contribution which would have resulted in no actuarial gains or losses.

The notation is as follows: the actual contribution to the plan is denoted $\hat{C}$, the contribution which would have produced no actuarial gains or losses is denoted C and the contribution which triggers a red flag with respect to deductibility is denoted $\hat{C}^*$. It is assumed the authorities do not investigate the assumptions to determine if they were appropriate unless $\hat{C} > \hat{C}^*$. If $\hat{C} > \hat{C}^*$, then the authorities determine whether $\hat{C} - C > D$, where D is an acceptable deviation.

If $\hat{C} > \hat{C}^*$ and $\hat{C} - C > D$, the actuary is penalized. The penalty is modeled here as a monetary penalty of P dollars. This could represent anything ranging from a fine to damage to one's reputation which would reduce earning power. An excise tax may be levied upon the plan sponsor which may have repercussions for the actuary in terms of compensation,

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[1]The case where, for example, the Pension Benefit Guaranty Corporation (PBGC) is concerned with the adequacy of plan funding, the range defined by a lower limit on the contribution may be equally important. In this model, concerns about plan solvency are captured by the conservativeness criterion. It is assumed that plan solvency is in the interest of the actuary and the plan sponsor and is imbedded in the utility function, and further pressure by the PBGC is not considered at this stage.

job security or future job prospects, or the actuary may face a lawsuit and possible loss of accreditation.[2] Furthermore, as will be stated more clearly in what follows, it is assumed that the damage to the actuary's reputation also leaves the actuary with a lower level of utility for any given wealth level in the event that the actuary is penalized.

The prudent actuary rule is characterized by a variable, ρ, representing the penalty, which takes on the following values:

$$\rho = 0 \qquad \text{if } \hat{C} \leq \hat{C}^{*}$$
$$\text{or}$$
$$\text{if } \hat{C} > \hat{C}^{*} \text{ and } \hat{C}\text{-}C \leq D$$

$$\rho = P \qquad \text{if } \hat{C} > \hat{C}^{*} \text{ and } \hat{C}\text{-}C > D.$$

Deductibility raises a perplexing problem. Solvency is one of the primary considerations underlying the funding of a pension plan, but the taxing authority may not explicitly allow a contingency reserve to protect this solvency. Additionally, as modeled above, there may be an arbitrarily limit to the maximum deductible contribution to a plan. Because of adverse experience, however, it may turn out that the deductible contribution is not sufficient to keep the plan solvent.

The second criterion, "conservativeness", follows from this concern about plan solvency. The actuary, as well as the plan sponsor, prefers to keep the probability that the plan has actuarial losses to a minimum. Therefore, the contribution is conservative if $\Pr[(\hat{C}-C)<0]<\epsilon$, or equivalently, $\Pr[C>\hat{C}]<\epsilon$, where ϵ is the tolerance level for conservativeness (that is, if the probability of actuarial losses is below ϵ). The actuary can use only his or

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[2]The simplifying assumption is made that the size of the monetary penalty is not a function of the size of the contribution or the experience of the plan. It is quite possible that these and other factors may have an impact on the size of the penalty, in which case the penalty would not be a constant.

her beliefs about the distribution of C to determine $\Pr[C>\hat{C}]$. Therefore, the actuary, concerned about conservativeness, prefers a contribution, $\hat{C}$, for which $[1-F(\hat{C})]<\epsilon$.

The final criterion which is incorporated into the model is the criterion of "best estimate". For this analysis, best estimate is interpreted to mean the estimate for which the expected value of the absolute deviation from the estimate is minimized as stated in Anderson (1985, p. 110). Again, the plan experience is characterized in terms of C, the contribution which would have resulted in no actuarial gains or losses. The actuary's "best estimate" of C is μ in that the actuary believes that, if C^E is defined as any estimate of C, then $E[|C^E-C|]$ is minimized when the estimate $C^E = \mu$. Furthermore, the actuary believes that C has probability density function f(C) and corresponding cumulative distribution function F(C). This is the actuary' subjective belief about the distribution of C and is necessary if expected utility theory is to be used.

3. The Expected Utility Model

At this point, the actuary's decision process is modeled explicitly using the theory of expected utility. It is assumed that the actuary obtains utility from two sources, wealth and the appropriateness of his or her assumptions. The appropriateness of the assumptions may affect wealth through a potential penalty, but apart from that, the actuary simply feels good about making an appropriate estimate and enjoys positive recognition from his or her employer.[3] The two aspects of the appropriateness of the assumptions that are modeled here are accuracy and conservativeness. The accuracy (or inaccuracy) of the assumptions is measured by $|\hat{C}-C|$, with smaller values representing greater accuracy. The larger the

[3]It is assumed that neither the actuary nor the employer is motivated to overfund the plan for the specific purpose of deferring taxation.

value of $|\hat{C}-C|$, the lower the actuary's utility, and this inaccuracy is weighted by a constant λ in the utility function. But, due to concerns about solvency and conservativeness, the actuary's utility is reduced further when there are actuarial losses. Therefore, when $\hat{C}-C<0$, the actuary has additional disutility equal to a constant γ. The actuary's disutility due to actuarial losses is characterized by a variable Γ which takes on the following values:

$\Gamma = 0$ if $\hat{C}-C \geq 0$

$\Gamma = \gamma$ if $\hat{C}-C<0$.

The actuary's utility function can be denoted $U(W-\rho,|\hat{C}-C|,\Gamma)$, where W represents his or her wealth before the penalty is determined.[4] Assuming additivity,[5] this utility function can be written more explicitly as:

$$(1) \quad U(W-\rho,|\hat{C}-C|,\Gamma) = u_s(W-\rho)-\lambda[|\hat{C}-C|]-\Gamma,$$

where $u_s(W-\rho)$ is the utility of wealth in state s, and where the state represents whether or not the actuary is penalized by the authorities. Let $s=0$ represent the state where the actuary is not penalized and $s=1$ represent the state where the actuary is penalized. Utility increases with wealth (so $u_s'(\cdot)>0$) and risk aversion with respect to wealth implies that $u_s''(\cdot)<0$. It is assumed that $u_0(w)>u_1(w)$ for any given wealth level w. This implies that the actuary suffers more than just a monetary fine when penalized by the authorities. Once sanctioned by the authorities, the actuary is worse off, in utility terms, at any given wealth level.

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[4]A more general representation of the actuary's utility function would be $U(Y(W,\rho),|\hat{C}-C|,\Gamma)$, where Y and W represent wealth after and before the penalty, respectively, and ρ represents an arbitrary penalty function.

[5]The authors are currently investigating a more general formulation of this utility function. However, the simplified version in the text is sufficient to convey the essence of the model.

In choosing the contribution, Ĉ, the actuary maximizes his or her expected utility where the expectation is taken over the distribution f(C). So, the actuary solves:

$$(2a) \quad Max_{\hat{C}} \quad E\left[u_s(W-\rho)-\lambda[|\hat{C}-C|]-\Gamma\right]$$

or equivalently,

$$(2b) \quad Max_{\hat{C}} \quad E[u_s(W-\rho)] - \lambda E\left[|\hat{C}-C|\right] - \gamma[1-F(\hat{C})]$$

where $\rho \epsilon \{0,P\}$.

Each of the three terms in expression (2b) represents one of the criteria which the actuary uses in making the funding decision. The second term represents the best estimate criterion. The contribution which minimizes this term, and hence maximizes its contribution to expected utility is the best estimate, μ. But the best estimate may not be the optimal contribution for the plan due to the offsetting effects of the two other criteria. The third term, representing conservativeness, is the probability of an actuarial loss weighted by the disutility such a loss brings and this term is subtracted from expected utility. To maximize this term's contribution to expected utility, the probability of an actuarial loss [1-F(Ĉ)] must be minimized. This provides an incentive for the actuary to choose a contribution which is above the best estimate in order to "play it safe". On the other hand, the first term is the expected utility of wealth which is dependent upon whether a penalty is received from the authorities for choosing a contribution which may not be deductible.

As just stated, the actuary has an incentive to choose a contribution which is higher than the best estimate due to concerns about solvency. But government officials may choose to interpret this behavior as an attempt to avoid current taxation. This exerts pressure on the actuary to choose a lower contribution than might otherwise be chosen. In particular, this first term is maximized when the chance of receiving a penalty and the subsequent damage to the actuary's reputation is eliminated (that is, when the contribution is below the authorities' upper bound). The relative weight with which each of the three criteria enter into expected utility determines the trade-off which must be made. Other factors which determine this tradeoff are initial wealth, W, the size of the acceptable deviation, D, and the size of the penalty, P. A final factor is the actuary's perception of the distribution of C, in particular, how probable it is that the deviation will turn out to be greater than zero and/or greater than D.

To analyze this more formally, assume that the actuary believes that $C_L \leq C \leq C_U$. There are two possible ranges within which the chosen contribution, $\hat{C}$, can fall, the prudent range where $\hat{C} \epsilon [C_L, \hat{C}^*]$, or the other range where $\hat{C} \epsilon (\hat{C}^*, C_U]$.[6] A penalty is imposed when $\hat{C} \epsilon (\hat{C}^*, C_U]$ and $\hat{C}$-$C > D$. On the other hand, when $\hat{C} \epsilon [C_L, \hat{C}^*]$ there is no possibility of receiving a penalty. Therefore, analysis of the decision requires that the maximization problem given by (2b) be separated into two steps since the expected utility function is discontinuous at the point $\hat{C} = \hat{C}^*$.

There are two expected utility functions which must be considered, the one which applies for values of $\hat{C} \leq \hat{C}^*$, and the one which applies for values of $\hat{C} > \hat{C}^*$. The approach taken here will be to graph both of these expected utility functions over the entire range of potential contributions, $\hat{C}$, and illustrate how the actuary's choice is affected by the tax authorities' choice of an upper limit on the prudent range, $\hat{C}^*$.

[6] It is assumed that $[C_L, \hat{C}^*]$ is non-empty.

The expected utility function which pertains to the range of contributions $\hat{C} \leq \hat{C}^*$ is:

$$(3a) \quad EU_{\hat{C} \leq \hat{C}^*} = u_0(W) - \lambda E\left[|\hat{C}-C|\right] - \gamma[1-F(\hat{C})],$$

or, more explicitly,

$$(3b) \quad EU_{\hat{C} \leq \hat{C}^*} = u_0(W) - \lambda\left[\int_{C_L}^{\hat{C}}(\hat{C}-C)f(C)dC + \int_{\hat{C}}^{C_U}(C-\hat{C})f(C)dC\right] - \gamma[1-F(\hat{C})].$$

The expected utility function which pertains to the range of contributions $\hat{C} > \hat{C}^*$ is:

$$(4a) \quad EU_{\hat{C} > \hat{C}^*} = u_1(W-P)F(\hat{C}-D) + u_0(W)[1-F(\hat{C}-D)] - \lambda E\left[|\hat{C}-C|\right] - \gamma[1-F(\hat{C})],$$

or, more explicitly,

$$(4b) \quad EU_{\hat{C} > \hat{C}^*} = u_1(W-P)F(\hat{C}-D) + u_0(W)[1-F(\hat{C}-D)]$$
$$- \lambda\left[\int_{C_L}^{\hat{C}}(\hat{C}-C)f(C)dC + \int_{\hat{C}}^{C_U}(C-\hat{C})f(C)dC\right] - \gamma[1-F(\hat{C})].$$

Comparison of (3a) and (4a) indicates that, for any given value of $\hat{C}$ greater than C_L+D, (3a) is greater than (4a) since $u_0(W) > u_1(W-P)$.[7] Furthermore, the gap between these two functions increases as $\hat{C}$ increases since more weight is given to $u_1(W-P)$ as $F(\hat{C}-D)$ increases.

Next, the contribution which provides the maximum level of expected utility must be determined for each of these functions. Differentiating (3b) and (4b) using Leibnitz's rule, and setting each equal to zero gives the conditions for the maximum values of $EU_{\hat{C} \leq \hat{C}^*}$ and

[7]Only contributions which are greater than C_L+D are potential choices as the maximization problem is set up. Intuitively, as long as the best estimate, μ, is greater than C_L+D, as will be assumed, contributions which are less than C_L+D would not be chosen since $\hat{C}$-C cannot be greater than D implying that there are no potential penalties in this range and, therefore, no benefits to be gained from reducing the contribution below C_L+D. This further implies that $\hat{C}-D>C_L$ for any possible solution, so $F(\hat{C}-D)>0$.

$EU_{\hat{C}>\hat{C}^*}$, respectively. These are labelled (3c) and (4c), respectively:

$$(3c)\quad \frac{dEU_{\hat{C}\leq\hat{C}^*}}{d\hat{C}} = -\lambda[2F(\hat{C})-1] + \gamma f(\hat{C}) = 0$$

$$(4c)\quad \frac{dEU_{\hat{C}>\hat{C}^*}}{d\hat{C}} = [u_1(W-P)-u_0(W)]f(\hat{C}-D) - \lambda[2F(\hat{C})-1] + \gamma f(\hat{C}) = 0$$

A useful point of comparison is the best estimate μ, which is obtained by minimizing $E[|\hat{C}-C|]$, that is, by solving:

$$(5a)\quad Min_{\hat{C}} \int_{C_L}^{\hat{C}}(\hat{C}-C)f(C)dC + \int_{\hat{C}}^{C_U}(C-\hat{C})f(C)dC.$$

The first order condition for this problem is:

$$(5b)\quad [2F(\hat{C})-1] = 0.$$

The value of $\hat{C}$ which solves equation (5b) is the best estimate and is denoted μ. Note that μ is the median since $F(\mu) = ½$.

4. An Example

For the purpose of example, assume that C is uniformly distributed on the interval $[C_L, C_U]$.[8] The best estimate is:

$$(5c)\quad \mu = \frac{C_U+C_L}{2}.$$

Furthermore, the contributions which maximize $EU_{\hat{C}\leq\hat{C}^*}$ and $EU_{\hat{C}>\hat{C}^*}$, respectively, as well

[8]This assumption is not meant to imply that this is the appropriate distribution for C, but rather is used to allow a clear characterization of the kind of solution which may be obtained using this type of model.

as the shapes of these expected utility functions, can be obtained by substituting for f(C) and F(C) into (3c) and (4c).[9] Equation (3c) becomes:

$$(3c') \quad \frac{dEU_{\hat{C}\leq\hat{C}^*}}{d\hat{C}} = -\lambda\left[\frac{2(\hat{C}-C_L)}{(C_U-C_L)}-1\right]+\frac{\gamma}{(C_U-C_L)} = 0$$

and equation (4c) becomes:

$$(4c') \quad \frac{dEU_{\hat{C}>\hat{C}^*}}{d\hat{C}} = \frac{[u_1(W-P)-u_0(W)]}{(C_U-C_L)}-\lambda\left[\frac{2(\hat{C}-C_L)}{(C_U-C_L)}-1\right]+\frac{\gamma}{(C_U-C_L)} = 0.$$

First note that both $EU_{\hat{C}\leq\hat{C}^*}$ and $EU_{\hat{C}>\hat{C}^*}$ are strictly concave, since the second derivative of each is negative. To see this, note that the derivatives of the left hand sides of (3c′) and (4c′) are identical and are equal to:

$$(6) \quad \frac{-2\lambda}{(C_U-C_L)} < 0.$$

Next, let the solution to (3c′) be denoted $\hat{C}_3$ and the solution to (4c′) be denoted $\hat{C}_4$.[10] Then $EU_{\hat{C}\leq\hat{C}^*}$ is maximized at:

$$(3d) \quad \hat{C}_3 = \mu+\frac{\gamma}{2\lambda}$$

and $EU_{\hat{C}>\hat{C}^*}$ is maximized at:

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[9]It is assumed that the contribution which maximizes $EU_{\hat{C}\leq\hat{C}^*}$ and the contribution which maximizes $EU_{\hat{C}>\hat{C}^*}$ are each an element of the interval $[C_L,C_U]$.

[10]Note that both of these points exist and are unique since the functions which are being maximized have been shown to be concave.

$$(4d)\quad \hat{C}_4 = \mu + \frac{\gamma}{2\lambda} - \frac{[u_0(W)-u_1(W-P)]}{2\lambda}.$$

So, in this example, when there is no concern about a penalty by the authorities, as is the case when $\hat{C} \leq \hat{C}^*$, the optimal contribution, $\hat{C}_3$, is the best estimate, μ, plus a contingency reserve equal to one half of the relative disutility of insolvency (that is, disutility, γ, relative to the weight given to accuracy, λ). When there is the possibility of a penalty, due to the fact that the contribution is not in the prudent range, then the optimal contribution, $\hat{C}_4$, is $\hat{C}_3$ reduced by one half of the relative disutility of being penalized (that is, the change in utility caused by a penalty, $[u_0(W)-u_1(W-P)]$, relative to the weight given to accuracy, λ).

It is now possible to graph the two expected utility functions, $EU_{\hat{C} \leq \hat{C}^*}$ and $EU_{\hat{C} > \hat{C}^*}$, over the range of potential contributions to illustrate how the value of $\hat{C}^*$ impacts upon the actuary's funding choice. The following characteristics of the expected utility functions have been determined from the above analysis. $EU_{\hat{C} \leq \hat{C}^*}$ is greater than $EU_{\hat{C} > \hat{C}^*}$ for any given value of $\hat{C}$, and the difference between these two functions increases as $\hat{C}$ increases. Both $EU_{\hat{C} \leq \hat{C}^*}$ and $EU_{\hat{C} > \hat{C}^*}$ are strictly concave, and $EU_{\hat{C} \leq \hat{C}^*}$ reaches its maximum value at a higher contribution level than $EU_{\hat{C} > \hat{C}^*}$ does (since $\hat{C}_3 > C_4$).

The graph in Figure 1 has been constructed from the foregoing observations. An important point on this graph is the lowest contribution level at which $EU_{\hat{C} \leq \hat{C}^*}$ is exactly equal to the maximum value of $EU_{\hat{C} > \hat{C}^*}$. This contribution level is denoted $\hat{C}_5$ (that is,

$EU_{\hat{C}\le\hat{C}^*}(\hat{C}_5) = EU_{\hat{C}>\hat{C}^*}(\hat{C}_4))$.[11] It is necessary to know the value of $\hat{C}^*$ to determine which expected utility function is relevant over which range of contributions. $EU_{\hat{C}\le\hat{C}^*}$ is the applicable expected utility function for all contributions below $\hat{C}^*$, and $EU_{\hat{C}>\hat{C}^*}$ is applicable for contributions above $\hat{C}^*$. If $\hat{C}^*$ greater than $\hat{C}_3$, then $\hat{C}_3$ is the contribution which is chosen by the actuary since expected utility is maximized at that point. If $\hat{C}_5 < \hat{C}^* \le \hat{C}_3$, then the optimal contribution is $\hat{C}^*$. If $\hat{C}^* = \hat{C}_5$, then the actuary is indifferent between $\hat{C}^*$ and $\hat{C}_4$. Finally, if $\hat{C}^* < \hat{C}_5$, then the optimal contribution is $\hat{C}_4$. These results are depicted in Figure 1, below the horizontal axis.

This graph indicates that when the upper bound on the prudent range is relatively high (that is, higher than the expected utility maximizing contribution in the absence of a penalty), then the constraint provided by the authorities is not binding and the actuary's choice ($\hat{C}_3$) is a tradeoff between the criteria of best estimate and conservativeness. When the upper bound on the prudent range is in some middle range (that is, $\hat{C}_5 < \hat{C}^* \le \hat{C}_3$), then the actuary chooses the upper bound as the optimal contribution since it is preferable to avoid the possibility of a penalty altogether. When the upper bound on the prudent range is relatively low (that is, below $\hat{C}_5$), then it is in the actuary's best interest to choose a contribution above the prudent range which makes a tradeoff between the criteria of conservativeness, prudence and best estimate. In this case, the chosen contribution is $\hat{C}_4$. If $\hat{C}_5$ is less than C_L then it is necessarily less than $\hat{C}^*$ implying that for any $\hat{C}^* \in [C_L, C_U]$, the

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[11] Solving for $\hat{C}_5$ explicitly gives:

$$\hat{C}_5 = \mu + \frac{\gamma}{2\lambda} - \sqrt{\left(\mu + \frac{\gamma}{2\lambda}\right)^2 - \hat{C}_4^2}.$$

It is possible that the value $\hat{C}_5$ is less than C_L, in which case it does not appear on the graph. The following analysis will make clear that, if this is the case, the optimal contribution must be in the prudent range.

optimal contribution, $\hat{C}$, is in the prudent range.

5. Closing Comments

The purpose of this paper has been to explore the use of expected utility theory to model the process by which an actuary chooses the appropriate contribution for a pension plan. Since this was just a first attempt, only a simple model was used and only a limited number of criterion were considered. Nonetheless, we were able to conceptualize the essence of some of the relationships. The paper will have served its purpose it provides a basis for further discussion and research.

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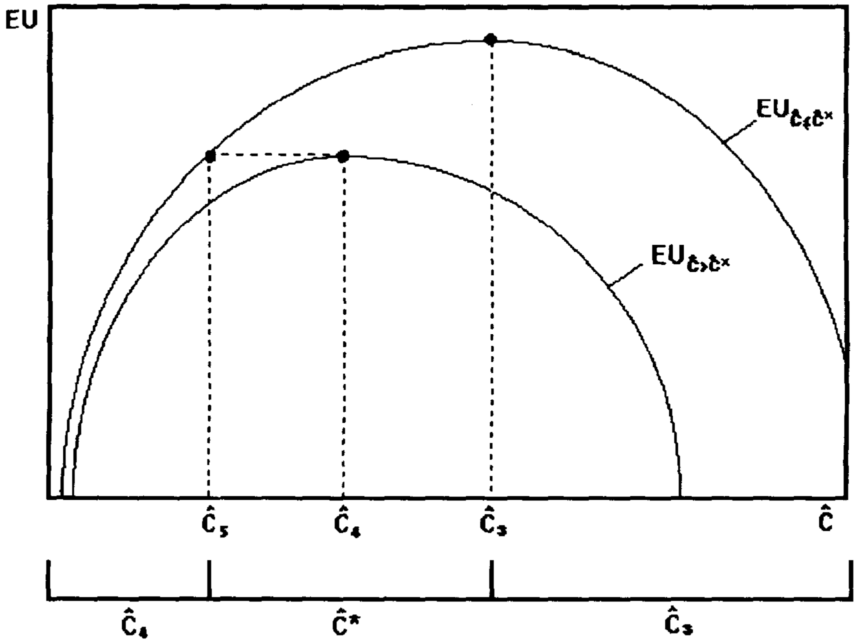
Figure 1



EU
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